Iranian Journal of Numerical Analysis and Optimization
Vol. 15, No. 2, 2025, pp 625–644
https://doi.org/10.22067/ijnao.2025.89967.1523
https://ijnao.um.ac.ir/

Research Article

An accurate numerical technique for solving a special case of fractional differential equations using the Khalouta transform of two different fractional derivatives

A. Khalouta*

Abstract

The aim of this paper is to present a novel coupling approach of the Khalouta transform method and the homotopy perturbation method in order to obtain an accurate and efficient method for solving a special case of fractional differential equations involving Caputo and Caputo-Fabrizio fractional derivatives. This method is called the fractional Khalouta homotopy perturbation method (FKHHPM). In particular, the FKHHPM is used to obtain a solution to the fractional reaction-diffusion-convection

*Corresponding author

Received 24 September 2024; revised 10 December 2024; accepted 2 March 2025

Ali Khalouta
Laboratory of Fundamental and Numerical Mathematics, Department of Mathematics, Faculty of Sciences, Setif 1 University-Ferhat ABBAS, Algeria.
e-mail: nadjibkh@yahoo.fr ; ali.khalouta@univ-setif.dz

How to cite this article
Khalouta, A., An accurate numerical technique for solving a special case of fractional differential equations using the Khalouta transform of two different fractional derivatives. *Iran. J. Numer. Anal. Optim.*, 2025; 15(2): 625-644. https://doi.org/10.22067/ijnao.2025.89967.1523

equations. The convergence analysis and a numerical example are presented. To evaluate the effectiveness of the proposed computational strategy, we examine the convergence of the series solution over different fractional values and evaluate the behavior of the solution as the time domain increases. The efficiency and originality of the FKHHPM are demonstrated by calculating the absolute error. This work is supported by two-dimensional and three-dimensional graphical representations made in accordance with MATLAB.

AMS subject classifications (2020): 35R11, 26A33, 34A45, 74G15.

Keywords: Fractional differential equation; Caputo fractional derivative; Caputo-Fabrizio fractional derivative; Khalouta transform; Homotopy perturbation method.

1 Introduction

Fractional calculus is an important mathematical field in the modeling of many natural and physical phenomena. Dating back to Leibniz's letter in the late 17th century, studies on fractional calculus have been constantly improving with some increase in the number in recent decades. The Riemann–Liouville, Grünwald–Letnikov, and Caputo fractional derivative definitions have been widely studied in the literature [20, 24, 26]. More recent definitions, such as the Atangana–Baleanu fractional derivative and the Caputo–Fabrizio fractional derivative, are also popular tools in modeling studies. Katugampola, Hilfer, and many other derivatives have been given in the literature [4, 6, 14, 15] along with conformable fractional derivative, which has become the basis of conformable calculus [1].

Fractional differential equations have been widely used to describe several phenomena in physics and help in their solutions, such as fluid mechanics, viscoelasticity, electrodynamics, quantum mechanics, rheology, damping laws, diffusion processes, economy, biology, geophysics, bioengineering, optimal control, and other areas of science; see [5, 8, 12, 13, 23, 27]. It is very difficult to obtain exact solutions to real-life problems using fractional differential equations, and usually complex mathematical techniques are required.

Recently, many mathematicians and physicists have studied the solutions to fractional differential equations involving different types of fractional derivatives using various analytical and numerical methods. For example, Kumawat et al. [21] treated the solutions of fractional differential equations using both Riemann–Liouville and Caputo fractional derivatives and used the Khalouta transform method. Alazman et al. [2] employed the Laplace decomposition technique to obtain the numerical solutions of the SIR model using the generalized fractional derivative. Yadav et al. [30] applied a precise and analytical method, namely the Shehu transform decomposition method, to examine the solutions of multi-dimensional fractional diffusion equations involving the Caputo–Fabrizio derivative. Dube, Mishraa, and Goswami [9] introduced the local fractional homotopy perturbation Sumudu transform method to solve the oxygen diffusion equation with a local fractional derivative. The author [17] presented two new semi-analytical methods called the Khalouta homotopy perturbation method and the Khalouta variational iteration method to find new approximate analytical solutions of nonlinear fractional partial differential equations via the Atangana–Baleanu fractional derivative. Also, the author [18] used a modification of a new general integral transform to study the properties of solutions of fractional differential equations containing a Caputo generalized fractional derivative. This method is called the ρ-Jafari transform method.

The main objective of the present paper is to propose an accurate numerical technique called the fractional Khalouta homotopy perturbation method (FKHHPM) for solving a special case of fractional differential equations, in particular, fractional reaction-diffusion-convection equations using Caputo and Caputo–Fabrizio derivatives of the forms

$$D_{\upsilon}^{\kappa}\chi = \left(\Psi(\chi)\chi_{\varrho}\right)_{\varrho} + \Phi(\chi)\chi_{\varrho} + \Pi(\chi), \tag{1}$$

subject to the condition

$$\chi(\varrho, 0) = \chi_0(\varrho), \tag{2}$$

where D_{υ}^{κ} is the fractional derivative of order $0 < \kappa \leq 1$ in the Caputo sense.

Moreover,

$$\mathcal{D}_{\upsilon}^{(\kappa)}\chi = \left(\Psi(\chi)\chi_{\varrho}\right)_{\varrho} + \Phi(\chi)\chi_{\varrho} + \Pi(\chi), \tag{3}$$

subject to the condition

$$\chi(\varrho,0)=\chi_0(\varrho), \tag{4}$$

where $\mathcal{D}_{\upsilon}^{(\kappa)}$ is is the fractional derivative of order $0<\kappa\leq 1$ in the Caputo–Fabrizio sense.

In (1) and (3), $\chi=\{\chi(\varrho,\upsilon),\varrho\in\mathbb{R},\upsilon\geq 0\}$ is an unknown function, where ϱ represents the space variable and υ represents the time variable, and the arbitrary smooth functions $\Psi(\chi),\Phi(\chi)$, and $\Pi(\chi)$ denote the diffusion term, the convection term, and the reaction term, respectively. The reaction-diffusion-convection problems is a very useful mathematical model in applied sciences such as biology modeling, physics, chemistry, astrophysics, hydrology, medicine, and engineering [3, 7, 10, 29].

The FKHHPM is used to find an approximate solution to the given equations. FKHHPM is a mixture of the Khalouta transform and the homotopy perturbation method. The Khalouta transform is an analytical method that provides exact solutions, while the homotopy perturbation is a semi-analytical method that provides approximate solutions. As a result, coupling these two methods would definitely yield a result extremely similar and highly convergent to the exact solution of the problem.

The advantages of the FKHHPM using the Khalouta transform lies in the capability of combining two powerful methods with the need of only initial conditions for obtaining an exact solution to the fractional reaction-diffusion-convection equations, and also the method is characterized by its rapid convergence and easy finding of the unknown coefficients of the series solution without any perturbation, discretization, and linearization, and this method is fast, simple, and adaptable to solving fractional differential equations and others.

The present paper is organized as follows. In section 2, we start by introducing the basic definitions and properties of fractional calculus and Khalouta transform. In section 3, we present the methodology of the FKHHPM for solving the fractional reaction-diffusion-convection equations (1) and (3). In addition, we establish its convergence analysis, and then provide a numerical example to demonstrate the effectiveness of the proposed method in section 4. A discussion of the results obtained is given in section 5. Finally, Section 6 contains the conclusions of our work.

2 Overview of fractional calculus theory

In this section, we present the necessary definitions of fractional calculus and new results concerning the theory of the Khalouta transform of fractional derivatives.

Definition 1. [20] The fractional integral of order $\kappa > 0$ in the Riemann–Liouville sense is expressed as

$$I_{\upsilon}^{\kappa}\chi(\upsilon) = \frac{1}{\Gamma(\kappa)} \int_0^{\upsilon} (\upsilon - \epsilon)^{\kappa-1} \chi(\epsilon) d\epsilon.$$

Definition 2. [20] The fractional derivative of order κ in the Caputo sense is expressed as

$$D_{\upsilon}^{\kappa}\chi(\upsilon) = \begin{cases} \dfrac{1}{\Gamma(n-\kappa)} \int_0^{\upsilon} (\upsilon - \epsilon)^{n-\kappa-1} \chi^{(n)}(\epsilon) d\epsilon, & n-1 < \kappa < n, \\ \chi^{(n)}(\upsilon), & \kappa = n. \end{cases}$$

Definition 3. [20] The Mittag-Leffler function is expressed as

$$E_{\kappa}(\upsilon) = \sum_{n=0}^{\infty} \frac{\upsilon^n}{\Gamma(n\kappa+1)}, \qquad \kappa > 0. \tag{5}$$

Definition 4. [6] The fractional derivative of order $0 < \kappa < 1$ in the Caputo–Fabrizio sense is expressed as

$$\mathcal{D}_{\upsilon}^{(\kappa)}\chi(\upsilon) = \frac{\mathfrak{N}(\kappa)}{1-\kappa} \int_0^{\upsilon} \chi'(\epsilon) \exp\left(-\frac{\kappa(\upsilon-\epsilon)}{1-\kappa}\right) d\epsilon, \tag{6}$$

where $\mathfrak{N}(\kappa)$ is a normalization function that satisfies $\mathfrak{N}(0) = \mathfrak{N}(1) = 1$ and $\chi(\upsilon) \in H^1(\mathbb{R}^+)$.

Definition 5. [6] The fractional derivative of order $\kappa + n$ with $0 < \kappa \leq 1$ and $n \geq 1$ in the Caputo–Fabrizio sense is expressed as

$$\mathcal{D}_{\upsilon}^{(\kappa+n)}\chi(\upsilon) = \mathcal{D}_{\upsilon}^{(\kappa)}(\mathcal{D}_{\upsilon}^{(n)}\chi(\upsilon)).$$

With the hypothesis that $\mathfrak{N}(\kappa) = 1$ in (6), the second definition of Losada and Nieto reads as follows.

Definition 6. [22] The fractional derivative of order $0 < \kappa \leq 1$ in the Caputo-Fabrizio sense is expressed as

$$\mathcal{D}_{\upsilon}^{(\kappa)}\chi(\upsilon)=\frac{1}{1-\kappa}\int_{0}^{\upsilon}\chi'(\epsilon)\exp\left(-\frac{\kappa(\upsilon-\epsilon)}{1-\kappa}\right)d\epsilon.$$

Definition 7. [16] The Khalouta transform is defined over the set of functions

$$\mathcal{S}=\big\{\chi(\upsilon):\text{there exist } K,\vartheta_1,\vartheta_2>0,|\chi(\upsilon)|<K\exp\left(\vartheta_j\,|\upsilon|\right), \\ \text{if } \upsilon\in(-1)^j\times\mathbb{R}^+\big\},$$

by the following integral

$$\mathbb{KH}\left[\chi(\upsilon)\right]=\mathcal{K}(s,\gamma,\eta)=\frac{s}{\gamma\eta}\int_{0}^{\infty}\exp\left(-\frac{s\upsilon}{\gamma\eta}\right)\chi(\upsilon)d\upsilon,$$

where $s,\gamma,\eta>0$ are the Khalouta transform variables.

Theorem 1. [16] The basic properties of the Khalouta transform are as follows.

1) If $\chi(\upsilon)$ and $\varpi(\upsilon)$ are defined on the set $\mathcal{S}$, then for all constants a and b, we have

$$\mathbb{KH}\left[a\chi(\upsilon)+b\varpi(\upsilon)\right]=a\mathbb{KH}\left[\chi(\upsilon)\right]+b\mathbb{KH}\left[\varpi(\upsilon)\right].$$

2) If the nth derivative of $\chi(\upsilon)$ is $\chi^{(n)}(\upsilon)$, then its Khalouta transform is given as

$$\mathbb{KH}\left[\chi^{(n)}(\upsilon)\right]=\left(\frac{s}{\gamma\eta}\right)^{n}\mathcal{K}(s,\gamma,\eta)-\sum_{r=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{n-r}\chi^{(r)}(0),\qquad n\geq 1.$$

3) If the Khalouta transform of $\chi(\upsilon)$ and $\varpi(\upsilon)$ are $\mathcal{K}(s,\gamma,\eta)$ and $\mathcal{H}(s,\gamma,\eta)$ respectively, defined on the set $\mathcal{S}$, then

$$\mathbb{KH}\left[\left(\chi*\varpi\right)(\upsilon)\right]=\int_{0}^{\infty}\chi(\upsilon)\varpi(\upsilon-\epsilon)d\epsilon=\frac{\gamma\eta}{s}\mathcal{K}(s,\gamma,\eta)\mathcal{H}(s,\gamma,\eta),$$

where $\mathbb{KH}\left[\left(\chi*\varpi\right)(\upsilon)\right]$ is the Khalouta convolution of the functions $\chi(\upsilon)$ and $\varpi(\upsilon)$.

4) The Khalouta transforms of some special functions are as follows.

$$\mathbb{KH}\left[1\right]=1,$$
$$\mathbb{KH}\left[\upsilon\right]=\frac{\gamma\eta}{s},$$

$$\mathbb{KH}\left[\frac{\upsilon^n}{n!}\right] = \left(\frac{\gamma\eta}{s}\right)^n, \quad n = 0,1,2,\ldots,$$
$$\mathbb{KH}\left[\frac{\upsilon^\kappa}{\Gamma(\kappa+1)}\right] = \left(\frac{\gamma\eta}{s}\right)^\kappa, \quad \kappa > -1,$$
$$\mathbb{KH}\left[\exp(a\upsilon\right] = \frac{s}{s-a\gamma\eta}, \quad a \in \mathbb{R}^+.$$

Theorem 2. [19] The Khalouta transform of fractional derivative in the Caputo sense is expressed as

$$\mathbb{KH}\left[D_\upsilon^\kappa \chi(\upsilon)\right] = \left(\frac{s}{\gamma\eta}\right)^\kappa \mathbb{KH}\left[\chi(\upsilon)\right] - \sum_{r=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{\kappa-r}\chi^{(r)}(0),$$

where $n-1 < \kappa \leq n$ and $n \in \mathbb{N}^*$.

Theorem 3. [19] The Khalouta transform of fractional derivative in the Caputo–Fabrizio sense is expressed as

$$\mathbb{KH}\left[\mathcal{D}_\upsilon^{(\kappa+n)}\chi(\upsilon)\right] = \frac{s}{s-\kappa(s-\gamma\eta)}\left(\left(\frac{s}{\gamma\eta}\right)^n \mathbb{KH}\left[\chi(\upsilon)\right] - \sum_{r=0}^{n}\left(\frac{s}{\gamma\eta}\right)^{n-r}\chi^{(r)}(0)\right),$$

where $0 < \kappa \leq 1$ and $n \geq 1$.

3 FKHHPM and its convergence

In this section, we explain the FKHHPM methodology for the fractional reaction-diffusion-convection equations (1) and (3) and study its convergence.

Theorem 4. Let the fractional reaction-diffusion-convection equations be of the form (1) and (3), respectively. According to FKHHPM, the solution to (1)) and (3) can be expressed as an infinite series as follows:

$$\chi(\varrho,\upsilon) = \lim_{p\to 1}\sum_{n=0}^{\infty}\mathfrak{p}^n\chi_n(\varrho,\upsilon) = \sum_{n=0}^{\infty}\chi_n(\varrho,\upsilon), \tag{7}$$

where $\mathfrak{p} \in [0,1]$ is the homotopy parameter.

Proof. To prove this result, we define

$$\mathfrak{M}_1\chi = \left(\Psi(\chi)\chi_\varrho\right)_\varrho,$$
$$\mathfrak{M}_2\chi = \Phi(\chi)\chi_\varrho,$$

$$\mathfrak{M}_3\chi = \Pi(\chi).$$

Case 1: Equation (1) becomes

$$D_\upsilon^\kappa \chi = \mathfrak{M}_1\chi + \mathfrak{M}_2\chi + \mathfrak{M}_3\chi. \tag{8}$$

Applying the Khalouta transform on (8) and using Theorem 2, we get

$$\mathbb{KH}\left[\chi\right] = \chi_0(\varrho) + \left(\frac{\gamma\eta}{s}\right)^\kappa \mathbb{KH}\left[\mathfrak{M}_1\chi + \mathfrak{M}_2\chi + \mathfrak{M}_3\chi\right]. \tag{9}$$

Operating with the inverse Khalouta transform on (9), we have

$$\chi = \chi_0(\varrho) + \mathbb{KH}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^\kappa \mathbb{KH}\left[\mathfrak{M}_1\chi + \mathfrak{M}_2\chi + \mathfrak{M}_3\chi\right]\right]. \tag{10}$$

Using the homotopy perturbation method (HPM) [11], we assume that the solution can be written as a power series of $\mathfrak{p}$ as follows:

$$\chi(\varrho, \upsilon) = \sum_{n=0}^{\infty} \mathfrak{p}^n \chi_n(\varrho, \upsilon), \tag{11}$$

where $\mathfrak{p} \in [0, 1]$.

The decomposition of the nonlinear terms is as follows:

$$\begin{aligned} \mathfrak{M}_1\chi &= \sum_{n=0}^{\infty} \mathfrak{p}^n \mathfrak{H}_n(\chi), \\ \mathfrak{M}_2\chi &= \sum_{n=0}^{\infty} \mathfrak{p}^n \mathfrak{K}_n(\chi), \\ \mathfrak{M}_3\chi &= \sum_{n=0}^{\infty} \mathfrak{p}^n \mathfrak{I}_n(\chi), \end{aligned} \tag{12}$$

where $\mathfrak{H}_n(\chi), \mathfrak{K}_n(\chi)$ and $\mathfrak{I}_n(\chi)$ are He's polynomials [25], of $\chi_0, \chi_1, \ldots, \chi_n$ and by using the following formulas they are computed:

$$\begin{aligned} \mathfrak{H}_n(\chi_0, \chi_1, \ldots, \chi_n) &= \frac{1}{n!}\frac{\partial^n}{\partial \mathfrak{p}^n}\left[\mathfrak{M}_1\left(\sum_{i=0}^{\infty} \mathfrak{p}^i \chi_i\right)\right]_{\mathfrak{p}=0}, \\ \mathfrak{K}_n(\chi_0, \chi_1, \ldots, \chi_n) &= \frac{1}{n!}\frac{\partial^n}{\partial \mathfrak{p}^n}\left[\mathfrak{M}_2\left(\sum_{i=0}^{\infty} \mathfrak{p}^i \chi_i\right)\right]_{\mathfrak{p}=0}, \\ \mathfrak{I}_n(\chi_0, \chi_1, \ldots, \chi_n) &= \frac{1}{n!}\frac{\partial^n}{\partial \mathfrak{p}^n}\left[\mathfrak{M}_3\left(\sum_{i=0}^{\infty} \mathfrak{p}^i \chi_i\right)\right]_{\mathfrak{p}=0}. \end{aligned} \tag{13}$$

Perform replacements of (11) and (12) in (10), we have

$$\sum_{n=0}^{\infty} \mathfrak{p}^n \chi_n(\varrho,\upsilon) = \chi_0(\varrho) + \mathbb{KH}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\kappa} \mathbb{KH}\left[\begin{array}{c} \sum_{n=0}^{\infty} \mathfrak{p}^n \mathfrak{H}_n(\chi) \\ + \sum_{n=0}^{\infty} \mathfrak{p}^n \mathfrak{K}_n(\chi) \\ + \sum_{n=0}^{\infty} \mathfrak{p}^n \mathfrak{J}_n(\chi) \end{array}\right]\right]. \qquad (14)$$

Using the coefficient of similar powers of $\mathfrak{p}$ in (14), we have

$$\begin{aligned}
\mathfrak{p}^0 &: \chi_0(\varrho,\upsilon) = \chi_0(\varrho),\\
\mathfrak{p}^1 &: \chi_1(\varrho,\upsilon) = \mathbb{KH}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\kappa} \mathbb{KH}\left[\mathfrak{H}_0(\chi) + \mathfrak{K}_0(\chi) + \mathfrak{J}_0(\chi)\right]\right],\\
\mathfrak{p}^2 &: \chi_2(\varrho,\upsilon) = \mathbb{KH}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\kappa} \mathbb{KH}\left[\mathfrak{H}_1(\chi) + \mathfrak{K}_1(\chi) + \mathfrak{J}_1(\chi)\right]\right],\\
\mathfrak{p}^3 &: \chi_3(\varrho,\upsilon) = \mathbb{KH}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\kappa} \mathbb{KH}\left[\mathfrak{H}_2(\chi) + \mathfrak{K}_2(\chi) + \mathfrak{J}_2(\chi)\right]\right],\\
&\vdots
\end{aligned}$$

Therefore, the series solution to (8) is found as $\mathfrak{p} \to 1$

$$\chi(\varrho,\upsilon) = \sum_{n=0}^{\infty} \chi_n(\varrho,\upsilon).$$

Case 2: Equation (3) becomes

$$\mathcal{D}_{\upsilon}^{(\kappa)}\chi = \mathfrak{M}_1\chi + \mathfrak{M}_2\chi + \mathfrak{M}_3\chi. \qquad (15)$$

Applying the Khalouta transform on (15) and using Theorem 3, we get

$$\mathbb{KH}[\chi] = \chi_0(\varrho) + \frac{s-\alpha(s-\gamma\eta)}{s}\mathbb{KH}\left[\mathfrak{M}_1\chi + \mathfrak{M}_2\chi + \mathfrak{M}_3\chi\right]. \qquad (16)$$

Operating with the inverse Khalouta transform on (16), we have

$$\chi = \chi_0(\varrho) + \mathbb{KH}^{-1}\left[\frac{s-\alpha(s-\gamma\eta)}{s}\mathbb{KH}\left[\mathfrak{M}_1\chi + \mathfrak{M}_2\chi + \mathfrak{M}_3\chi\right]\right]. \qquad (17)$$

Using the HPM [11], we assume that the solution can be written as a power series of $\mathfrak{p}$ as follows:

$$\chi(\varrho,\upsilon) = \sum_{n=0}^{\infty} p^n \chi_n(\varrho,\upsilon),$$

and by using the formulas (13), the decomposition of the nonlinear terms are computed.

Perform replacements of equations (11) and (12) in (17), we have

$$\sum_{n=0}^{\infty}\mathfrak{p}^n\chi_n(\varrho,\upsilon)=\chi_0(\varrho)+\mathbb{KH}^{-1}\left[\frac{s-\alpha\left(s-\gamma\eta\right)}{s}\mathbb{KH}\left[\begin{array}{c}\sum\limits_{n=0}^{\infty}\mathfrak{p}^n\mathfrak{H}_n(\chi)\\+\sum\limits_{n=0}^{\infty}\mathfrak{p}^n\mathfrak{K}_n(\chi)\\+\sum\limits_{n=0}^{\infty}\mathfrak{p}^n\mathfrak{J}_n(\chi)\end{array}\right]\right]. \tag{18}$$

Using the coefficient of similar powers of $\mathfrak{p}$ in (18), we have

$$\begin{aligned}
&\mathfrak{p}^0:\chi_0(\varrho,\upsilon)=\chi_0(\varrho),\\
&\mathfrak{p}^1:\chi_1(\varrho,\upsilon)=\mathbb{KH}^{-1}\left[\frac{s-\alpha\left(s-\gamma\eta\right)}{s}\mathbb{KH}\left[\mathfrak{H}_0(\chi)+\mathfrak{K}_0(\chi)+\mathfrak{J}_0(\chi)\right]\right],\\
&\mathfrak{p}^2:\chi_2(\varrho,\upsilon)=\mathbb{KH}^{-1}\left[\frac{s-\alpha\left(s-\gamma\eta\right)}{s}\mathbb{KH}\left[\mathfrak{H}_1(\chi)+\mathfrak{K}_1(\chi)+\mathfrak{J}_1(\chi)\right]\right],\\
&\mathfrak{p}^3:\chi_3(\varrho,\upsilon)=\mathbb{KH}^{-1}\left[\frac{s-\alpha\left(s-\gamma\eta\right)}{s}\mathbb{KH}\left[\mathfrak{H}_2(\chi)+\mathfrak{K}_2(\chi)+\mathfrak{J}_2(\chi)\right]\right],\\
&\qquad\vdots
\end{aligned}$$

Therefore, the series solution of (15) is found as $\mathfrak{p}\to 1$

$$\chi(\varrho,\upsilon)=\sum_{n=0}^{\infty}\chi_n(\varrho,\upsilon).$$

□

Theorem 5. Let $\mathfrak{B}$ be a Banach space. Then the FKHHPM solution is convergent if $\chi_0\in\mathfrak{B}$ is bounded and

$$\left\|\chi_n\right\|\leq\varepsilon\left\|\chi_{n-1}\right\|,\qquad \text{for all } n\in\mathbb{N}^*,$$

where $0<\varepsilon<1$.

Proof. Let $\left\{\mathfrak{S}_n\right\}_{n\geq 0}$ be the sequence as partial sum of (7) as

$$\mathfrak{S}_n(\varrho,\upsilon)=\chi_0(\varrho,\upsilon)+\chi_1(\varrho,\upsilon)+\chi_2(\varrho,\upsilon)+\cdots+\chi_n(\varrho,\upsilon).$$

To obtain the desired result, we must establish that $\{\mathfrak{S}_n\}_{n\geq 0}$ forms a Cauchy sequence in the Banach space $\mathfrak{B}$. Furthermore, let

$$\|\mathfrak{S}_{n+1}-\mathfrak{S}_n\| \leq \|\chi_{n+1}\| \leq \varepsilon\|\chi_n\| \leq \varepsilon^2\|\chi_{n-1}\| \leq \cdots \leq \varepsilon^{n+1}\|\chi_0\|.$$

For all $n, m \in \mathbb{N}$, with $n \geq m$, we have

$$\begin{aligned}\|\mathfrak{S}_n-\mathfrak{S}_m\| &= \|\mathfrak{S}_{m+1}-\mathfrak{S}_m+\mathfrak{S}_{m+2}-\mathfrak{S}_{m+1}+\cdots+\mathfrak{S}_n-\mathfrak{S}_{n-1}\| \\ &\leq \|\mathfrak{S}_{m+1}-\mathfrak{S}_m\|+\|\mathfrak{S}_{m+2}-\mathfrak{S}_{m+1}\|+\cdots+\|\mathfrak{S}_n-\mathfrak{S}_{n-1}\| \\ &\leq \varepsilon^{m+1}\|\chi_0\|+\varepsilon^{m+2}\|\chi_0\|+\cdots+\varepsilon^{n}\|\chi_0\| \\ &= \gamma^{m+1}\left(1+\varepsilon+\cdots+\varepsilon^{n-m-1}\right)\|\chi_0\| \\ &\leq \varepsilon^{m+1}\left(\frac{1-\varepsilon^{n-m}}{1-\varepsilon}\right)\|\chi_0\|.\end{aligned}$$

Since $0<\varepsilon<1$, we have $1-\varepsilon^{n-m}<1$; then

$$\|\mathfrak{S}_n-\mathfrak{S}_m\| \leq \frac{\varepsilon^{m+1}}{1-\varepsilon}\|\chi_0\|.$$

Since χ_0 is bounded, then $\|\chi_0\|<\infty$, which means

$$\lim_{n\longrightarrow\infty}\|\mathfrak{S}_n-\mathfrak{S}_m\|=0,$$

for any finite value of m.

Consequently, $\{\mathfrak{S}_n\}_{n\geq 0}$ is a Cauchy sequence in $\mathfrak{B}$. It follows that the series solution of (1) and (3) as (7) is convergent. □

Theorem 6. Suppose that $\sum_{k=0}^{m}\chi_k(\varrho,\upsilon)<\infty$ and that $\chi(\varrho,\upsilon)$ is its truncated solution. Then the maximum absolute error is presented as

$$\left\|\chi(\varrho,\upsilon)-\sum_{k=0}^{m}\chi_k(\varrho,\upsilon)\right\| \leq \frac{\varepsilon^{m+1}}{1-\varepsilon}\|\chi_0(\varrho,\upsilon)\|.$$

Proof. From Theorem 5, we have

$$\|\mathfrak{S}_n-\mathfrak{S}_m\| \leq \frac{\varepsilon^{m+1}}{1-\varepsilon}\|\chi_0(\varrho,\upsilon)\|. \tag{19}$$

Indeed $\mathfrak{S}_n=\sum_{k=0}^{n}\chi_k(\varrho,\upsilon)$ and considering $n\to\infty$, then $\mathfrak{S}_n\to\chi(\varrho,\upsilon)$; therefore (19) becomes

$$\|\chi(\varrho,\upsilon)-\mathfrak{S}_m\| = \left\|\chi(\varrho,\upsilon)-\sum_{k=0}^{m}\chi_k(\varrho,\upsilon)\right\| \leq \frac{\varepsilon^{m+1}}{1-\varepsilon}\left\|\chi_0(\varrho,\upsilon)\right\|.$$

□

4 Numerical application

Consider the fractional reaction-diffusion-convection equation under Caputo fractional derivative

$$D_\upsilon^\kappa \chi = \left(\chi\chi_\varrho\right)_\varrho + 3\chi\chi_\varrho + 2\left(\chi-\chi^2\right), 0<\kappa\leq 1, \tag{20}$$

subject to the condition

$$\chi(\varrho,0) = 2\sqrt{e^\varrho - e^{-4\varrho}}. \tag{21}$$

Here $\chi = \left\{\chi(\varrho,\upsilon) : (\varrho,\upsilon)\in\mathbb{R}^+\times\mathbb{R}^+\right\}$.

Using the methodology of the FKHHPM presented in Section 3, we get the following result:

$$\begin{aligned}
\mathfrak{p}^0 &: \chi_0(\varrho,\upsilon) = 2\sqrt{e^\varrho - e^{-4\varrho}},\\
\mathfrak{p}^1 &: \chi_1(\varrho,\upsilon) = \frac{4\upsilon^\kappa}{\Gamma(\kappa+1)}\sqrt{e^\varrho - e^{-4\varrho}},\\
\mathfrak{p}^2 &: \chi_2(\varrho,\upsilon) = \frac{8\upsilon^{2\kappa}}{\Gamma(2\kappa+1)}\sqrt{e^\varrho - e^{-4\varrho}},\\
\mathfrak{p}^3 &: \chi_3(\varrho,\upsilon) = \frac{16\upsilon^{3\kappa}}{\Gamma(3\kappa+1)}\sqrt{e^\varrho - e^{-4\varrho}},\\
&\vdots
\end{aligned}$$

Therefore, the series solution to (20) and (21) is

$$\begin{aligned}
\chi(\varrho,\upsilon) &= \lim_{p\to 1}\sum_{n=0}^{\infty}\mathfrak{p}^n\chi_n(\varrho,\upsilon) = \sum_{n=0}^{\infty}\chi_n(\varrho,\upsilon)\\
&= \chi_0(\varrho,\upsilon)+\chi_1(\varrho,\upsilon)+\chi_2(\varrho,\upsilon)+\chi_3(\varrho,\upsilon)+\cdots\\
&= 2\sqrt{e^\varrho - e^{-4\varrho}}\left(1+\frac{2\upsilon^\kappa}{\Gamma(\kappa+1)}+\frac{2^2\upsilon^{2\kappa}}{\Gamma(2\kappa+1)}+\frac{2^3\upsilon^{3\kappa}}{\Gamma(3\kappa+1)}+\cdots\right)\\
&= 2\sqrt{e^\varrho - e^{-4\varrho}}\sum_{n=0}^{\infty}\frac{(2\upsilon^\kappa)^n}{\Gamma(n\kappa+1)}
\end{aligned}$$

$$= 2\sqrt{e^{x} - e^{-4x}} E_{\alpha}\left(2\upsilon^{\kappa}\right),$$

where $E_{\alpha}\left(2\upsilon^{\kappa}\right)$ is the Mittag-Leffler function defined by (5).

At $\kappa \longrightarrow 1$, the exact solution obtained is

$$\chi(\varrho, \upsilon) = 2e^{2\upsilon}\sqrt{e^{\varrho} - e^{-4\varrho}}.$$

which is the same result, when we use the homotopy analysis method (HAM) [28].

Now, we consider the fractional reaction-diffusion-convection equation under Caputo–Fabrizio fractional derivative

$$\mathcal{D}_{\upsilon}^{(\kappa)}\chi = \left(\chi\chi_{\varrho}\right)_{\varrho} + 3\chi\chi_{\varrho} + 2\left(\chi - \chi^{2}\right), 0 < \kappa \leq 1, \tag{22}$$

subject to the condition

$$\chi(\varrho, 0) = 2\sqrt{e^{\varrho} - e^{-4\varrho}}. \tag{23}$$

Here $\chi = \left\{\chi(\varrho, \upsilon) : (\varrho, \upsilon) \in \mathbb{R}^{+} \times \mathbb{R}^{+}\right\}$.

Using the methodology of the FKHHPM presented in Section 3, we get the following result:

$$\begin{aligned}
&\mathfrak{p}^{0} : \chi_{0}(\varrho, \upsilon) = 2\sqrt{e^{\varrho} - e^{-4\varrho}},\\
&\mathfrak{p}^{1} : \chi_{1}(\varrho, \upsilon) = 4\left(1 - \kappa + \kappa\upsilon\right)\sqrt{e^{\varrho} - e^{-4\varrho}},\\
&\mathfrak{p}^{2} : \chi_{2}(\varrho, \upsilon) = 8\left((1 - \kappa)^{2} + 2\kappa(1 - \kappa)\upsilon + \kappa^{2}\frac{\upsilon^{2}}{2!}\right)\sqrt{e^{\varrho} - e^{-4\varrho}},\\
&\mathfrak{p}^{3} : \chi_{3}(\varrho, \upsilon) = 16\left((1 - \kappa)^{3} + 3\kappa(1 - \kappa)^{2}\upsilon + 3\kappa^{2}(1 - \kappa)\frac{\upsilon^{2}}{2!} + \kappa^{3}\frac{\upsilon^{3}}{3!}\right)\sqrt{e^{\varrho} - e^{-4\varrho}},\\
&\quad\vdots
\end{aligned}$$

Therefore, the approximate series solution to equations (22) and (23) is

$$\begin{aligned}
\chi(\varrho, \upsilon) &= \lim_{p \to 1}\sum_{n=0}^{\infty}\mathfrak{p}^{n}\chi_{n}(\varrho, \upsilon) = \sum_{n=0}^{\infty}\chi_{n}(\varrho, \upsilon)\\
&= \chi_{0}(\varrho, \upsilon) + \chi_{1}(\varrho, \upsilon) + \chi_{2}(\varrho, \upsilon) + \chi_{3}(\varrho, \upsilon) + \cdots.
\end{aligned}$$

At $\kappa \longrightarrow 1$, the exact solution obtained is

$$\begin{aligned}\chi(\varrho,\upsilon) &= 2\sqrt{e^{\varrho}-e^{-4\varrho}}\left(1+2\upsilon+\frac{(2\upsilon)^2}{2!}+\frac{(3\upsilon)^3}{3!}+\cdots\right)\\ &= 2e^{2\upsilon}\sqrt{e^{\varrho}-e^{-4\varrho}}.\end{aligned}$$

which is the same result when we use the HAM [28].

5 Results and discussion

In this section, we examine the figures and tables. The results are calculated graphically and numerically using MATLAB R2016a to demonstrate the approximate analytical solution compared to the exact solution. The simulation results reveal that the results obtained by the proposed method closely approximate the exact solution to (20) and (22). Figure 1 represents the surface graph of the 4-term approximate solutions by Caputo-FKHHPM at $\kappa = 0.8, 0.9, 1$ and the exact solution for (20). Figure 2 represents the surface graph of the 4-term approximate solutions by Caputo–Fabrizio-FKHHPM at $\kappa = 0.8, 0.9, 1$ and the exact solution for (22). Figure 3 represents the behavior of the 4-approximate solutions at $\varrho = 1$ for different values of κ and the exact solution for equations (20) and (22), respectively. According to these figures, we can say that when the order of the fractional derivative tends to 1, the approximate solutions obtained by FKHHPM tend continuously to the exact solutions. Tables 1 and 2 show the numerical values of the 4-term approximate solutions and the exact solution for different values of κ and υ at $\varrho = 0.1$ for equations (20) and (22), respectively. From the above, numerical results confirm that the approximate solutions are in good agreement with exact solutions at $\kappa = 1$ for (20) and (22), respectively, and the high precision of the proposed method. Also, it is to be noted that the accuracy can be improved by computing more terms of approximated solutions. In addition, it can be observed that an increase in υ does not significantly affect the accuracy of FKHHPM and that our method still approximates the exact solution sufficiently due to the fractional parameter.

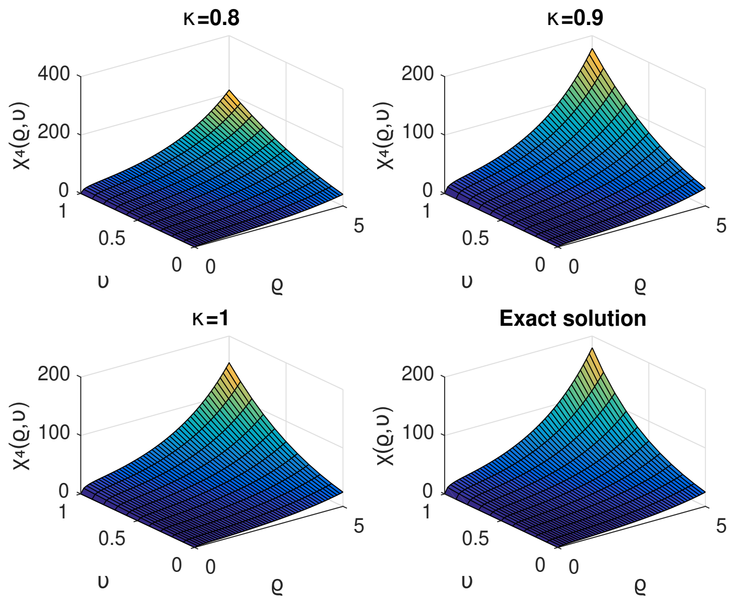

Figure 1: Surface graphs of the Caputo-FKHHPM solution and exact solution for (20).

Table 1: The values of the exact and numerical solutions of (20) for different values of κ and υ at $\varrho = 0.1$

υ	$\kappa = 0.7$	$\kappa = 0.8$	$\kappa = 0.9$	$\kappa = 1$	χ_{exact}	$\lvert\chi_{exact} - \chi_{FKHHPM-C}\rvert$
0.01	1.4415	1.3924	1.3631	1.3455	1.3455	8.8277×10^{-9}
0.03	1.6026	1.5045	1.4416	1.4004	1.4004	7.2082×10^{-7}
0.05	1.7484	1.6099	1.5190	1.4576	1.4576	5.6070×10^{-6}
0.07	1.8908	1.7147	1.5975	1.5170	1.5171	2.1716×10^{-5}
0.09	2.0334	1.8207	1.6780	1.5789	1.5790	5.9828×10^{-5}

6 Conclusion

In this study, we proposed a novel combination process known as the FKHHPM to solve the fractional reaction-diffusion-convection equations with Caputo and Caputo–Fabrizio fractional derivatives. A numerical application is solved to illustrate that the current method is efficient, accurate, and converges rapidly to the exact solution. Therefore, we can say that the proposed

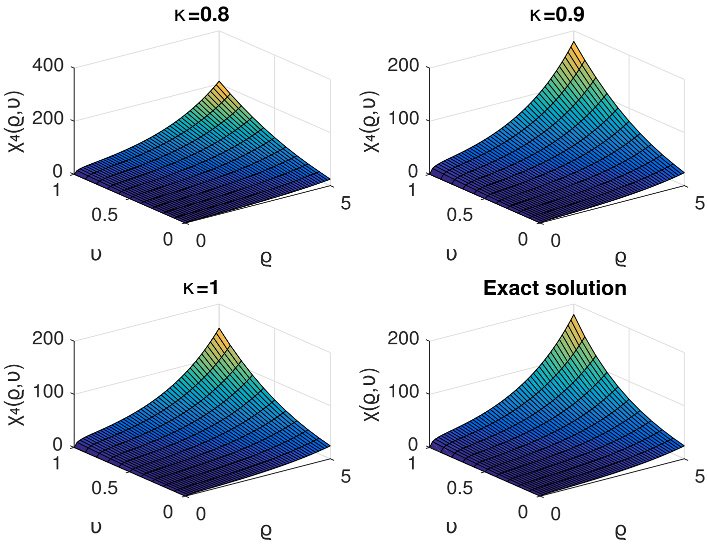

Figure 2: Surface graphs of the Caputo–Fabrizio-FKHHPM solution and exact solution for (22).

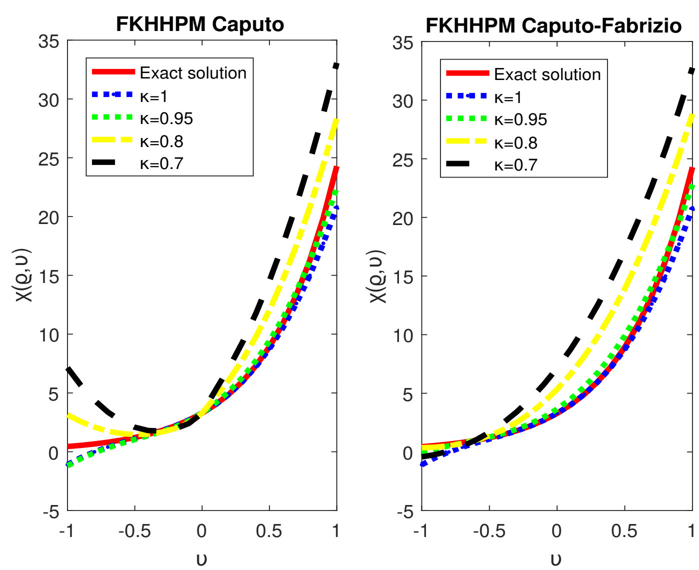

Figure 3: 2D plots of the numerical and exact solutions for (20) and (22) for different values of κ at $\varrho = 1$.

Table 2: The values of the exact and numerical solutions of (22) for different values of κ and υ at $\varrho = 0.1$

υ	$\kappa = 0.7$	$\kappa = 0.8$	$\kappa = 0.9$	$\kappa = 1$	χ_{exact}	$\lvert\chi_{exact} - \chi_{FKHHPM-CF}\rvert$
0.01	2.9308	2.1903	1.6824	1.3455	1.3455	8.8277×10^{-9}
0.03	3.0548	2.2895	1.7573	1.4004	1.4004	7.2082×10^{-7}
0.05	3.1818	2.4971	1.8351	1.4576	1.4576	5.6070×10^{-6}
0.07	3.3117	2.5510	1.9157	1.5170	1.5171	2.1716×10^{-5}
0.09	3.4447	2.6056	1.9993	1.5789	1.5790	5.9828×10^{-5}

method is a reliable and robust method for solving fractional differential equations, which has wide applications in natural sciences and engineering.

Acknowledgements

The author thanks the anonymous reviewers and the editor for their constructive comments.

References

[1] Abdeljawad, T. *On conformable fractional calculus*, Int. J. Comput. Appl. Math. 279 (2015) 57–66.

[2] Alazman, I., Mishra, M.N., Alkahtani, B.S. and Dubey, R.S *Analysis of infection and diffusion coefficient in an SIR model by using generalized fractional derivative*, Fractal Fract. 2024(8) (2024) 1–21.

[3] Allaire, G. and Raphael, A.L. *Homogenization of a convection-diffusion model with reaction in a porous medium*, C. R. Acad. Sci. Ser. I. 344 (2007) 523–528.

[4] Atangana, A. and Baleanu, D. *New fractional derivatives with non-local and non-singular kernel theory and application to heat transfer model*, Therm. Sci. 20(2) (2016), 763–769.

[5] Baskonus, H.M., Mekkaoui, T., Hammouch, Z. and Bulut, H. *Active control of a chaotic fractional order economic system*, Entropy. 17(8) (2015) 5771–5783.

[6] Caputo, M. and Fabrizio, M. *A new definition of fractional derivative without singular kernel*, Prog. Fract. Differ. Appl. 1(2) (2015) 73–85.

[7] Crauste, F., Lhassan, M. and Kacha, A. *A delay reaction–diffusion model of the dynamics of botulinum in fish*, Math. Biosci. 216 (2008) 17–29.

[8] Dastjerdi, R.H. and Ahmadi, G. *Designing the sinc neural networks to solve the fractional optimal control problem*, Iran. J. Numer. Anal. Optim. 14(4) (2024) 1016–1036.

[9] Dubeya, R.S., Mishraa, M.N. and Goswami, P. *Systematic analysis of oxygen diffusion problem having local fractional derivative*, Applied Innovative Research, 3(1-4) (2022) 113–120.

[10] Ferreira, S., Martins, M. and Vilela, M. *Reaction-diffusion model for the growth of avascular tumor*, Phys. Rev. 65(2) (2002) 1–12.

[11] He, J.H. *Application of homotopy perturbation method to nonlinear wave equations*, Chaos Soliton. Fract. 26 (2005) 695–700.

[12] Hilton, H.H. *Generalized fractional derivative anisotropic viscoelastic characterization*, Mater. 5 (2012) 169–191.

[13] Iaffaldano, G., Caputo, M. and Martino, S. *Experimental and theoretical memory diffusion of water in sand*, ; Hydrol. Earth Syst. Sci. 10(1) (2006) 93–100.

[14] Kamocki, R. *A new representation formula for the Hilfer fractional derivative and its application*, Int. J. Comput. Appl. Math. 308 (2016) 39–45

[15] Katugampola, U.N. *New approach to generalized fractional integral*, Appl. Math. Comput. 218 (2011) 860–865.

[16] Khalouta, A. *A new exponential type kernel integral transform: Khalouta transform and its applications*, Math. Montisnigri. 57 (2023) 5–23.

[17] Khalouta, A. *The study of nonlinear fractional partial differential equations via the Khalouta-Atangana-Baleanu operator*, J. Appl. Anal. Comput. 14(6) (2024) 3175–3196.

[18] Khalouta, A. *New results of the ρ-Jafari transform and their application to linear and nonlinear generalized fractional differential equations*, Rev. Colomb. Mat. 58(1) (2024), 25–46.

[19] Khalouta, A. *Khalouta transform via different fractional derivative operators,* Vestn. Samar. Gos. Tekhn. Univ., Ser. Fiz.-Mat. Nauki. 28(3) (2024), 407–425.

[20] Kilbas, A., Srivastava, H.M. and Trujillo, J.J. *Theory and application of fractional differential equations*, Elsevier, Amsterdam, 2006.

[21] Kumawat, N., Shukla, A., Mishra, M.N., Sharma, R. and Dubey, R.S. *Khalouta transform and applications to Caputo-fractional differential equations*, Front. Appl. Math. Stat. 10 (2024) 1351526.

[22] Losada, J. and Nieto, J.J. *Properties of a new fractional derivative without singular kernel*. Prog. Fract. Differ. Appl. 1(2) (2015) 87–92.

[23] Magin, R.L. *Fractional calculus in bioengineering*, Begell House Inc. Publishers, 2006.

[24] Miller, K.S. and Ross, B. *An introduction to the fractional calculus and fractional differential equations*, Wiley, New York, 1993.

[25] Mohyud-Din, S.T. Noor, M.A. and Noor, K.I. *Traveling wave solutions of seventh-order generalized KdV equation using He's polynomials*, Int. J. Nonlinear Sci. Numer. Simul. 10 (2009) 227–233.

[26] Podlubny, I. *Fractional differential equations*, Academic Press, New York, 1999.

[27] Sandev, T., Metzler, R. and Tomovski, Z. *Fractional diffusion equation with a generalized Riemann–Liouville time fractional derivative*, J. Phys. A: Math. Theor. 44 (2011) 1–21.

[28] Shidfar, A., Babaei, A. Molabahrami, A. and Alinejadmofradi, M. *Approximate analytical solutions of nonlinear reaction-diffusion-convection problem*, Math. Comput. Model. 53(1-2) (2011) 261–268.

[29] Siddheshwar, P.G. and Manjunath, S. *Unsteady convective-diffusion with heterogeneous chemical reaction in a plane-Poseuille flow of a micropolar fluid*, Internat. J. Engrg. Sci. 38 (2000) 765–783.

[30] Yadav, S.K., Purohit, M., Gour, M.M., Yadav, L.K. and Mishra, M.N., *Hybrid technique for multi-dimensional fractional diffusion problems involving Caputo–Fabrizio derivative*, Int. J. Math. Ind. 16(01) (2024) 2450020.