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Research Article

Bilinear optimal control of a reaction-diffusion equation: overcoming boundedness constraints on controls

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Abstract

We study an optimal control problem for bilinear reaction-diffusion equations. The novelty of our approach lies in considering controls from the space of essentially bounded functions without imposing a priori bounds on the admissible set. We introduce an auxiliary problem with controls in L^p spaces ($p > 1 + N/2$, where N is the spatial dimension) and demonstrate that both formulations are equivalent in terms of optimal solutions.

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Under suitable assumptions on the system parameters, we establish the existence of optimal controls and derive first-order optimality conditions. Our theoretical findings are supported by numerical simulations that validate the practical effectiveness of the proposed approach. This work provides a new framework for handling bilinear control problems without artificial constraints, offering potential applications in population dynamics and ecological management.

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1 Introduction

In this paper, we address the control problem of bilinear reaction-diffusion equations with essentially bounded controls. Specifically, we consider the following evolution problem:

$$\begin{cases} \dot{y}(x,t) + Ay(x,t) = u(x,t)L(y(x,t)) + f(x,t) \quad \text{in} \quad \Omega \times (0,T), \\ \dfrac{\partial y}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega, \\ y(0) = y_0. \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded Lipschitz domain, $A := -\Delta$ is the Laplacian operator, $T > 0$ is a fixed final time, $Q = \Omega \times (0,T)$, and $\Sigma = \partial\Omega \times (0,T)$. The Neumann boundary condition $\frac{\partial y}{\partial \mathbf{n}} = 0$ ensures that there is no flux across the boundary $\partial\Omega$, where $\mathbf{n}$ denotes the outward unit normal vector to $\partial\Omega$.

The optimization problem we study is as follows:

$$\text{Find } \ \bar{u} \ \text{ such that } \ J(\bar{u}) = \inf_{u \in L^\infty(Q)} J(u), \tag{P}$$

where the cost function is expressed as

$$J(u) := \frac{1}{2} \iint_Q (y(x,t) - y_d(x,t))^2 \,\mathrm{d}x\mathrm{d}t + \frac{\varepsilon}{p} \iint_Q |u(x,t)|^p \,\mathrm{d}x\mathrm{d}t, \tag{2}$$

with y_d representing a target state, y being the solution to (1) associated with u, and $\varepsilon > 0$ being a regularization parameter.

Throughout this paper, we assume the following assumptions:

Assumption 1. *The spatial dimension N satisfies $N \geq 2$.*
This condition ensures that the inclusion $H^1(\Omega) \subset L^2(\Omega)$ is compact. However, for the one-dimensional system, we can also consider homogeneous Dirichlet boundary condition where the injection $H_0^1(\Omega) \subset L^2(\Omega)$ is already compact.

Assumption 2. *The exponent p is chosen such that $1 + N/2 < p < \infty$. Additionally, y_d, $f \in L^p(Q)$, and $y_0 \in C(\bar{\Omega})$.*
These conditions are essential for deriving L^∞-norm bounds for both the state and adjoint state. The problem is also discussed under weaker assumptions (see Section 5).

Assumption 3. *The operator $L : L^{p^*}(\Omega) \to L^\infty(\Omega)$ is assumed to be bounded and linear, where p^* is the conjugate exponent of p, that is, $p^* = \frac{p}{p-1}$.*
Examples of such operators include spectral projections [32], Bochner–Riesz means and spectral measures [30, 10], as well as spectral multipliers of self-adjoint operators [29, 25]. Spectral multipliers, in particular, enable us to control the behavior of the original operator by modifying its spectrum through a specific function.

Under these assumptions, (1) serves as a fundamental model in population dynamics, describing the spatio-temporal evolution of a single species within a geographically isolated domain. The equation captures the combined effects of diffusive movement, represented by the term $Ay(x,t)$, and population growth dynamics, influenced by the bilinear control term $u(x,t)L(y(x,t))$, where $y(x,t)$ denotes the species density at position x and time t. The bilinear control structure is particularly notable, as the control $u(x,t)$ acts as a spatial distribution kernel that modulates reproduction rates, offering significant applications in ecological management and optimization problems. The non-local nature of the reaction term, $u(x,t)L(y(x,t))$, reflects the realistic scenario where population changes at a given point depend on the entire species distribution. Furthermore, the Neumann boundary conditions impose population conservation across the domain, ensuring no flux at the boundary. This model's formulation is applicable in various ecological contexts and control theory frameworks, with further details discussed in [17, 3, 20, 28, 2].

Traditional approaches for optimization with essentially bounded controls require imposing a priori bounds on the set of admissible controls to guarantee the existence of solutions (e.g., [4, 23, 13, 16, 14, 1, 26]). However, this practice presents significant drawbacks. Determining appropriate bounds can be challenging in practical applications. Moreover, choosing an upper bound that is not large enough can lead to suboptimal solutions, potentially compromising the effectiveness of the control strategy. The methods used in these references are therefore not suitable for solving the problem at hand.

Our work aims to overcome these limitations by developing a framework that does not require such constraints. We formulate the control problem with admissible controls in $L^\infty(Q)$ without directly imposing any explicit boundedness conditions on the controls. To tackle this challenge, we introduce an auxiliary optimal control problem, where the control set is the space $L^p(Q)$, as defined at the beginning of Section 3. Through rigorous analysis, we establish in Proposition 3 that these problems are equivalent in terms of optimality: A control solves the auxiliary problem if and only if it solves the original problem in $L^\infty(Q)$. This fundamental equivalence enables us to solve the original optimization problem without requiring a priori bounds.

This equivalence holds when both the optimal state $\bar{y}$ and optimal adjoint state $\bar{\varphi}$ belong to $L^\infty(Q)$. Notably, when $\bar{u} \in L^p(Q)$ solves the auxiliary problem, the necessary optimality condition yields

$$|\bar{u}(x,t)| = C|L(\bar{y}(x,t))\bar{\varphi}(x,t)|^{\frac{1}{p-1}} \quad \text{a.e. in } Q,$$

from which we deduce $\bar{u} \in L^\infty(Q)$. While the well-posedness of system (1) in $C([0,T];L^2(\Omega))$ is known [22, p. 66], our optimal control framework requires enhanced regularity properties. We establish the existence and uniqueness of solutions in $L^\infty(Q)$ for both state and adjoint equations through fixed-point analysis under Assumptions (2) and (3), where the interplay between L and p plays a crucial role in achieving this improved regularity.

Related problems have been studied in the literature for systems governed by an additive control operator [9, 8]. For instance, Casas and Kunisch [8] investigated the minimization of a quadratic cost function ($p = 2$) subject to an infinite-horizon semilinear system of dimension up to $N = 3$. For a bilinear control operator, when considering a quadratic cost function, the problem can only be addressed for one-dimensional systems (see section 5 for more details). Therefore, strong assumptions are required to handle bilinear optimal control, compared to additive control within the framework of minimization over $L^\infty(Q)$.

Considerable attention has been devoted in the literature to systems governed by semilinear equations with multiplicative controls. For instance, optimization problems involving purely time-dependent controls have been extensively explored [5, 34, 35], as have problems involving distributed controls, as discussed in [18, 31]. However, the methods developed in these studies are not directly applicable to the bilinear control problem addressed here, as they typically consider controls belonging to Hilbert spaces, while our framework necessitates a different approach. Additionally, studies on the controllability of such systems can be found in [27, 7].

The cost function analyzed in this work has been studied in various contexts. For instance, similar functionals have been explored in bilinear control problems involving chemotaxis-consumption systems in bounded domains [19, 11]. However, the methods used in these studies are not suitable for the specific bilinear optimal control problem considered here, as they rely on the assumption that optimal controls exist within Banach spaces of the form $L^r(Q)$, for some $r > 1$. Additionally, analogous functionals have been examined in optimal control problems involving purely time-dependent controls within bilinear infinite-dimensional systems [36]. This method is not applicable to our problem, as the admissible control set in our framework extends beyond pure time-dependent controls.

The primary contribution of this work is the formulation of a new optimal control problem with essentially bounded controls, where no pre-defined upper bounds are placed on the admissible controls. We also demonstrate the existence of optimal controls and derive their structure using first-order optimality conditions. Numerical simulations are provided to verify the theoretical findings and emphasize the practical relevance of the proposed approach.

The organization of the paper is as follows: In Section 2, we establish the well-posedness of the state equation. Section 3 proves the existence of an optimal control and derives the first-order necessary conditions for optimality. Section 4 presents numerical examples to illustrate the theoretical results. Section 5 offers additional remarks and discussions. Finally, section 6 concludes the paper.

2 Well-posedness of the state equation

This section focuses on establishing the well-posedness of the state equation. We start by defining the appropriate function spaces and discussing their properties. Next, we introduce the variational formulation and provide the definition of a weak solution to (1).
Let $W(0,T)$ denote the Hilbert space

$$W(0,T)=\left\{y\in L^2(0,T;H^1(\Omega)):\dot{y}\in H^1(\Omega)^*\right\},$$

endowed with the norm

$$\|y\|_{W(0,T)}=\left(\int_0^T\left(\|y(t)\|^2_{L^2(\Omega)}+\sum_{i=1}^N\left\|\frac{\partial y(t)}{\partial x_i}\right\|^2_{L^2(\Omega)}+\|\dot{y}(t)\|^2_{H^1(\Omega)^*}\right)\mathrm{d}t\right)^{1/2}.$$

This space admits a continuous embedding into $C([0,T];L^2(\Omega))$ [24], with embedding constant M_0:

$$\|y\|_{L^\infty(0,T,L^2(\Omega))}\leq M_0\|y\|_{W(0,T)}. \tag{3}$$

Moreover, by the Rellich–Kondrachov Theorem [15] and the Aubin–Lions Lemma [24], we have the compact embedding $W(0,T)\subset L^2(Q)$. For the variational formulation, we introduce the space $W_2^{1,1}(Q)$:

$$W_2^{1,1}(Q)=\left\{y\in L^2(Q):\dot{y}\in L^2(Q)\text{ and }\frac{\partial y}{\partial x_i}\in L^2(Q),\quad\text{for all }i=1,\dots,N\right\},$$

endowed with the norm

$$\|y\|_{W_2^{1,1}(Q)}=\left(\int_0^T\left(\|y(t)\|^2_{L^2(\Omega)}+\sum_{i=1}^N\left\|\frac{\partial y(t)}{\partial x_i}\right\|^2_{L^2(\Omega)}+\|\dot{y}(t)\|^2_{L^2(\Omega)}\right)\mathrm{d}t\right)^{1/2}.$$

The test function space $\mathcal{W}$ is defined as

$$\mathcal{W} = \left\{ v \in W_2^{1,1}(Q) : v(x,T) = 0 \text{ for almost all } x \in \Omega \right\}.$$

For the operator L, we define

$$|L| = \max \left\{ \|L\|_{\mathcal{L}(L^2(\Omega), L^\infty(\Omega))}, \|L\|_{\mathcal{L}(L^{p^*}(\Omega), L^\infty(\Omega))} \right\}.$$

The variational formulation is obtained by multiplying the state equation with a test function $v \in \mathcal{W}$ and integrating over Q:

$$\iint_Q \Big(-y\dot{v} + \nabla y \nabla v \Big) \mathrm{d}x\mathrm{d}t \quad = \iint_Q \Big(uL(y)v + fv \Big) \mathrm{d}x\mathrm{d}t + \int_\Omega y_0\, v(0)\, \mathrm{d}x. \quad (4)$$

Definition 1. A function $y \in W(0,T)$ is called a weak solution to the state equation (1) if it satisfies the variational equality (4) for all test functions $v \in \mathcal{W}$.

The following enhanced regularity result holds for linear systems.

Lemma 1. [33, Ch.V] Let $g \in L^p(Q)$. For any $z_0 \in C(\bar{\Omega})$, the system

$$\begin{cases} \dot{z} + Az = g & \text{in} \quad Q, \\ \frac{\partial z}{\partial \mathbf{n}} = 0 & \text{on} \quad \Sigma, \\ z(0) = z_0 & \text{in} \quad \Omega, \end{cases} \tag{5}$$

admits a unique solution $z \in W(0,T) \cap C(\bar{Q})$ satisfying

$$\|z\|_{W(0,T)} + \|z\|_{L^\infty(Q)} \leq c_\infty \Big(\|z_0\|_{L^\infty(\Omega)} + \|g\|_{L^p(Q)} \Big), \tag{6}$$

where c_∞ is a positive constant, non-decreasing with respect to $|Q|$ and independent of g and z_0.

Remark 1. The constant c_∞ in Lemma 1 depends only on $|Q|$, as detailed in [21, Ch.III]. Moreover, the estimate (6) remains valid with the same constant c_∞ when considering (5) over any subinterval $(\tau_1, \tau_2) \subseteq (0,T)$. This time-independence of c_∞ will be crucial in our subsequent analysis.

Building on this result, we establish the well-posedness of the state equation.

Proposition 1. For every $u \in L^p(Q)$, the state equation (1) possesses a unique weak solution $y \in W(0,T) \cap C(\bar{Q})$. In addition, this solution satisfies the following priori bound:

$$\|y\|_{W(0,T)} + \|y\|_{L^\infty(Q)} \leq C_u\Big(\|y_0\|_{L^\infty(\Omega)} + \|f\|_{L^p(Q)}\Big), \tag{7}$$

where C_u is a constant that does not depend on y_0 and f, and non-decreasing with respect to $\|u\|_{L^p(Q)}$.

Proof. Fix an arbitrary $\tau \in (0,T]$. Let $Q_\tau := \Omega \times (0,\tau)$, $\Sigma_\tau := \partial\Omega \times (0,\tau)$, and define the Banach space $X_\tau := W(0,\tau) \cap C(\bar{Q}_\tau)$ equipped with the norm $\|y\|_{X_\tau} = \|y\|_{W(0,\tau)} + \|y\|_{L^\infty(Q_\tau)}$.
Consider the mapping $\mathcal{Y} : X_\tau \to X_\tau$, where $\mathcal{Y}(w) = y$ solves

$$\begin{cases} \dot{y} + Ay = uL(w) + f & \text{in} \quad Q_\tau, \\ \frac{\partial y}{\partial \mathbf{n}} = 0 & \text{on} \quad \Sigma_\tau, \\ y(0) = y_0 & \text{in} \quad \Omega. \end{cases} \tag{8}$$

By Lemma 1, $\mathcal{Y}$ is well-defined since $uL(w) + f \in L^p(Q_\tau)$. To show $\mathcal{Y}$ is a contraction, take $w_1, w_2 \in X_\tau$ and let $z = \mathcal{Y}(w_1) - \mathcal{Y}(w_2)$. Then z satisfies

$$\begin{cases} \dot{z} + Az = uL(w_1 - w_2) & \text{in} \quad Q_\tau, \\ \frac{\partial z}{\partial \mathbf{n}} = 0 & \text{on} \quad \Sigma_\tau, \\ z(0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{9}$$

Applying Lemma 1 and using (3), we obtain

$$\begin{aligned} \|\mathcal{Y}(w_1) - \mathcal{Y}(w_2)\|_{X_\tau} &\leq c_\infty \Big\| \big\| u(t)L(w_1(t) - w_2(t)) \big\|_{L^p(\Omega)} \Big\|_{L^p(0,\tau)} \\ &\leq c_\infty |L| \, \Big\| \|u(t)\|_{L^p(\Omega)} \, \|w_1(t) - w_2(t)\|_{L^2(\Omega)} \Big\|_{L^p(0,\tau)} \\ &\leq c_\infty |L| \, \|u\|_{L^p(Q_\tau)} \|w_1 - w_2\|_{L^\infty(0,\tau;L^2(\Omega))} \\ &\leq c_\infty M_0 |L| \|u\|_{L^p(Q_\tau)} \|w_1 - w_2\|_{X_\tau}. \end{aligned}$$

Therefore, $\mathcal{Y}$ is a contraction mapping if

$$c_\infty M_0 |L| \|u\|_{L^p(Q_\tau)} < 1.$$

If the above condition is not met, we subdivide the interval $[0, \tau]$ into subintervals $[0, \tau_1], [\tau_1, \tau_2], \ldots, [\tau_{I-1}, \tau_I] = [\tau_{I-1}, \tau]$ such that

$$c_\infty M_0 |L| \|u\|_{L^p(\tau_i, \tau_{i+1}; L^p(\Omega))} \leq 1/2, \quad \text{for all } i = 0, \ldots, I-1,$$

and we consider the following systems:

$$(\mathcal{S}_0) \begin{cases} \dot{y}^0 + Ay^0 = uL(y^0) + f & \text{in} \quad \Omega \times (0, \tau_1)\,, \\ \frac{\partial y^0}{\partial \mathbf{n}} = 0 & \text{on} \quad \partial\Omega \times (0, \tau_1)\,, \\ y^0\,(0) = y_0 & \text{in} \quad \Omega, \end{cases}$$

and for $i = 1, \ldots, I-1$,

$$(\mathcal{S}_i) \begin{cases} \dot{y}^i + Ay^i = uL(y^i) + f & \text{in} \quad \Omega \times (\tau_i, \tau_{i+1})\,, \\ \frac{\partial y^i}{\partial \mathbf{n}} = 0 & \text{on} \quad \partial\Omega \times (\tau_i, \tau_{i+1})\,, \\ y^i\,(\tau_i) = y^{i-1}\big(\tau_i\big) & \text{in} \quad \Omega. \end{cases}$$

We then define solutions on each subinterval using a similar approach as before. This leads to a solution y for the entire interval $[0, \tau]$ by piecing together the solutions on subintervals.
Moreover, this solution satisfies

$$\|y\|_{X_\tau} \leq c_\infty \Big(\|y_0\|_{L^\infty(\Omega)} + \|uL(y) + f\|_{L^p(Q_\tau)} \Big). \tag{10}$$

This estimate can be further refined to obtain

$$\|y\|_{X_\tau} \leq c_\infty \Big(\|y_0\|_{L^\infty(\Omega)} + \|f\|_{L^p(Q_\tau)} + |L| \sqrt{|\Omega|}\, \|u\|_{L^p(Q_\tau)}\, \|y\|_{L^\infty(Q_\tau)} \Big). \tag{11}$$

We derive also from (10)

$$\begin{aligned} \|y(\tau)\|_{L^\infty(\Omega)} \leq c_\infty &\Bigg(\|y_0\|_{L^\infty(\Omega)} + \|f\|_{L^p(Q_\tau)} \\ &\quad + |L| \left(\int_0^\tau \|u(t)\|^p_{L^p(\Omega)} \|y(t)\|^p_{L^2(\Omega)} \,\mathrm{d}t \right)^{\frac{1}{p}} \Bigg) \\ \leq c_\infty &\Bigg(\|y_0\|_{L^\infty(\Omega)} + \|f\|_{L^p(Q_\tau)} \\ &\quad + |L| \sqrt{|\Omega|} \left(\int_0^\tau \|u(t)\|^p_{L^p(\Omega)} \|y(t)\|^p_{L^\infty(\Omega)} \,\mathrm{d}t \right)^{\frac{1}{p}} \Bigg). \end{aligned}$$

Taking both sides of this inequality to the p-th power, and using the fact $(a+b)^p \le 2^{p-1}(a^p+b^p)$ for any non-negative a and b, we get

$$\|y(\tau)\|^p_{L^\infty(\Omega)} \le 2^{p-1}c^p_\infty\Bigg(\|y_0\|^p_{L^\infty(\Omega)} + \|f\|^p_{L^p(Q_\tau)} + \left(|L|\sqrt{|\Omega|}\right)^p \int_0^\tau \|u(t)\|^p_{L^p(\Omega)}\|y(t)\|^p_{L^\infty(\Omega)}\,\mathrm{d}t\Bigg).$$

By applying Gronwall's lemma, we derive

$$\begin{aligned}\|y(\tau)\|^p_{L^\infty(\Omega)} &\le 2^{p-1}c^p_\infty\left(\|y_0\|^p_{L^\infty(\Omega)} + \|f\|^p_{L^p(Q_\tau)}\right)e^{2^{p-1}c^p_\infty\left(|L|\sqrt{|\Omega|}\right)^p\|u\|^p_{L^p(Q_\tau)}}\\ &\le 2^{p-1}c^p_\infty\left(\|y_0\|_{L^\infty(\Omega)} + \|f\|_{L^p(Q)}\right)^p e^{2^{p-1}\left(c_\infty|L|\sqrt{|\Omega|}\right)^p\|u\|^p_{L^p(Q)}}.\end{aligned}$$

This inequality is valid for all $\tau\in[0,T]$. Consequently, we have

$$\|y\|_{L^\infty(Q)} \le 2^{\frac{p-1}{p}}c_\infty\left(\|y_0\|_{L^\infty(\Omega)} + \|f\|_{L^p(Q)}\right)e^{\frac{2^{p-1}\left(c_\infty|L|\sqrt{|\Omega|}\right)^p}{p}\|u\|^p_{L^p(Q)}}. \quad (12)$$

Combining estimates (11) and (12), we get

$$\begin{aligned}&\|y\|_{W(0,T)} + \|y\|_{L^\infty(Q)}\\ &\le c_\infty\left(\|y_0\|_{L^\infty(\Omega)} + \|f\|_{L^p(Q)}\right)\\ &\quad + c^2_\infty|L|\sqrt{|\Omega|}\,\|u\|_{L^p(Q)}2^{\frac{p-1}{p}}\left(\|y_0\|_{L^\infty(\Omega)} + \|f\|_{L^p(Q)}\right)e^{\frac{2^{p-1}\left(c_\infty|L|\sqrt{|\Omega|}\right)^p}{p}\|u\|^p_{L^p(Q)}}.\end{aligned}$$

Finally, this estimate leads to the desired estimate (7) with C_u defined as

$$C_u = c_\infty + c^2_\infty|L|\sqrt{|\Omega|}\,\|u\|_{L^p(Q)}2^{\frac{p-1}{p}}e^{\frac{2^{p-1}\left(c_\infty|L|\sqrt{|\Omega|}\right)^p}{p}\|u\|^p_{L^p(Q)}}.$$

□

3 Existence and characterization of optimal controls

In this section, we analyze the optimal control problem. We introduce an auxiliary optimization problem with controls from the function space $L^p(Q)$. Using standard methods from the calculus of variations, we establish the existence of a solution and provide its characterization via an optimality system. Finally, we show that any solution to the auxiliary problem is also a solution to the original problem (P).

Consider the following auxiliary problem:

$$\text{Find } \bar{u} \text{ such that } J(\bar{u}) = \inf_{u \in L^p(Q)} J(u). \tag{P$_A$}$$

In the following proposition, we show that this problem has a solution.

Proposition 2. The minimization problem (P_A) admits at least one solution.

Proof. Let us begin by selecting a minimizing sequence (u_n) in $L^p(Q)$. The coercivity property of J over $L^p(Q)$ ensures that this sequence is bounded. By the Banach-Alaoglu theorem, we can extract a subsequence that converges weakly to some element $\bar{u}$ in $L^p(Q)$.
For each control u_n, let y_n denote the corresponding state solution to (1). From the boundedness properties established in Proposition 1, we know that (y_n) forms a bounded sequence in $W(0,T)$. The reflexivity of this space allows us to extract a subsequence that converges weakly to some $\bar{y}$ in $W(0,T)$. Moreover, the compact embedding theorem yields strong convergence of a subsequence to $\bar{y}$ in $L^2(Q)$.
We must now verify that $\bar{y}$ is indeed the state associated with the control $\bar{u}$. For any test function $v \in \mathcal{W}$, we have the weak convergences

$$\begin{aligned} \dot{y}_n &\rightharpoonup \dot{\bar{y}} \quad \text{weakly in } L^2(0,T;H^1(\Omega)^*), \\ Ay_n &\rightharpoonup A\bar{y} \text{ weakly in } L^2(0,T;H^1(\Omega)^*). \end{aligned}$$

These convergences imply

$$\iint_Q (-y_n \dot{v} + \nabla y_n \nabla v)\, \mathrm{d}x\mathrm{d}t \underset{n\to\infty}{\longrightarrow} \iint_Q (-\bar{y}\dot{v} + \nabla \bar{y} \nabla v)\, \mathrm{d}x\mathrm{d}t. \tag{13}$$

For the remaining terms, we have the following decomposition:

$$u_n L(y_n) - \bar{u} L(\bar{y}) = (u_n - u) L(\bar{y}) + u_n L(y_n - \bar{y}). \tag{14}$$

The mapping $u \mapsto uL(\bar{y})$ is continuous from $L^p(Q)$ to $L^2(Q)$, yielding (see [6, Theorem III-3.10])

$$(u_n - \bar{u})L(\bar{y}) \rightharpoonup 0 \quad \text{weakly in} \quad L^2(Q). \tag{15}$$

Furthermore, we derive

$$\begin{aligned}
&\iint_Q u_n L(y_n - \bar{y}) v \,\mathrm{d}x\mathrm{d}t \\
&\leq \int_0^T \|u_n(t) L(y_n(t) - \bar{y}(t))\|_{L^2(\Omega)} \|v(t)\|_{L^2(\Omega)} \,\mathrm{d}t \\
&\leq |L| |\Omega|^{\frac{p+2}{2p}} \int_0^T \|u(t)\|_{L^p(\Omega)} \|y_n(t) - \bar{y}(t)\|_{L^2(\Omega)} \|v(t)\|_{L^2(\Omega)} \,\mathrm{d}t \\
&\leq |L| (|\Omega| T)^{\frac{p+2}{2p}} \|u_n\|_{L^p(Q)} \|v\|_{L^\infty(0,T;L^2(\Omega))} \|y_n - \bar{y}\|_{L^2(Q)} \underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Combining this with (13) and (15), we conclude that $\bar{y}$ is indeed the solution to (1) corresponding to the control $\bar{u}$.

Finally, the lower semicontinuity of norms yields

$$J(\bar{u}) \leq \liminf_n J(u_n) \leq \inf_{u \in L^p(Q)} J(u),$$

which proves the optimality of $\bar{u}$ in $L^p(Q)$. □

We introduce the control-to-state map G, defined as $G : L^p(Q) \longrightarrow W(0,T)$, where $G(u) = y$ represents the solution to (1) corresponding to u. The following lemma establishes the differentiability property of G.

Lemma 2. The operator G is Fréchet differentiable. Its derivative at $u \in L^p(Q)$ in the direction of $h \in L^p(Q)$, denoted by z_h, is defined as follows:

$$\begin{cases}
\dot{z}_h + A z_h = u L(z_h) + h L(G(u)), \\
\frac{\partial z_h}{\partial \mathbf{n}} = 0, \\
z_h(0) = 0.
\end{cases} \tag{16}$$

Proof. Let us consider arbitrary controls u and h in $L^p(Q)$. According to Proposition 1, both $G(u)$ and $G(u+h)$ exist in $W(0,T)$ and satisfy the following inequalities:

$$\begin{aligned}
\|G(u)\|_{W(0,T)} &\leq C_u \left(\|y_0\|_{L^\infty(\Omega)} + \|f\|_{L^p(Q)} \right), \\
\|G(u+h)\|_{W(0,T)} &\leq C_{u+h} \left(\|y_0\|_{L^\infty(\Omega)} + \|f\|_{L^p(Q)} \right).
\end{aligned}$$

The mapping $h \mapsto z_h$ is well-defined and bounded:

$$\begin{aligned}\|z_h\|_{W(0,T)} &\leq C_u \|hL(G(u))\|_{L^p(Q)}\\ &\leq C_u |L| \|y\|_{L^\infty(0,T;L^2(\Omega))} \|h\|_{L^p(Q)}\\ &\leq C_u |L| M_0 \|y\|_{W(0,T)} \|h\|_{L^p(Q)}\\ &\leq c_1 \|h\|_{L^p(Q)}.\end{aligned}$$

Here, $c_1 = C_u^2 |L| M_0 \left(\|y_0\|_{L^\infty(\Omega)} + \|f\|_{L^p(Q)}\right)$.
We define $w_1 = G(u+h) - G(u)$. Then, w_1 satisfies the following system:

$$\begin{cases} \dot{w}_1 + Aw_1 = uL(w_1) + hL(G(u+h)),\\ \frac{\partial w_1}{\partial \mathbf{n}} = 0,\\ w_1(0) = 0. \end{cases} \tag{17}$$

This system has a solution satisfying

$$\begin{aligned}\|w_1\|_{W(0,T)} &\leq C_u \|hL(G(u+h))\|_{L^p(Q)}\\ &\leq c_2 \|h\|_{L^p(Q)}.\end{aligned}$$

Here, $c_2 = C_u C_{u+h} |L| M_0 \left(\|y_0\|_{L^\infty(\Omega)} + \|f\|_{L^p(Q)}\right)$.
Next, we define $w_2 = w_1 - z_h$. Then, w_2 satisfies

$$\begin{cases} \dot{w}_2 + Aw_2 = uL(w_2) + hL(w_1),\\ \frac{\partial w_2}{\partial \mathbf{n}} = 0,\\ w_2(0) = 0. \end{cases} \tag{18}$$

Furthermore, we have

$$\begin{aligned}\|w_2\|_{W(0,T)} &\leq C_u \|hL(w_1)\|_{L^p(Q)}\\ &\leq c_3 \|h\|^2_{L^p(Q)}.\end{aligned}$$

Here, $c_3 = C_u |L| M_0 c_2$. Consequently,

$$\frac{\|G(u+h) - G(u) - z_h\|_{W(0,T)}}{\|h\|_{L^p(Q)}} \leq c_3 \|h\|_{L^p(Q)} \longrightarrow 0 \quad \text{as} \quad \|h\|_{L^p(Q)} \to 0.$$

□

The Fréchet differentiability of the control-to-state map and the smoothness of the L^2 and L^p norms ensure the differentiability of the cost function.

Lemma 3. The cost function is Fréchet differentiable. Its derivative at u in the direction of h is given by

$$J'(u)h = \iint_Q \Big((y - y_d)z_h \; + \varepsilon \,\mathrm{sgn}(u)\, |u|^{p-1} h\Big)\, \mathrm{d}x\mathrm{d}t, \tag{19}$$

where z_h is the solution to (16), and sgn stands for the signum function.

Since the operator $L : L^{p^*}(\Omega) \to L^\infty(\Omega)$ is bounded and linear, we define its adjoint $L^* : L^1(\Omega) \to L^p(\Omega)$ as follows:

$$\langle L(y_1), y_2\rangle_{L^\infty(\Omega),L^1(\Omega)} = \langle y_1, L^*(y_2)\rangle_{L^{p^*}(\Omega),L^p(\Omega)}, \qquad \text{for all } y_1 \in L^{p^*}(\Omega),\ y_2 \in L^1(\Omega).$$

This adjoint operator L^* is also bounded and linear.

We now prove the existence of optimal solutions and provide their characterization for the auxiliary problem (P_A).

Theorem 1. Problem (P_A) has at least one solution. Moreover, for every solution $\bar{u}$, there exist $\bar{y}, \bar{\varphi} \in W(0,T) \cap C(\bar{Q})$, such that the following conditions hold:

$$\begin{cases} \dot{\bar{y}} + A\bar{y} = \bar{u}L(\bar{y}) + f & \text{in} \quad Q, \\ \frac{\partial \bar{y}}{\partial \mathbf{n}} = 0 & \text{on} \quad \Sigma, \\ \bar{y}(0) = y_0 & \text{in} \quad \Omega, \end{cases} \tag{20}$$

$$\begin{cases} \dot{\bar{\varphi}} - A\bar{\varphi} = -L^*(\bar{u}\bar{\varphi}) - (\bar{y} - y_d) & \text{in } Q, \\ \frac{\partial \bar{\varphi}}{\partial \mathbf{n}} = 0 & \text{on } \Sigma, \\ \bar{\varphi}(T) = 0 & \text{in } \ \Omega, \end{cases} \tag{21}$$

$$\iint_Q \Big(L(\bar{y})\bar{\varphi} + \varepsilon \,\mathrm{sgn}(\bar{u})\, |\bar{u}|^{p-1}\Big)(u - \bar{u})\, \mathrm{d}x\mathrm{d}t \geq 0, \qquad \text{for all } u \in L^p(Q), \tag{22}$$

and

$$\bar{u}(x,t) = -\frac{\mathrm{sgn}(L(\bar{y})\bar{\varphi})}{\varepsilon^{\frac{1}{p-1}}}\, |L(\bar{y})\bar{\varphi}|^{\frac{1}{p-1}} \quad \text{a.e. in} \quad Q. \tag{23}$$

Proof. By Proposition 2, the problem (P_A) has a solution $\bar{u}$. Let $\bar{y}$ denote the corresponding trajectory that satisfies (20). We will first prove the existence of $\bar{\varphi}$ that satisfies (21). Then derive the variational inequality (22), and conclude by obtaining the optimal control characterization in (23).

Step 1: Existence and uniqueness of $\bar{\varphi}$.

We proceed to prove that (21) admits a unique solution. To this end, we consider the transformation

$$q(t) = \bar{\varphi}(T-t), \quad v(t) = \bar{u}(T-t), \quad g(t) = \bar{y}(T-t) - y_d(T-t),$$

for almost all $t \in [0,T]$. This transforms (21) into

$$\begin{cases} \dot{q} + Aq = L^*(vq) + g & \text{in } Q, \\ \frac{\partial q}{\partial \mathbf{n}} = 0 & \text{on } \Sigma, \\ q(0) = 0 & \text{in } \Omega. \end{cases}$$

Since $\bar{y} - y_d \in L^p(Q)$, Proposition 1 ensures a unique solution $q \in W(0,T) \cap C(\bar{Q})$. Thus, there exists a unique $\bar{\varphi} \in W(0,T) \cap C(\bar{Q})$ satisfying (21).

Step 2: Derivation of the inequality (22).

As $\bar{u}$ is optimal for problem (P_A), we have $J'(\bar{u})(u-\bar{u}) \geq 0$ for all $u \in L^p(Q)$. Let $h = u - \bar{u}$. Using Lemma 3, we can express this as

$$J'(\bar{u})h = \iint_Q \Big((\bar{y} - y_d) z_h + \varepsilon \operatorname{sgn}(\bar{u}) \, |\bar{u}|^{p-1} h \Big) \, \mathrm{d}x\mathrm{d}t \geq 0, \tag{24}$$

where z_h satisfies (16).

Now, let us focus on simplifying the term $\iint_Q (\bar{y} - y_d) z_h \, \mathrm{d}x\mathrm{d}t$:

$$\begin{aligned} \iint_Q (\bar{y} - y_d) z_h \, \mathrm{d}x\mathrm{d}t &= \iint_Q (-\dot{\bar{\varphi}} + A\bar{\varphi} - L^*(\bar{u}\bar{\varphi})) z_h \, \mathrm{d}x\mathrm{d}t \quad (\text{using (21)}) \\ &= \iint_Q (-\dot{\bar{\varphi}} z_h + \bar{\varphi} A z_h - L^*(\bar{u}\bar{\varphi}) z_h) \, \mathrm{d}x\mathrm{d}t \\ &= \iint_Q (\bar{\varphi} \dot{z}_h + \bar{\varphi} A z_h - L^*(\bar{u}\bar{\varphi}) z_h) \, \mathrm{d}x\mathrm{d}t \\ &\quad - \int_\Omega \bar{\varphi}(T) z_h(T) \mathrm{d}x \\ &\quad + \int_\Omega \bar{\varphi}(0) z_h(0) \, \mathrm{d}x. \qquad (\text{integration by parts}) \end{aligned}$$

Using the equation for z_h, and noting that $\bar{\varphi}(T) = 0$ and $z_h(0) = 0$, we get

$$\iint_Q (\bar{y} - y_d) z_h \, \mathrm{d}x\mathrm{d}t = \iint_Q (\bar{\varphi} \bar{u} L(z_h) + \bar{\varphi} h L(\bar{y}) - L^*(\bar{u}\bar{\varphi}) z_h) \, \mathrm{d}x\mathrm{d}t.$$

By definition of L^*, we infer

$$\iint_Q (\bar{y} - y_d) z_h \, \mathrm{d}x\mathrm{d}t = \iint_Q h \bar{\varphi} L(\bar{y}) \, \mathrm{d}x\mathrm{d}t.$$

Substituting this back into (24), we obtain

$$\iint_Q \left(h\bar{\varphi}L(\bar{y}) + \varepsilon\,\mathrm{sgn}(\bar{u})\,|\bar{u}|^{p-1}h \right) \mathrm{d}x\mathrm{d}t \geq 0,$$

which gives the variational inequality (22).

Step 3: Derivation of characterization (23).

From the variational inequality (22), we can deduce the following pointwise relationship:

$$L(\bar{y})\bar{\varphi} + \varepsilon\,\mathrm{sgn}(\bar{u})\,|\bar{u}|^{p-1} = 0 \quad \text{a.e. in } Q. \tag{25}$$

The derivation of the pointwise condition (25) from the variational inequality (22) is standard, see for instance the method outlined in [33, Section 2.8]. Rearranging this equation and taking the $(p-1)$-th root, we obtain

$$\bar{u} = -\frac{\mathrm{sgn}(L(\bar{y})\bar{\varphi})}{\varepsilon^{\frac{1}{p-1}}}|L(\bar{y})\bar{\varphi}|^{\frac{1}{p-1}} \quad \text{a.e. in } Q,$$

which is precisely the optimal control characterization (23). □

Remark 2. It is worth noting that for $p = 2$, a variant of the previous result can be obtained under weaker assumptions: Replacing (2) with $y_d, f \in L^2(Q)$ and $y_0 \in L^2(\Omega)$, as discussed in Zerrik and Boukhari [36] using a semigroup framework. However, our subsequent main result maintains the original assumptions to address the problem at hand.

Now, we state our main result.

Theorem 2. Problem (P) admits a solution $\bar{u}$. Moreover, for any solution $\bar{u}$, there exist $\bar{y}, \bar{\varphi} \in W(0,T) \cap C(\bar{Q})$ satisfying the following conditions:

$$\begin{cases} \dot{\bar{y}} + A\bar{y} = \bar{u}L(\bar{y}) + f & \text{in} \quad Q, \\ \frac{\partial \bar{y}}{\partial \mathbf{n}} = 0 & \text{on} \quad \Sigma, \\ \bar{y}(0) = y_0 & \text{in} \quad \Omega, \end{cases} \tag{26}$$

$$\begin{cases} \dot{\bar{\varphi}} - A\bar{\varphi} = -L^*(\bar{u}\bar{\varphi}) - (\bar{y} - y_d) & \text{in } Q, \\ \frac{\partial \bar{\varphi}}{\partial \mathbf{n}} = 0 & \text{on } \Sigma, \\ \bar{\varphi}(T) = 0 & \text{in } \Omega, \end{cases} \tag{27}$$

$$\iint_Q \left(L(\bar{y})\bar{\varphi} + \varepsilon \operatorname{sgn}(\bar{u})\,|\bar{u}|^{p-1}\right)(h-\bar{u})\,\mathrm{d}x\mathrm{d}t \geq 0, \qquad \text{for all } h \in L^\infty(Q), \tag{28}$$

and

$$\bar{u}(x,t) = -\frac{\operatorname{sgn}(L(\bar{y})\bar{\varphi})}{\varepsilon^{\frac{1}{p-1}}}\,|L(\bar{y})\bar{\varphi}|^{\frac{1}{p-1}} \quad \text{a.e. in} \quad Q. \tag{29}$$

Proof. Let $\bar{u}$ be a solution to problem (P_A), with associated state and adjoint variables $\bar{y}, \bar{\varphi} \in W(0,T) \cap C(\bar{Q})$. Let us prove that $\bar{u}$ is also a solution to problem (P).

First, let $\mathring{y}$ denote the solution to (26) under the null control. By Lemma 1, we conclude that

$$\|\mathring{y}\|_{W(0,T)} + \|\mathring{y}\|_{L^\infty(Q)} \leq c_\infty(\|y_0\|_{L^\infty(\Omega)} + \|f\|_{L^p(Q)}),$$

which implies

$$\|\mathring{y}\|_{L^2(Q)} \leq K_0 := c_\infty(\|y_0\|_{L^\infty(\Omega)} + \|f\|_{L^p(Q)}).$$

Since $\bar{u}$ is optimal for problem (P_A), we have $J(\bar{u}) \leq J(0)$, leading to

$$\|\bar{u}\|^p_{L^p(Q)} \leq \frac{p}{2\varepsilon}\left(\|\mathring{y} - y_d\|^2_{L^2(Q)}\right) \leq \frac{p}{\varepsilon}(K_0^2 + \|y_d\|^2_{L^2(Q)}),$$

which gives

$$\|\bar{u}\|_{L^p(Q)} \leq K_1 := \left(\frac{p}{\varepsilon}(K_0^2 + \|y_d\|^2_{L^2(Q)})\right)^{\frac{1}{p}}.$$

Using Proposition 1, we estimate $\bar{y}$ as

$$\|\bar{y}\|_{W(0,T)} + \|\bar{y}\|_{L^\infty(Q)} \leq C_{K_1}(\|y_0\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)}),$$

where C_{K_1} depends on K_1. The embedding $C(\bar{Q}) \subset L^p(Q)$ yields

$$\|\bar{y}\|_{L^p(Q)} \leq |Q|^{\frac{1}{p}} C_{K_1}(\|y_0\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)}).$$

For the operator L, we have

$$\|L(\bar{y})\|_{L^\infty(Q)} \leq K_2 := |L| M_0 C_{K_1}(\|y_0\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)}).$$

For $\bar{\varphi}$ satisfying (27), Theorem 1 implies

$$\|\bar{\varphi}\|_{L^\infty(Q)} \leq K_3 := C_{K_1}\left(|Q|^{\frac{1}{p}} C_{K_1}(\|y_0\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)}) + \|y_d\|_{L^p(Q)}\right).$$

Finally, using (29) and the bounds on $L(\bar{y})$ and $\bar{\varphi}$, we conclude

$$|\bar{u}(x,t)| \leq \frac{1}{\varepsilon^{\frac{1}{p-1}}}(K_2K_3)^{\frac{1}{p-1}} =: K_4 \quad \text{a.e. in } Q.$$

This shows that $\bar{u} \in L^\infty(Q)$ with $\|\bar{u}\|_{L^\infty(Q)} \leq K_4$. Since $L^\infty(Q) \subset L^p(Q)$, we conclude that $\bar{u}$ is also a solution of problem (P). $\square$

The next result shows that both optimization problems share identical solution sets.

Proposition 3. Problems (P) and (P_A) are equivalent; that is, they have the same solutions.

Proof. Suppose that $\bar{u}$ is a solution of problem (P_A). By using the proof of Theorem 2, we conclude that $\bar{u}$ is also a solution of problem (P).

For the reverse implication, let $\bar{u}$ be a solution to problem (P), and consider any $u \in L^p(Q)$. We construct a sequence by defining $u_n = \Pi_{[-n,n]}(u)$ for each $n \in \mathbb{N}$, where $\Pi_{[-n,n]}$ represents the pointwise projection onto the interval $[-n,n]$. This sequence has two key properties:

- $u_n \in L^\infty(Q)$ for each n,
- $u_n \to u$ in $L^p(Q)$ as $n \to \infty$.

Since $\bar{u}$ is optimal for problem (P), we have

$$J(\bar{u}) \leq J(u_n) \quad \text{for all } n \in \mathbb{N}.$$

The continuity of J allows us to pass to the limit as $n \to \infty$, yielding

$$J(\bar{u}) \leq J(u).$$

As u was chosen arbitrarily in $L^p(Q)$, this proves that $\bar{u}$ solves problem (P_A). $\square$

4 Simulations and numerical results

In this section, we provide numerical examples to illustrate our theoretical results and examine the behavior of the bilinear optimal control problem in both one- and two-dimensional settings.

4.1 Problem setup

A practical scenario motivating this study arises in ecological management, where the objective is to regulate the population of a species within a geographically bounded region Ω (where Ω could be a linear habitat in one-dimensional, or a surface in two-dimensional). The species density $y(x,t)$ evolves according to the dynamics:

$$\frac{\partial y}{\partial t} - \Delta y = u(x,t)\int_\Omega y(x,t)\,\mathrm{d}x, \quad (x,t) \in \Omega \times (0,T),$$

with the initial distribution $y(x,0) = y_0(x)$. Here, the control $u(x,t)$ represents interventions, such as resource allocation or habitat modification, which influence the reproduction dynamics. The goal is to steer the population density $y(x,t)$ to track a desired trajectory distribution $y_d(x,t)$ over a specified period T (e.g., one week). No explicit constraints are imposed on the control input $u(x,t)$, as it is not known whether such constraints exist, and the focus is on determining the optimal control strategy while minimizing interventions.

This problem can be reformulated as problem (P), with the operator $L : L^1(\Omega) \to L^\infty(\Omega)$ defined as

$$L(y) = \int_\Omega y(x)\,\mathrm{d}x.$$

and $f = 0$. We also consider the parameters $T = 1$ and $\varepsilon = 10^{-4}$.

A solution $\bar{u}$ to this problem must satisfy the following optimality system:

$$\frac{\partial \bar{y}}{\partial t} - \Delta \bar{y} = \bar{u}L(\bar{y}), \quad \bar{y}(0) = y_0, \tag{30}$$

$$-\frac{\partial \bar{\varphi}}{\partial t} - \Delta \bar{\varphi} = (\bar{y} - y_d) + L(\bar{u}\bar{\varphi}), \quad \bar{\varphi}(T) = 0, \tag{31}$$

and

$$\bar{u}(x,t) = -\frac{\operatorname{sgn}(L(\bar{y})\bar{\varphi})}{\varepsilon^{\frac{1}{p-1}}} \left|L(\bar{y})\bar{\varphi}\right|^{\frac{1}{p-1}}. \tag{32}$$

The boundary conditions for (30) and (31) will be specified later. We choose p such that $1 + \frac{N}{2} < p < \infty$, in accordance with Assumption (2) of our theoretical framework.

4.2 Numerical method

We employ the thresholding operator

$$u_{m+1} = S_\alpha(u_m, L(y_m)\varphi_m), \tag{33}$$

where

$$S_\alpha(u,v) = \frac{1}{\alpha + \varepsilon^{\frac{1}{p-1}}} \left(\alpha u - \operatorname{sgn}(v)|v|^{\frac{1}{p-1}}\right). \tag{34}$$

Here, $\alpha > 0$ is an accuracy threshold. The convergence of this algorithm has been rigorously established in the literature (see, for instance, [12]).

The iterative procedure for solving the optimization problem is as follows:

Algorithm 1 Optimal Control Algorithm

1: Initialize $u_0 = 0$, set $m = 0$, choose $\alpha > 0$, and tol > 0 a chosen tolerance
2: **repeat**
3: Compute y_m by solving forward the state equation.
4: Determine φ_m by solving backward the adjoint equation
5: Update control via the thresholding operator:
6: $u_{m+1} = S_\alpha(u_m, L(y_m)\varphi_m)$,
7: $m \leftarrow m + 1$
8: **until** $\|u_m - u_{m-1}\|_{L^\infty(Q)} <$ tol.

4.3 One-dimensional case

For the one-dimensional case, we examine a system on the unit interval $\Omega = (0,1)$ with $N = 1$ and $p = 2$. The system is governed by homogeneous Dirichlet boundary conditions, representing a scenario where the population density vanishes at the domain boundaries. The complete system is formulated as

$$\begin{cases} \frac{\partial y}{\partial t} - \frac{\partial^2 y}{\partial x^2} = u(x,t) \int_0^1 y(x,t)\,\mathrm{d}x, \\ y(0,t) = y(1,t) = 0, \\ y(x,0) = x - x^2. \end{cases} \tag{35}$$

We choose a target state $y_d(x,t) = \frac{1}{2}\sin(\pi x)$, which represents a desired sinusoidal distribution pattern for the population density. This choice of target state is particularly relevant for ecological applications as it maintains zero density at the boundaries while achieving maximum population density at the center of the domain.

The numerical results, presented in Figure 1, reveal several key aspects of the control system's behavior. The optimal control profile demonstrates smooth temporal evolution while maintaining essential boundedness, despite no explicit bounds being imposed. This natural boundedness validates our theoretical framework's core prediction. The corresponding state trajectory shows effective tracking of the target profile, successfully counteracting the system's natural dissipative tendency that would otherwise drive the population to extinction ($u = 0$ case). The tracking error exhibits monotonic decrease, indicating stable convergence of the control algorithm, while the cost functional's evolution demonstrates the effective balance between tracking accuracy and control effort maintained by our chosen cost structure. These results provide strong numerical evidence for the practical viability of our theoretical framework in the one-dimensional setting.

4.4 Two-dimensional case

For the two-dimensional case, we consider a more complex setting with $\Omega = (0,1)^2$ and parameters $N = 2$ and $p = 3$, where the higher value of p satisfies our theoretical requirement $p > 1 + N/2$. The system is subject to Neumann boundary conditions, representing no-flux conditions at the domain boundaries, a natural choice for population dynamics in enclosed habitats. The complete system takes the form:

$$\begin{cases} \frac{\partial y}{\partial t} - \left(\frac{\partial^2 y}{\partial x_1^2} + \frac{\partial^2 y}{\partial x_2^2}\right) = u(x_1,x_2,t)\int_\Omega y(x_1,x_2,t)\,\mathrm{d}x_1\mathrm{d}x_2, \\ \frac{\partial y}{\partial n} = 0 \text{ on } \partial\Omega, \\ y(x_1,x_2,0) = 2x_1^3 - 3x_1^2. \end{cases} \tag{36}$$

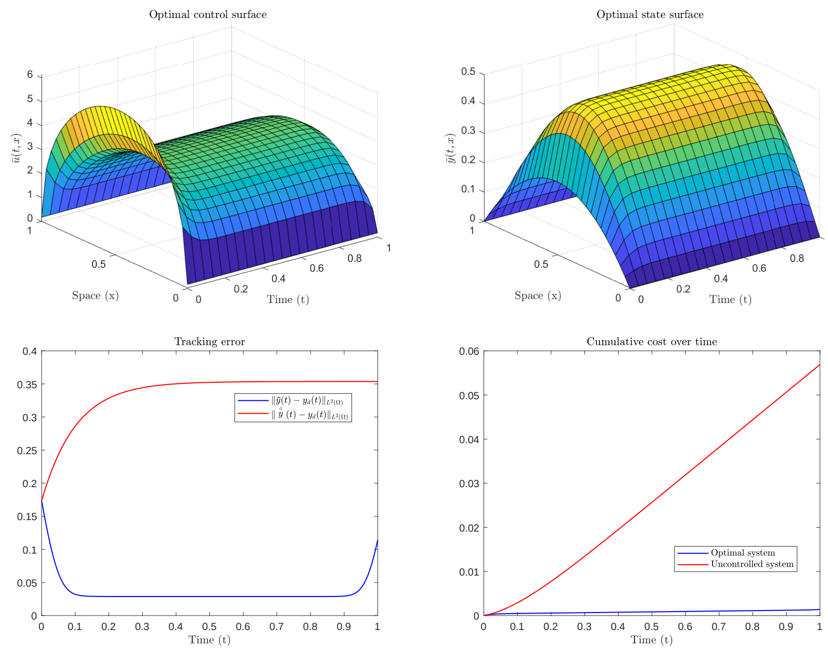

Figure 1: Optimal control (top left), corresponding state (top right), evolution of tracking error (bottom left), and cost functional (bottom right) for the one-dimensional case.

The target state is chosen as $y_d(x_1, x_2, t) = \frac{1}{2}\cos(\pi x_1)\cos(\pi x_2)\sin(\pi t)$, representing a time-varying spatial pattern that tests the system's ability to track complex spatio-temporal distributions.

The numerical results, displayed in Figure 2, demonstrate the framework's robustness in higher dimensions. The evolution of control norms confirms the maintenance of essential boundedness without explicit constraints, even in this more complex setting. The state and adjoint state norms exhibit stable behavior throughout the time horizon, indicating well-regulated dynamics. The tracking error and cumulative cost profiles show consistent convergence, though with slightly different characteristics compared to the one-dimensional case due to the increased spatial dimension. The success in maintaining the desired state pattern, particularly notable given the time-varying nature of the target, validates our theoretical framework's applicability to higher-dimensional scenarios. The effective control performance without a priori bounds demonstrates the framework's practical utility for real-world applications in spatial population dynamics.

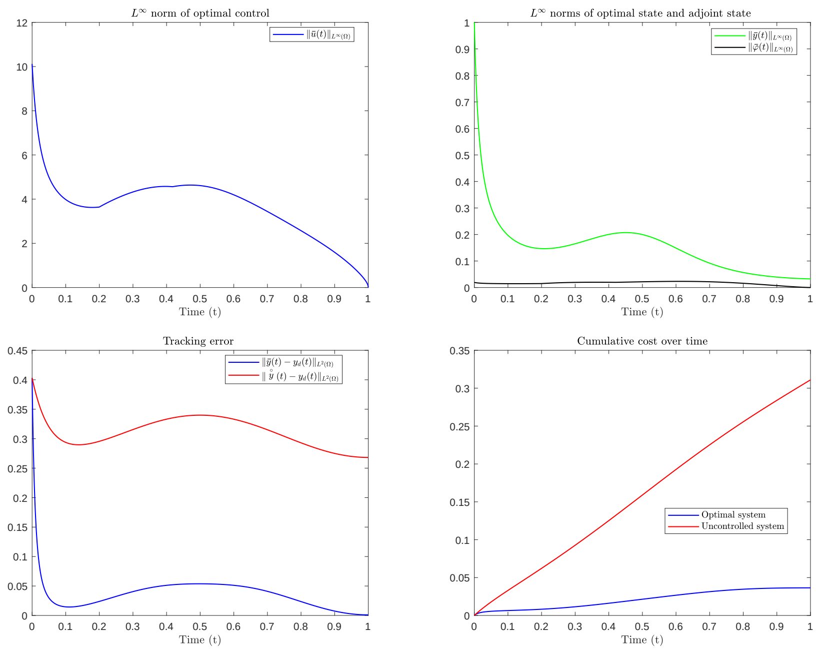

Figure 2: Results for the two-dimensional case: Optimal control norms (top left), state and adjoint state norms (top right), tracking error (bottom left), and cumulative cost (bottom right).

Both the one-dimensional and two-dimensional results demonstrate our algorithm's capability to compute essentially bounded controls that successfully achieve the desired state tracking objectives, while maintaining numerical stability and convergence properties predicted by the theoretical framework.

5 Further comments

Using an additive control operator instead of a multiplicative one can allow for weaker assumptions in solving the optimization problem. To illustrate, let's consider a quadratic cost function (i.e., $p = 2$) and investigate in which spatial dimensions the problem can be addressed for both additive and multiplicative control operators.

To bypass the restriction imposed in (1), we consider state equations with homogeneous Dirichlet boundary conditions while maintaining Assumption (3) unchanged.

For the optimal control of a linear system, we consider the problem

$$\min_{u\in L^\infty(Q)} J(u) := \frac{1}{2}\|y-y_d\|^2_{L^2(Q)} + \frac{\varepsilon}{2}\|u\|^2_{L^2(Q)}, \tag{PL}$$

subject to

$$\begin{cases} \dot{y} + Ay = u \quad \text{in} \quad Q, \\ y = 0 \quad \text{on} \quad \partial\Omega, \\ y(0) = y_0. \end{cases} \tag{37}$$

It is well-known that the cost function J possesses a minimizer $\bar{u}$ in $L^2(Q)$, with a corresponding state $\bar{y} \in C([0,T];L^2(\Omega))$ and an adjoint state $\bar{\varphi} \in C([0,T];L^2(\Omega))$ satisfying

$$\begin{cases} -\dot{\bar{\varphi}} + A\bar{\varphi} = \bar{y} - y_d \quad \text{in} \quad Q, \\ \bar{\varphi} = 0 \quad \text{on} \quad \partial\Omega, \\ \bar{\varphi}(T) = 0, \end{cases} \tag{38}$$

$$\bar{u}(x,t) = \frac{-1}{\varepsilon}\bar{\varphi}(x,t) \quad \text{a.e. in } Q. \tag{39}$$

Assuming $y_d \in L^\infty(0,T;L^2(Q))$, we have $\bar{y} - y_d \in L^\infty(0,T;L^2(Q))$. According to Ladyženskaja et al., if the dimension N satisfies $N/4 < 1$, then $\bar{\varphi} \in L^\infty(Q)$. Consequently, $\bar{u} \in L^\infty(Q)$, and problem (PL) admits an optimal triple $(\bar{u},\bar{y},\bar{\varphi})$ satisfying (37), (38), and (39) for dimensions $N = 1,2,3$, as established by Casas.

In contrast, for the optimal control of a bilinear system, as discussed in Section 3, we have an optimal triple $(\bar{u},\bar{y},\bar{\varphi}) \in L^2(Q) \times C([0,T];L^2(\Omega))^2$ solving problem (P_A). To prove that $\bar{u}$ solves problem (P), we must show that $\bar{\varphi} \in L^\infty(Q)$.

The dynamics of $\bar{\varphi}$ are given by $-\dot{\bar{\varphi}} = A\bar{\varphi} + L^*(\bar{u}\bar{\varphi}) + (\bar{y} - y_d)$. For a given $\varphi \in L^\infty(Q)$, we have $L^*(\bar{u}\varphi) + (\bar{y} - y_d) \in L^2(Q)$. Again, by Ladyženskaja et al., a solution to (38) has the regularity $\bar{\varphi} \in L^\infty(Q)$ if $1/2 + N/4 < 1$. Therefore, we can solve problem (P) only in the one-dimensional case.

Our study has focused on optimal bilinear control with a basic differential operator (Laplacian) and a trivial external forcing f. However, the problem can be explored for a wider class of reaction-diffusion systems. Specifically, consider a differential operator:

$$Ay := -\sum_{i,j=1}^{N} \partial_{x_j} \left(a_{ij}(x) \partial_{x_i} y \right),$$

satisfying

$$a_{ij} \in L^\infty(\Omega) \quad \text{and there exists } m > 0 \text{ such that}$$
$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq m|\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^N, \text{ for all } x \in \Omega,$$

and a Carathéodory function $f = f(x,t,y) : Q \times \mathbb{R}$ of class C^1 with respect to y, where $f(\cdot,\cdot,0) \in L^p(Q)$, satisfying

$$\text{there exists } C_f \in \mathbb{R} \text{ s.t. } \frac{\partial f}{\partial y}(x,t,y) \leq C_f, \quad \text{for all } y \in \mathbb{R},$$

$$\text{for all } M > 0, \text{ there exists } C_{f,M} > 0 \text{ s.t. } \left| \frac{\partial f}{\partial y}(x,t,y) \right| \leq C_{f,M}, \quad \text{for all } |y| \leq M.$$

We can then consider the evolution problem

$$\begin{cases} \dot{y} + Ay = uL(y) + f(x,t,y) & \text{in} \quad Q, \\ \frac{\partial y}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial\Omega, \\ y(0) = y_0, \end{cases}$$

with the associated variational formulation:

$$\iint_Q \Big(-y\dot{v} + \sum_{i,j=1}^{N} a_{ij}(x)\partial_{x_i} y \partial_{x_j} v \Big) \mathrm{d}x\mathrm{d}t = \iint_Q \Big(uL(y)v + f(y)v \Big) \mathrm{d}x\mathrm{d}t + \int_\Omega y_0\, v(0)\, \mathrm{d}x.$$

Assuming the other conditions remain unchanged, we can follow the same approach to deduce that this state equation has a unique solution $y_u \in W(0,T) \cap C(\bar{Q})$. Referring to Tröltzsch [33, Ch.V] and Theorem 2, we can establish the existence of an optimal control. Furthermore, we can deduce that for each solution $\bar{u}$, there exists a pair $\bar{y}, \bar{\varphi} \in W(0,T) \cap C(\bar{Q})$ satisfying the following system:

$$\begin{cases} \dot{\bar{y}} + A\bar{y} = \bar{u}L(\bar{y}) + f(\cdot,\cdot,\bar{y}) \quad \text{in} \quad Q, \\ \frac{\partial \bar{y}}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial\Omega, \\ \bar{y}(0) = y_0, \end{cases}$$

$$\begin{cases} -\dot{\bar{\varphi}} + A^*\bar{\varphi} = L^*(\bar{u}\bar{\varphi}) + f_y(\cdot,\cdot,\bar{y})\bar{\varphi} + (\bar{y} - y_d) \text{ in } Q, \\ \frac{\partial \bar{\varphi}}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial\Omega, \\ \bar{\varphi}(T) = 0, \end{cases}$$

$$\iint_Q \Big(L(\bar{y})\bar{\varphi} + \varepsilon \operatorname{sgn}(\bar{u}) \, |\bar{u}|^{p-1} \Big)(h - \bar{u}) \, \mathrm{d}x\mathrm{d}t \geq 0,$$

and

$$\bar{u}(x,t) = -\frac{\operatorname{sgn}(L(\bar{y})\bar{\varphi})}{\varepsilon^{\frac{1}{p-1}}} \, |L(\bar{y})\bar{\varphi}|^{\frac{1}{p-1}} \qquad \text{a.e. in } \ Q.$$

Here, $f_y(\cdot,\cdot,\bar{y})$ represents the derivative of f with respect to the second variable evaluated at $\bar{y}$.
These generalizations demonstrate the versatility of our approach, allowing for application to a wider class of reaction-diffusion systems while maintaining the key insights gained from the simpler model.

6 Conclusion

In this paper, we have presented a comprehensive study of the optimal control of reaction-diffusion systems subject to essentially bounded functions. Our primary findings include the proof of existence of optimal controls and the derivation of their first-order optimality conditions. We also illustrated the theoretical results with numerical examples for both one- and two-dimensional cases, demonstrating the applicability of our approach.

A key contribution of this work is the introduction of a novel formulation for the optimal control problem that does not require a priori bounds on the admissible controls. This approach effectively addresses a significant challenge in practical applications, where defining control bounds in advance is often difficult and potentially leads to suboptimal solutions. By working in the $L^\infty(Q)$ space, we have established an equivalence between the original problem and an auxiliary optimization problem in $L^p(Q)$, proving that solutions to the auxiliary problem are inherently essentially bounded.

The results presented here contribute significantly to the theory of bilinear optimal control. They provide a robust framework for practical applications, particularly in scenarios where traditional bounded control approaches are limiting. Our findings are expected to have broad relevance, particularly in areas where control bounds are not easily determined.

This work also opens several promising directions for future research. One potential extension is to analyze the problem in an infinite horizon setting. Additionally, exploring the sensitivity of optimal control strategies to changes in system parameters or initial conditions could yield valuable insights. Investigating alternative cost functionals may offer the opportunity to tailor control characteristics for specific applications. Furthermore, the development of more efficient numerical methods to solve the optimality system could enhance the practical implementation of this approach. Lastly, this framework could be adapted and applied to other types of partial differential equations involving bilinear control terms, broadening its applicability across various domains.

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