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Research Article

Quasilinearization method to solve several classes of nonlinear Lane–Emden equations by Gegenbauer polynomials

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Abstract

This paper presents a comprehensive numerical approach for solving various types of singular nonlinear Lane–Emden equations. The proposed method begins by applying the quasilinearization method, to transform the nonlinear differential equation into a sequence of linear equations. Subsequently, these linear equations are solved using the collocation method with Gegenbauer functions. A notable advantage of Gegenbauer polynomials lies in their adjustable parameter, when appropriately selected; this parameter enhances their effectiveness in obtaining numerical solutions for specific problems. After establishing the convergence of the method, we present numerical examples to demonstrate its effectiveness. A comparison of the results with those obtained using other basis functions highlights the superior performance and applicability of these polynomials.

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1 Introduction

We investigate the following nonlinear differential equations, which fall under the classification of Lane–Emden type equations:

$$u''(x) + \frac{\tau}{x}u'(x) + g(x,u) = z(x), \qquad x \geq 0,\ \tau \geq 1, \tag{1}$$

subject to the initial conditions

$$u(0) = \lambda, \qquad u'(0) = 0. \tag{2}$$

It should be noted that $g(x,u)$ is continuous and z belongs to $C[0,1]$. These equations incorporate various specific forms of $g(x,u)$, which model a wide range of problems in mathematics, physics, and astrophysics, including phenomena such as thermal explosions and stellar structures, among others [5, 18]. This type of equation is named after the astrophysicists Jonathan Homer Lane and Robert Emden. It describes the variation of density as a function of radial distance for a polytrope.

Nowadays, various scientific problems are formulated in infinite or semi-infinite domains [2]. To address these challenges, several spectral methods have been proposed [2, 3]. Additionally, a variety of approaches have been introduced in the literature for solving these types of equations. The emphasis on numerical methods stems from the challenges associated with obtaining accurate and efficient analytical solutions. In many scientific applications, the advantages of these methods over analytical or semi-analytical approaches have been clearly demonstrated.

Researchers have employed various numerical methods to solve these equations. For instance, Wazwaz [21] applied the Adomian decomposition method; Mandelzweig and Tabakin [13] utilized the quasilinearization method; Yousefi [22] implemented the Legendre wavelets method; Parand and Khaleghi [16] used the rational Chebyshev (second kind) collocation method; Bhrawy and Alofi [1] applied the Jacobi–Gauss collocation method, and Nasab et al. [12] employed the Chebyshev wavelet finite difference method.

The advantages of Gegenbauer polynomials include their flexibility, high accuracy, and adaptability to a wide range of problems.

• These polynomials are characterized by a tunable parameter, which allows them to be tailored

to specific applications, such as solving differential equations or approximating functions.
• Their orthogonal properties make them particularly effective for spectral methods, ensuring rapid convergence.
• Additionally, Gegenbauer polynomials can handle singularities and boundary conditions more effectively than many other basis functions.
They are a powerful tool in computational mathematics and physics [14, 19].

Since no prior work has explored the use of Gegenbauer polynomials as basis functions in the quasilinearization method (QLM), we propose a novel approach for solving Lane–Emden equations by integrating the QLM with these polynomials as basis functions.

Our approach involves a systematic pattern for solving Lane–Emden equations. Initially, the parameter of the Gegenbauer polynomials is estimated to achieve high-accuracy approximate solutions for problems with known exact solutions. Subsequently, this parameter, in conjunction with the QLM, is employed to derive approximate solutions for a broader class of equations. After establishing the convergence of the method, we present several illustrative examples to demonstrate its effectiveness. Finally, we compare our results with those obtained using alternative methods, highlighting the efficiency and applicability of the proposed scheme.

This study is structured as follows:
In Section 2, the definitions and preliminary concepts used throughout the article are briefly outlined. The proposed technique is described in Section 3. In Section 4, our scheme is applied to various numerical examples. Finally, the conclusion is presented in the last section.

2 Preliminaries

We provide a brief introduction to the definitions and formulations of polynomials used in the method outlined in this article. Furthermore, we discuss several types of Lane–Emden equations.

2.1 Some types of polynomials

2.1.1 Bernoulli polynomials

Initially, Bernoulli numbers [6] can be derived using the following expression:

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}\delta_n\frac{x^n}{n!}. \tag{3}$$

It is also evident that

$$\delta_{2k+1}=0, \qquad k=1,2,3,\ldots.$$

The Bernoulli polynomials, named after Jacob Bernoulli, of order n are combinations of Bernoulli numbers and binomial coefficients, and are expressed as follows [17, 4]:

$$\beta_n(x)=\sum_{k=0}^{n}\binom{n}{k}\delta_{n-k}x^k, \qquad n=0,1,\ldots, \tag{4}$$

where δ_{n-k} represents the Bernoulli numbers.

2.1.2 Fractional order of rational Bernoulli functions

The rational Bernoulli functions, for $L>0$, are defined as follows: [4]:

$$RB_m(x;L)=\beta_m\Big(\frac{x}{x+L}\Big)=\sum_{k=0}^{n}\binom{n}{k}\delta_{n-k}(\frac{x}{x+L})^k, \qquad m=0,1,2,\ldots. \tag{5}$$

By applying the algebraic mapping $x{\to}x^{\alpha}$, these functions can be expressed in fractional order as follows [17]:

$$FRB_n(x,L)=R_n(x^{\alpha},L)=\sum_{k=0}^{n}\binom{n}{k}\beta_k\Big(\frac{x^{\alpha}}{x^{\alpha}+L}\Big)^{n-k}, \qquad n=0,1,\ldots. \tag{6}$$

2.1.3 Rational Bessel functions

The solutions to Bessel's differential equation of order n are known as Bessel functions. The equation is expressed as follows:

$$x^2\frac{d^2u(x)}{dx^2}+x\frac{du(x)}{dx}+(x^2-n^2)u(x)=0, \qquad x{\in}(-\infty,\infty). \tag{7}$$

The Bessel polynomials, which form an orthogonal sequence of polynomials [15], are defined as follows:

$$B_n(x)=\sum_{r=0}^{\lfloor\frac{N-n}{2}\rfloor}\frac{(-1)^r}{r!(n+r)!}\Big(\frac{x}{2}\Big)^{2r+n}, \qquad x{\in}[0,1], \tag{8}$$

where $n \in \mathbb{N}$, and the number of bases of the Bessel polynomials is denoted as N.

Rational Bessel functions (RBs), derived using the algebraic mapping $x \rightarrow \dfrac{x}{x+L}$ in (8), are expressed as follows:

$$RB_n(x;L) = B_n(\frac{x}{x+L}) = \sum_{r=0}^{\lfloor \frac{N-n}{2} \rfloor} \frac{(-1)^r}{r!(n+r)!}\Big(\frac{x}{2(x+L)}\Big)^{2r+n}, \quad n = 0,1,\ldots,N. \tag{9}$$

2.1.4 Exponential Bessel functions

The exponential Bessel functions (EBs), similar to the previous definition [15], are derived from (8) using the algebraic mapping $x \rightarrow 1 - e^{-\dfrac{x}{L}}$. They are defined as follows:

$$EB_n(x;L) = B_n(1 - e^{-\dfrac{x}{L}}) = \sum_{r=0}^{\lfloor \frac{N-n}{2} \rfloor} \frac{(-1)^r}{r!(n+r)!}\Big(1 - e^{-\dfrac{x}{L}}\Big)^{2r+n}, \quad n = 0,1,\ldots,N. \tag{10}$$

In both definitions, the parameter L is a constant and acts as the scaling factor for the mapping.

2.1.5 Gegenbauer polynomials

Gegenbauer polynomials, also known as ultra-spherical polynomials $C_j^{\alpha}(x)$, are defined for non-negative integers $j \in \mathbb{Z}^+$ and are associated with the parameter $\alpha > -1/2$. These polynomials are named after Leopold Gegenbauer. Their analytical form is given by

$$C_j^{\alpha}(x) = \sum_{k=0}^{j} (-1)^{j-k} \frac{j!\Gamma(\alpha+1/n)\Gamma(j+k+2\alpha)}{\Gamma(k+\alpha+1/2)\Gamma(j+2\alpha)(j-k)!k!L^k}(\frac{x+1}{2})^k, \tag{11}$$

where $j = 0,1,2,\ldots$.
When $\alpha = 0$, the Gegenbauer polynomial $C_j^0(x)$ reduces to the Chebyshev polynomials of the first kind, denoted as $T_j(x)$. For $\alpha = \frac{1}{2}$, the polynomial $C_j^{\frac{1}{2}}(x)$ corresponds to the Legendre polynomials $L_j(x)$. Furthermore, when $\alpha = 1$, the polynomial $C_j^1(x)$ is equal to $\frac{1}{j+1}U_j(x)$, where $U_j(x)$ represents the Chebyshev polynomials of the second type [9].

2.2 Some types of Lane–Emden equations

Lane–Emden type equations play a significant role in various fields of science and technology, leading to different forms of $g(x,u)$ and $z(x)$. For the case where $g(x,u) = u^m(x)$, $\tau = 2$, $z(x) = 0$, $\lambda = 1$ in (1) and (2), we obtain the standard Lane–Emden equation. This equation is used to model the thermal behavior of a spherical gas cloud due to the mutual attraction of its molecules, following the principles of classical thermodynamics. It is expressed as follows:

$$u''(x) + \frac{2}{x}u'(x) + u^m(x) = 0, \qquad u(0) = 1, \qquad u'(0) = 0.$$

These equations have analytical solutions for $m = 0, 1, 5$. Another variant of the Lane–Emden equation, introduced by Bonner, is characterized by $\tau = 2$, $g(x,u) = e^{u(x)}$, $z(x) = 0$, and $\lambda = 0$. This variant is used to model isothermal gas spheres and is expressed as follows:

$$u''(x) + \frac{2}{x}u'(x) + e^{u(x)} = 0, \qquad u(0) = 0, \qquad u'(0) = 0.$$

This equation models frictional gas spheres embedded in a pressurized medium, representing the maximum mass for hydrostatic equilibrium. Due to its physical similarities with the first type of Lane–Emden equation, it is often referred to as the second type of Lane–Emden equation.

3 Description of the present technique

3.1 The quasilinearization method

QLM is a powerful method for solving nonlinear differential equations. The process involves the following key steps:

• **Linearization:** At each iteration, the nonlinear components of the differential equation are linearized around the current approximate solution, often using a Taylor series expansion.

• **Iteration:** The linearized equation is solved to obtain a new approximate solution. This method is applied iteratively, generating a sequence of approximations that progressively converge to the accurate solution of the nonlinear problem.

• **Convergence:** Under certain conditions, the sequence of solutions converges to the exact solution of the original nonlinear differential equation.

• **Applications:** QLM is widely used in fields such as physics, engineering, and applied mathematics, especially for problems where direct methods are challenging to apply.

This method is particularly effective for solving initial value or boundary value problems and can be efficiently implemented using numerical techniques.

Kalaba and Bellman [11] were the pioneers of the QLM. They developed this method as a systematic strategy to address nonlinear problems by leveraging the tools of linear analysis. Over the decades, the technique has found widespread application in fields such as engineering, physics, and economics, particularly in scenarios where nonlinear behaviors are dominant [3].

In this section, we focus our discussion on nonlinear ordinary differential equations involving a single variable, defined over the semi-infinite interval $[0,\infty)$, as presented below:

$$L^n u(x) = f\Big(u(x), u^{(1)}(x), u^{(2)}(x), \dots, u^{(n-1)}(x), x\Big), \tag{12}$$

under the initial conditions

$$p_k\Big(u(0), u^{(1)}(0), \dots, u^{(n-1)}(0)\Big) = 0, \qquad k = 1, \dots, l, \tag{13}$$

and

$$p_k\Big(u(b), u^{(1)}(b), \dots, u^{(n-1)}(b)\Big) = 0, \qquad k = l+1, \dots, n. \tag{14}$$

In this context, L^n denotes a linear nth order ordinary differential operator, while f and p_i, for $i = 1, \dots, n$, are nonlinear functions of $u(x)$ and its derivatives. This technique generates the iterative estimate $u_{r+1}(x)$ to the solution of (12) as a solution to the subsequent linear differential equation:

$$\begin{aligned} L^n u_{r+1}(x) =& f\Big(u_r(x), u_r^{(1)}(x), \dots, u_r^{(n-1)}(x), x\Big) \\ &+ \sum_{s=0}^{n-1} \Big(u_{r+1}^{(s)}(x) - u_r^{(s)}(x)\Big) f_{u^{(s)}}\Big(u_r(x), u_r^{(1)}(x), \dots, u_r^{(n-1)}(x), x\Big), \end{aligned} \tag{15}$$

with the following linearized initial conditions:

$$\begin{aligned} \sum_{s=0}^{n-1} \Big(u_{r+1}^{(s)}(0) - u_r^{(s)}(0)\Big) p_{k_{u^{(s)}}}\Big(u_r(0), u_r^{(1)}(0), \dots, u_r^{(n-1)}(0), 0\Big) = 0,& \\ k = 1, \dots, l,& \end{aligned} \tag{16}$$

and

$$\begin{aligned} \sum_{s=0}^{n-1} \Big(u_{r+1}^{(s)}(b) - u_r^{(s)}(b)\Big) p_{k_{u^{(s)}}}\Big(u_r(b), u_r^{(1)}(b), \dots, u_r^{(n-1)}(b), b\Big) = 0,& \\ k = l+1, \dots, n,& \end{aligned} \tag{17}$$

where $u_r^{(0)}(x) = u_r(x)$. The initial approximation $u_0(x)$ is chosen based on mathematical or physical principles. Additionally, the following functions

$$f_{u^{(s)}} = \frac{\partial^s f}{\partial u^{(s)}},$$

and

$$p_{k_{u^{(s)}}} = \frac{\partial^s p_k}{\partial u^{(s)}},$$

where $s = 1, 2, \ldots, n-1$, represent the partial derivatives of the respective functions as follows:

$$f(u_r(x), u_r^{(1)}(x), \ldots, u_r^{(n-1)}(x), x),$$

$$p_{k_{u^{(s)}}}(u_r(x), u_r^{(1)}(x), \ldots, u_r^{(n-1)}(x), x).$$

3.2 Application of the methods

Consider the Lane–Emden equation (1), subject to the initial conditions specified in (2). The proposed approach is based on the QLM, which transforms the original nonlinear equation (1) into a sequence of linear equations.
The Lane–Emden type equations are formulated as follows:

$$u''(x) + \frac{\tau}{x}u'(x) + g(x,u) = z(x), \qquad \tau > 1,\ x \geq 0, \tag{18}$$

under the initial conditions

$$u(0) = \lambda, \qquad u'(0) = 0. \tag{19}$$

Applying the QLM to the Lane–Emden type equations yields an iterative estimate, denoted as $u_{r+1}(x)$, which is expressed as follows:

$$\begin{aligned} &u''_{r+1}(x) + \frac{\tau}{x}u'_r(x) + f(x, u_r(x)) \\ &- (u_{r+1}(x) - u_r(x)) f_u(x, u_r(x)) + \frac{\tau}{x}(u'_{r+1}(x) - u'_r(x)) = 0, \end{aligned} \tag{20}$$

under the initial conditions:

$$u_{r+1}(0) = \lambda, \qquad u'_{r+1}(0) = 0, \tag{21}$$

We begin by selecting a reasonable initial approximation for the function $y(x)$, denoted as $y_0(x)$. Here, we set $y_0(x) = A$. Since $y_0(x)$ is a known function of x, (20) is reformulated into a linear differential equation with variable coefficients. The proposed solution for the $(r+1)$th estimate is initially defined as follows:

$$S_{n,r}(x) = \sum_{i=0}^{n} a_{i,r} f_i(x; L), \tag{22}$$

where $f_i(x; L)$ represent the basic functions, n is a positive integer, and $a_{i,r}$ are the unknown coefficients. The basis functions, defined in Section 2, are employed in the collocation method.

Our goal is to determine the coefficients $a_{i,r}$ as Gegenbauer functions. The initial conditions in (20) are then satisfied through the following relation:

$$u_{n,r+1}(x) = \lambda + x^2 S_{n,r}(x). \tag{23}$$

By substituting (23) in (20), the residual function at each iteration of the QLM is formulated as follows:

$$\begin{aligned} Res_r(x) =& u''_{n,r+1}(x) + \frac{\tau}{x} u'_{n,r}(x) + f(x, u_{n,r}(x)) - z(x) \\ &- (u_{n,r+1}(x) - u_{n,r}(x)) f_u(x, u(x)) + \frac{\tau}{x}(u'_{n,r+1}(x) - u'_{n,r}(x)). \end{aligned} \tag{24}$$

Next, we collocate (24) at the points provided in [17]:

$$x_t = L \left(\frac{1 - \cos(\frac{(2j-1)\pi}{(N+1)2})}{2} \right)^{\frac{1}{\gamma}}, \qquad t = 1, 2, \ldots, N+1, \tag{25}$$

where these points correspond to the roots of the generalized first-kind Chebyshev functions on the range $[0, L]$, with the parameter γ for the collocation points set to $\frac{1}{2}$. By substituting (25) into (24), $N+1$ algebraic equations can be obtained and solved. We obtain $N+1$ algebraic equations, which can then be solved. The primary objective of the collocation method is to determine the unknown coefficients $a_{i,r}$, by minimizing the residual function.

3.3 Convergence and existence of the method

In this section, the issues of convergence and existence will be briefly discussed [20]. It has been demonstrated that the sequence of functions $u_r(x)$ converges to the exact solution of the original equation when the value of x_c, or the interval $[0, x_c]$, is sufficiently small.

To demonstrate that the presented technique achieves quadratic convergence to the solution of (12) with the boundary conditions specified in (13), we simplify our analysis to the second-order differential equation specified given by

$$u''_{r+1}(x) = f(u_r(x), x) + (u_{r+1}(x) - u_r(x)) f_{u_r}(u_r(x), x). \tag{26}$$

The boundary conditions for this equation are defined as follows:

$$u_{r+1}(0) = u_{r+1}(x_c) = 0. \tag{27}$$

For further details, refer to [11].

3.4 Existence

To prove the existence of the sequence defined by (15), we use an integral representation to express the solution to this system. This can be achieved by applying Green's function. Specifically, the Green's function $G(x, \theta)$ for the homogeneous system described by

$$u''_{r+1}(x) = 0, \qquad u_{r+1}(0) = u_{r+1}(x_c) = 0, \tag{28}$$

can be easily derived as follows [7, 8]:

$$G(x, \theta) = \frac{x(\theta - x_c)}{x_c}, \qquad 0 \leq x \leq \theta, \tag{29}$$

$$G(x, \theta) = \frac{\theta(x - x_c)}{x_c}, \qquad \theta \leq x \leq x_c. \tag{30}$$

If we express (26) in the form

$$u''_{r+1}(x) = f(x), \tag{31}$$

then (26) can be represented by the following linear integral equation:

$$u_{r+1}(x) = \int_0^{x_c} G(x, \theta)[f(u_r(x), x) + (u_{r+1}(x) - u_r(x)) f_{u_r}(u_r(x), x)] d\theta. \tag{32}$$

To estimate the bound on $u_{r+1}(x)$, we consider the absolute values in the integral representation (32) as follows:

$$\begin{aligned} |u_{r+1}(x)| \leq \int_0^{x_c} & |G(x, \theta)| [|f(u_r(x), x)| + |u_r(x)||f_{u_r}(u_r(x), x)| \\ & + |u_{r+1}(x)||f_{u_r}(u_r(x), x)|] d\theta. \end{aligned} \tag{33}$$

Let us denote the maximum values of $\max|f(u_r(x),x)|$ and $\max|f_{u_r}(u_r(x),x)|$ by M. Thus, we can rewrite the inequality as follows:

$$|u_{r+1}(x)|\leq \int_0^{x_c} |G(x,\theta)|[M+M|u_r(x)|+M|u_{r+1}(x)|]d\theta. \tag{34}$$

Now, assuming that $|u_0(x)|\leq 1$ for $r=0$, we can simplify (34) to

$$|u_1(x)|\leq \int_0^{x_c} |G(x,\theta)|[2M+M|u_1(x)|]d\theta. \tag{35}$$

This inequality provides a way to estimate the bound on $|u_1(x)|$ in terms of the Green's function and the maximum values of the functions involved. To further analyze this, we can isolate $|u_1(x)|$ on one side of the inequality and derive a more explicit bound.

From (29) and (30), we can determine that

$$max|G(x,\theta)| = \frac{x_c}{4}. \tag{36}$$

Using this result, we can rewrite the bound on $|u_1(x)|$ as follows:

$$|u_1(x)|\leq \frac{x_c}{4}\int_0^{x_c} [2M+M|u_1(x)|]d\theta. \tag{37}$$

Integrating both sides of (37) and solving it for $|u_1(x)|$, give

$$|u_1(x)|\leq \frac{M\dfrac{x_c^2}{2}}{1-M\dfrac{x_c^2}{4}}. \tag{38}$$

If we choose x_c to be sufficiently small, then we can ensure that $|u_1(x)|$ remains bounded. Specifically, if

$$x_c^2\leq \frac{4}{3M},$$

then it follows that

$$|u_1(x)|\leq 1.$$

We can apply the same procedure iteratively to $u_2(x), u_3(x), \ldots,$ and conclude that, under the condition $x_c^2\leq \dfrac{4}{3M}$,

$$u_r(x)\leq 1, \qquad 0\leq x\leq x_c.$$

Thus, we can conclude that the sequence $u_r(x)$ is bounded for sufficiently small intervals $[0,x_c]$.

3.5 Convergence

To prove the quadratic convergence to the solution of (26) and (27), we rewrite (26) in a more explicit form. Start from

$$u_r''(x) = f(u_{r-1}(x), x) + f_{u_{r-1}}(u_{r-1}(x), x)(u_r(x) - u_{r-1}(x)). \tag{39}$$

By subtracting (39) from (26), the following expression for the second derivative of $u_{r-1}(x)$ is derived:

$$\begin{aligned} u_{r+1}''(x) =& f(u_r(x), x) - f(u_{r-1}(x), x) - f_{u_{r-1}}(u_{r-1}(x), x)(u_r(x) - u_{r-1}(x)) \\ &+ f_{u_r}(u_r(x), x)(u_{r+1}(x) - u_r(x)), \end{aligned} \tag{40}$$

Here, $(u_{r+1}(x) - u_r(x))$ is the unknown quantity we want to analyze.

By applying the mean value theorem, the difference $f(u_r(x), x) - f(u_{r-1}(x), x)$ can be expressed as follows:

$$\begin{aligned} f(u_r(x), x) - f(u_{r-1}(x), x) - (u_r(x) - u_{r-1}(x))f_{u_{r-1}}(u_{r-1}(x), x) \\ = \frac{1}{2}(u_r(x) - u_{r-1}(x))^2 f_{vv}(v), \end{aligned} \tag{41}$$

where v lies between $u_{r-1}(x)$ and $u_r(x)$. Substituting (41) into (40), we obtain

$$\begin{aligned} u_{r+1}(x) - u_r(x) = \int_0^{x_c} G(x,\theta)[\frac{1}{2}(u_r(x) - u_{r-1}(x))^2 f_{vv}(v) \\ + (u_{r+1}(x) - u_r(x))f_{u_r}(u_r(x), x)]d\theta. \end{aligned} \tag{42}$$

Let $m_1 = max|f_{vv}(v)|$. From this, we can derive the following inequality:

$$|u_{r+1}(x) - u_r(x)| \leq \frac{x_c}{4}\int_0^{x_c} [\frac{m_1}{2}(u_r(x) - u_{r-1}(x)^2 + m|u_{r+1}(x) - u_r(x)|]d\theta. \tag{43}$$

Rearranging (43), we find

$$|u_{r+1}(x) - u_r(x)| \leq \frac{m_1\frac{x_c^2}{8}}{1 - m\frac{x_c^2}{4}}(u_r(x) - u_{r-1}(x))^2 = k(|u_r(x) - u_{r-1}(x)|)^2. \tag{44}$$

Equation (44) indicates that if there is convergence, then it is quadratic. To further illustrate this, we can rewrite the inequality recursively:

$$|u_{r+1}(x)-u_r(x)|\le k(|u_r(x)-u_{r-1}(x)|)^2,$$

which can be expanded as

$$|u_r(x)-u_{r-1}(x)|\le k(|u_{r-1}(x)-u_{r-2}(x)|)^2,$$

$$|u_{r-1}(x)-u_{r-2}(x)|\le k(|u_{r-2}(x)-u_{r-3}(x)|)^2,$$

and so on, leading to:

$$|u_2(x)-u_1(x)|\le k(|u_1(x)-u_0(x)|)^2.$$

This recursive relationship shows that each successive error term is bounded by a constant k multiplied by the square of the previous error term.

To proceed with the analysis using simple substitution, we introduce a new variable to simplify the expressions. Let us define

$$|u_{r+1}(x)-u_r(x)|\le k(|u_r(x)-u_{r-1}(x)|)^2\le k(k(|u_{r-1}(x)-u_{r-2}(x)|)^2)^2\le\cdots.$$

To derive the inequality, we start from the established recursive relationship:

$$|u_{r+1}(x)-u_r(x)|\le k[k^{2^r-2}(|u_1(x)-u_0(x)|)^{2^r}]=\frac{[k(u_1(x)-u_0(x)]^{2^r}}{k}. \tag{45}$$

If the quantity $[k(u_1(x)-u_0(x))]<1$, then as r increases. The right-hand side of (45) will approach zero. Consequently, $u_r(x)$ will approach to a function $u(x)$ and (32) reduces to

$$u(x)=\int_0^{x_c}G(x,\theta)f(u(x))d\theta. \tag{46}$$

Therefore, the function $u(x)$ satisfies the original equation. Note that if the value of x or the interval $[0,x]$ is sufficiently small, the quantity $k(|u_1(x)-u_0(x)|)$ will be less than one. It is also important to note that this quantity depends on the maximum value of $|u_1(x)-u_0(x)|$. Therefore, if the interval $[0,x_c]$ is excessively large, then we can theoretically select a better initial approximation $u_0(x)$ to ensure that the maximum difference between the absolute values of $u_1(x)$ and $u_0(x)$ is small enough to keep $[k(|u_1(x)-u_0(x)|)]$ less than one.

4 Illustrative examples

In this section, we address various nonlinear Lane–Emden differential equations using the QLM and collocation technique with Gegenbauer polynomials (with parameter $\alpha = 0.45$). Subsequently, we compare the results obtained from the proposed method with those derived from alternative approaches and exact solutions.

Example 1. For $\tau = 2$, $g(x,u) = u^m(x)$, $z(x) = 0$, $\lambda = 1$, (1) and (2) reduce to the standard Lane–Emden equations:

$$u''(x) + \frac{2}{x}u'(x) + u^m(x) = 0, \qquad x > 0, \tag{47}$$

under the initial conditions:

$$u(0) = 1, \qquad u'(0) = 0,$$

where m is a constant. Replacing $m = 0, 1, 5$ into (47) yields the following exact solutions:
1. $m = 0$:

$$u(x) = 1 - \frac{1}{3!}x^2,$$

2. $m = 1$:

$$u(x) = \frac{\sin(x)}{x},$$

and
3. $m = 5$:

$$u(x) = \frac{1}{\sqrt{1 + \frac{x^2}{3}}}.$$

In cases where an exact analytical solution is unavailable, we use (20) and (21) to obtain

$$u''_{r+1}(x) + \frac{2}{x}u'_{r+1}(x) + mu_{r+1}(x)u_r^{(m-1)}(x) - (m-1)u_r^{(m)}(x) = 0, \tag{48}$$

under the following initial conditions:

$$u_{r+1}(0) = 1, \qquad u'_{r+1}(0) = 0, \qquad r = 0, 1, \ldots, \text{max-iteration}. \tag{49}$$

We assume an initial guess $u_0(x) = 1$. Based on the current approach, we define the following residual function:

Table 1: Analysis of the first zeros of $u(x)$ obtained for $m = 2, 3$ and 4, and $N = 50$ of Example 1.

m	x	FRB [17]	RBs [15]	EBs [15]	$C^{\alpha}, \alpha = 0.45$	[10]	[16]
2	0.1	0.9983349985	0.9983349985	0.9983349985	0.9983349985	0.9983350	0.99833499
	0.5	0.9593527158	0.9593527158	0.9593527158	0.9593527158	0.9593527	0.95935270
	1.0	0.8486541114	0.8486541114	0.8486541114	0.8486541114	0.8486541	0.84865410
	3.0	0.2418240830	0.2418240830	0.2418240830	0.2418240830	0.2418241	0.24182408
	4.0	0.0488401499	0.0488401499	0.0488401499	0.0488401499	0.0488401	0.04884013
	4.3	0.0068109	0.006810943	0.0068109432	0.00681094	0.0068109	0.00681093
3	0.1	0.9983358295	0.9983358295	0.9983358295	0.9983358295	0.9983358	0.99833582
	0.5	0.9598390699	0.9598390699	0.9598390699	0.9598390699	0.9598391	0.95983906
	1.0	0.8550575685	0.8550575685	0.8550575685	0.8550575685	0.8550576	0.85505756
	3.0	0.3592265006	0.3592265006	0.35922650	0.3592265006	0.35922650	0.35922650
	4.0	0.2092816133	0.2092816133	0.20928161	0.2092816133	0.20928161	0.20928161
	4.3	0.1754138726	0.1754138726	0.17541387	0.1754138726	0.17541387	0.17541387
4	0.1	0.9983366613	0.9983366595	0.9983366595	0.9983366595	0.9.983367	1.0158508
	0.5	0.9603108972	0.9603109023	0.9603109023	0.9603109023	0.9.603109	0.9603108
	1.0	0.8608141488	0.8608138122	0.8608138122	0.8608138122	0.8608138	0.8608138
	3.0	0.4400560799	0.4400506915	0.4400506915	0.4400506923	0.44005068	0.4400555
	4.0	0.3180503821	0.3180424290	0.3180424290	0.3180424441	0.31804312	0.3180442
	4.3	0.2900786481	0.2900624969	0.2900624969	0.2900621154	0.29006281	0.2900622

$$Res_r(x) = u''_{n,r+1}(x) + \frac{2}{x}u'_{n,r+1}(x) + mu_{n,r+1}(x)u^{(m-1)}_{n,r}(x) - (m-1)u^{(m)}_{n,r}(x). \tag{50}$$

We apply the collocation method to (50) at $N+1$ points, as specified in (25), to construct a system of $N+1$ algebraic equations. These equations are solved using Newton's iterative method. Numerical results approximated for $n = 50$ with nonlinear exponent values $m = 2, 3, 4$, are reported to 10 decimal places in Table 1 and compared with references [15, 17, 10, 16]. Additionally, the absolute errors and maximum absolute errors of the approximate solutions for $L = 4.35$, and exponents $m = 0, 1, 5$, are presented in Tables 2 and 3, respectively. Finally, CPU times of $U_{10}(x)$ for different values of x and m are presented in Table 4.

Example 2. For $\tau = 2$, $g(x, u) = e^{u(x)}$, $z(x) = 0$, $\lambda = 0$, (1) and (2) reduce to

$$u''(x) + \frac{2}{x}u'(x) + e^{u(x)} = 0, \qquad x\geq 0 \tag{51}$$

under the following initial conditions:

$$u(0) = 0, \qquad u'(0) = 0. \tag{52}$$

Using (20) and (21), we obtain

$$u''_{r+1}(x) + \frac{2}{x}u'_{r+1}(x) + e^{u_r(x)}\Big(u_{r+1}(x) - u_r(x) + 1\Big) = 0, \tag{53}$$

under the initial conditions,

Table 2: Analysis of the absolute errors in the approximate solutions of Example 1 for $N = 50$, $L = 4.35$, and 10th iteration.

m	x	FRB [17]	RBs [15]	EBs [15]	$C^{\alpha}, \alpha = 0.45$
0	0.1	0	$1.65\,10^{-16}$	$4.04\,10^{-17}$	$1.69\,10^{-53}$
	0.5	$3.00\,10^{-60}$	$1.17\,10^{-17}$	$1.22\,10^{-17}$	$1.66\,10^{-46}$
	1.0	$2.00\,10^{-60}$	$7.93\,10^{-18}$	$3.30\,10^{-19}$	$6.73\,10^{-42}$
	3.0	$2.95\,10^{-58}$	$7.90\,10^{-18}$	$9.22\,10^{-19}$	$2.18\,10^{-34}$
	4.0	$6.18\,10^{-58}$	$7.89\,10^{-18}$	$9.21\,10^{-19}$	$2.33\,10^{-32}$
	4.3	$9.1\,10^{-58}$	$7.89\,10^{-18}$	$9.21\,10^{-19}$	$2.90\,10^{-34}$
1	0.1	$1.36\,10^{-23}$	$1.65\,10^{-16}$	$4.04\,10^{-17}$	$1.13\,10^{-53}$
	0.5	$3.39\,10^{-22}$	$1.13\,10^{-17}$	$1.22\,10^{-17}$	$1.11\,10^{-46}$
	1.0	$1.97\,10^{-21}$	$6.65\,10^{-18}$	$1.81\,10^{-19}$	$4.52\,10^{-42}$
	3.0	$1.07\,10^{-20}$	$3.45\,10^{-19}$	$4.05\,10^{-20}$	$1.46\,10^{-34}$
	4.0	$3.05\,10^{-20}$	$1.50\,10^{-18}$	$1.75\,10^{-19}$	$1.56\,10^{-32}$
	4.3	$8.60\,10^{-21}$	$1.68\,10^{-18}$	$1.96\,10^{-19}$	$1.31\,10^{-34}$
5	0.1	$1.56\,10^{-16}$	$1.65\,10^{-16}$	$4.04\,10^{-17}$	$4.14\,10^{-33}$
	0.5	$8.04\,10^{-15}$	$1.01\,10^{-17}$	$1.21\,10^{-17}$	$1.09\,10^{-25}$
	1.0	$8.52\,10^{-14}$	$3.35\,10^{-18}$	$2.03\,10^{-19}$	$1.69\,10^{-20}$
	3.0	$3.84\,10^{-13}$	$1.99\,10^{-18}$	$2.32\,10^{-19}$	$7.40\,10^{-13}$
	4.0	$1.71\,10^{-12}$	$2.14\,10^{-18}$	$2.50\,10^{-19}$	$7.61\,10^{-11}$
	4.3	$2.38\,10^{-12}$	$2.11\,10^{-18}$	$2.47\,10^{-19}$	$1.98\,10^{-12}$

Table 3: The maximum absolute errors of Example 1 for $N = 50$ and $m = 0, 1, 5$.

m	FRB [17]	RBs [15]	EBs [15]	$C^{\alpha}, \alpha = 0.45$
0	$6.18\,10^{-58}$	$1.65\,10^{-16}$	$4.04\,10^{-17}$	$2.33\,10^{-32}$
1	$3.05\,10^{-20}$	$1.65\,10^{-16}$	$4.04\,10^{-17}$	$1.56\,10^{-32}$
5	$4.773\,10^{-12}$	$1.65\,10^{-16}$	$4.04\,10^{-17}$	$7.61\,10^{-11}$

Table 4: The CPU times of Example 1 by the presented method for $u_{10}(x)$ in different values of x and m.

x	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
0.1	$2.88\,ms^{*}$	$3.52\,ms$	$3.16\,ms$	$3.59\,ms$	$3.00\,ms$	$2.29\,ms$
0.5	$3.48\,ms$	$4.45\,ms$	$3.55\,ms$	$3.71\,ms$	$3.53\,ms$	$2.8\,ms$
1.0	$875.00\,us^{**}$	$1.07\,ms$	$828.10\,us$	$853.10\,us$	$857.80\,us$	$760.90\,us$
3.0	$2.78\,ms$	$3.01\,ms$	$2.90\,ms$	$3.67\,ms$	$3.02\,ms$	$2.38\,ms$
4.0	$3.23\,ms$	$3.19\,ms$	$3.36\,ms$	$3.21\,ms$	$3.18\,ms$	$2.55\,ms$
4.3	$5.27\,ms$	$5.07\,ms$	$4.88\,ms$	$4.97\,ms$	$5.18\,ms$	$4.16\,ms$

* ms (millisecond),
** us (microsecond).

Table 5: The derived values of $u(x)$ for Example 2 via the presented method.

x	FRB [17]	RBs [15]	EBs [15]	$C^{\alpha}, \alpha = 0.45$	[21]	[16]	$CPUTime$
0.1	−0.001665833	−0.001665833	−0.001665833	−0.001665833	−0.001665833	−0.001666418	1.60 ms
0.2	−0.006653367	−0.006653367	−0.006653367	−0.006653367	−0.006653367	−0.006653971	2.03 ms
0.5	−0.041153957	−0.041153957	−0.041153957	−0.041153957	−0.041153956	−0.041154515	1.86 ms
1.0	−0.158827677	−0.158827677	−0.158827677	−0.158827677	−0.158827353	−0.158828173	568.70 us
1.5	−0.338019424	−0.338019424	−0.338019424	−0.338019424	−0.338013110	−0.338019830	2.20 ms
2.0	−0.559823004	−0.559823004	−0.559823004	−0.559823004	−0.559962660	−0.559823312	1.55 ms
2.5	−0.806340870	−0.806340870	−0.806340870	−0.806340870	−0.810019671	−0.806341084	2.35 ms

$$u_{r+1}(0) = 0, \qquad u'_{r+1}(0) = 0, \qquad r = 0, 1, \ldots, \text{max-iteration}. \tag{54}$$

We set $u_0(x) = 0$, which implies that the initial guess satisfies the boundary conditions. Using the proposed method, we construct the following residual function:

$$Res_r(x) = u''_{n,r+1}(x) + \frac{2}{x}u'_{n,r+1}(x) + e^{u_{n,r+1}(x)}\Big(u_{n,r+1}(x) - u_{n,r}(x) + 1\Big). \tag{55}$$

In Table 5, the CPU times and approximate values of $u(x)$ obtained from the presented method with $n = 50$, at the 10th iteration, and $L = 2.5$, are compared with those obtained in [17, 15, 21, 16].

Example 3. For $\tau = 2$, $g(x, u) = \sinh(u(x))$, $\lambda = 1$ and $z(x) = 0$, (1) and (2) reduce to

$$u''(x) + \frac{2}{x}u'(x) + \sinh(u(x)) = 0, \qquad x \geq 0, \tag{56}$$

under the following initial conditions:

$$u(0) = 1, \qquad u'(0) = 0.$$

Using the QLM in (20) and (21), the iterative approximation $u_{r+1}(x)$ is determined as the solution to the following differential equation:

$$u''_{r+1}(x) + \frac{2}{x}u'_{r+1}(x) + \sinh(u_r(x)) + (u_{r+1}(x) - u_r(x))\cosh(u_r(x)) = 0, \tag{57}$$

under the initial conditions:

$$u_{r+1}(0) = 1, \qquad u'_{r+1}(0) = 0, \qquad r = 0, 1, \ldots, \text{max-iteration}. \tag{58}$$

We assume $u_0(x) = 1$. Using the proposed approach, we define the following residual function:

Table 6: The derived values of $u(x)$ for Example 3 via the presented method.

x	FRB [17]	RBs [15]	EBs [15]	$C^{\alpha}, \alpha = 0.45$	[21]	[16]	$CPUTime$
0.1	0.998042841	0.998042841	0.998042841	0.998042841	0.9980428	0.998042841	2.49 ms
0.2	0.992189434	0.992189434	0.992189434	0.992189434	0.9921894	0.992189434	2.80 ms
0.5	0.951961092	0.951961092	0.951961092	0.951961092	0.9519611	0.951961092	2.91 ms
1.0	0.818242928	0.818242928	0.040263157	0.818242928	0.8182516	0.818242928	592.20 us
1.5	0.625438763	0.625438763	0.625438763	0.625438763	0.62589160	0.625438763	3.03 ms
2.0	0.406622887	0.40662288	0.406622887	0.406622887	0.41366910	0.406622887	786.00 us

$$\begin{aligned} Res_r(x) =& u''_{r+1}(x) + \frac{2}{x} u'_{r+1}(x) + \sinh(u_r(x)) \\ &+ (u_{r+1}(x) - u_r(x)) \cosh(u_r(x)) = 0. \end{aligned} \tag{59}$$

In Table 6, the CPU times and approximate values of $u(x)$ obtained from the presented method with $n = 50$, at the 10th iteration, and $L = 2$, are compared with those obtained in references [17, 15, 21, 16].

Example 4. If $\tau = 2$, $g(x, u) = \sin(u(x))$, $\lambda = 1$ and $z(x) = 0$, then (1) and (2) have been defined as follows:

$$u''(x) + \frac{2}{x} u'(x) + \sin(u(x)) = 0, \tag{60}$$

under the following initial conditions:

$$u(0) = 1, \qquad u'(0) = 1.$$

By using the QLM, we obtain

$$u''_{r+1}(x) + \frac{2}{x} u'_{r+1}(x) + \sin(u_r(x)) + \cos(u_r(x)) \Big(u_{r+1}(x) - u_r(x) \Big) = 0, \tag{61}$$

under the initial conditions:

$$u_{r+1}(0) = 1, \qquad u'_{r+1}(0) = 0, \qquad r = 0, 1, 2, \ldots, \text{max-iteration}. \tag{62}$$

We consider $u_0(x) = 1$. Based on the present technique, we formulate the residual function as follows:

$$\begin{aligned} Res_r(x) =& u''_{n,r+1}(x) + \frac{2}{x} u'_{n,r+1}(x) + \sin(y_{n,r}(x)) \\ &+ \cos(u_{n,r}(x)) \Big(u_{n,r+1}(x) - u_{n,r}(x) \Big). \end{aligned} \tag{63}$$

In Table 7, the CPU times and approximate values of $u(x)$ obtained from the presented method with $n = 40$, at the 10th iteration, and $L = 2$, are compared with those obtained in [17, 15, 21, 16].

Table 7: The derived values of $u(x)$ for Example 4 via the presented method.

x	FRB [17]	RBs [15]	EBs [15]	$C^{\alpha}, \alpha = 0.45$	[21]	[16]	$CPUTime$
0.1	0.998597927	0.998597927	0.998597927	0.998597927	0.9985979	0.998605142	1.21 ms
0.2	0.994396264	0.994396264	0.994396264	0.994396264	0.9943962	0.994406270	1.28 ms
0.5	0.965177780	0.965177780	0.965177780	0.965177780	0.96517778	0.965188168	1.40 ms
1.0	0.863681125	0.863681125	0.863681125	0.863681125	0.86368110	0.863688130	184.30 us
1.5	0.705045233	0.705045233	0.705045233	0.705045233	0.70504192	0.705052410	1.49 ms
2.0	0.506463627	0.506463627	0.506463627	0.506463627	0.50637203	0.50646875	484.40 us

5 Conclusion

We have proposed an efficient and accurate numerical technique to approximate solutions for singular nonlinear Lane–Emden equations with initial conditions. Specifically, we determined the parameter of Gegenbauer functions and combined the QLM with the collocation technique to solve various types of Lane–Emden equations.

The primary advantage of this approach lies in its ability to transform the nonlinear differential equation into a system of linear algebraic equations, which are simpler to solve. Furthermore, the collocation method minimizes the residual functions, enabling the determination of unknown coefficients. To compute the solutions and absolute errors, we developed a code in Maple software, which efficiently handled all computations.

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Conflicts of Interest

The authors declare no conflicts of interest.

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