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Research Article

Exploring hyperchaotic synchronization of a fractional-order system without equilibrium points: A sliding mode control approach

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Abstract

Recently, constructing hidden attractors of chaotic systems without equilibrium point has become a key discussion point in the application fields of chaos and hyperchaos science. This paper introduces a novel hyperchaotic system without equilibrium points, distinct from existing systems that rely on the Shilnikov criterion for demonstrating hyperchaos. This study investigates the qualitative properties of the system, including its hyperchaotic

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attractors, Poincare map, Lyapunov exponents, and Kaplan-Yorke dimension. To enhance the practical applicability of this system, an integral sliding mode control method for synchronization is proposed. Lyapunov theory ensures the stability and effectiveness of the synchronization scheme. The efficiency of the approach is demonstrated by numerical simulations, which validate the potential of the system for various applications.

AMS subject classifications (2020): 34C28, 93C10, 93D05, 93D20

Keywords: Fractional derivative; Synchronization; Lyapunov stability theory; Lyapunov exponents; Sliding mode control.

1 Introduction

Fractional calculus, a mathematical discipline with over 300 years of history, extends traditional calculus to noninteger order derivatives and integrals. The study of fractional calculus has attracted significant attention due to its potential applications in various fields. Fractional-order systems provide an explicit description and further insight into physical processes. They also serve as valuable tools in modeling many phenomena. The use of fractional derivatives has been investigated in various research fields, including electromagnetism [41], secure communication [19], cryptography [34], viscolelasticty [15], diffusion [18], and chemical processing [16], due to their elegant description of interdisciplinary applications.

Chaos theory is the study of systems that obey deterministic rules but exhibit seemingly random behavior. Small differences in initial conditions can lead to very different outcomes, creating a fascinating interplay between order and randomness. The study of chaos allows us to understand and potentially predict complex behavior in a variety of scientific disciplines. Hyperchaotic systems, a subset of chaotic systems, exhibit even more complex dynamics with multiple positive Lyapunov exponents [42]. Many chaotic systems, including the Chua circuit [17], Duffing system [32, 46], Chen system [30], Lu system [35], and Rossler system [31], can be elegantly modeled using fractional-order equations.

Synchronization refers to a fascinating phenomenon in which two systems adjust their dynamics to become identical, essentially when one of two systems adapts its trajectory to follow that of the other [27, 45]. In 1990, Pecora and Carroll [38] established a pioneering scheme for achieving synchronization, known as the drive-response method. Nowadays, various synchronization techniques, such as complete synchronization [5, 8], phase synchronization [40], projective synchronization [7, 19, 47], inverse matrix projective synchronization [23], modified projective synchronization [21], modified hybrid synchronization [25], generalized synchronization [22, 53], $Q-S$ synchronization [9], fixed-time synchronization [58], and combination synchronization [3, 2, 20, 24, 26] have been described. Recently, due to the wide range of applications, many different control schemes have been applied to synchronize fractional-order chaotic systems, such as active control [4, 14], adaptive control [56], scalar transmitted signal methods [36], impulsive control [52], event-triggered controller [43, 55], and sliding mode control (SMC) [1, 29, 54], among others.

The SMC is a powerful control strategy renowned for its robustness to parameter variations [6]. Additionally, SMC offers several advantages, including excellent transient performance, rapid dynamic responses, and disturbance rejection capability.

The analysis of chaotic systems has traditionally been conducted through the lens of equilibrium points, due to the sensitivity of these systems to initial conditions and their unpredictable long-term behavior. However, recent research has revealed a fascinating class of chaotic systems that challenge this convention: nonequilibrium chaotic systems [13, 44, 51, 57]. Despite the absence of stable equilibrium points, nonequilibrium chaotic systems are capable of generating intricate, confined trajectories, designated as hidden attractors. This counterintuitive behavior offers promising opportunities for potential applications in a range of fields. As research progresses, the challenge of synchronizing these systems remains a captivating challenge [39, 48, 49, 50].

This work makes a contribution to the field of hyperchaotic systems with hidden attractors by introducing a novel fractional-order system that lacks equilibrium points. A comprehensive study of the system's dynamical behavior is conducted with the objective of achieving synchronization using integral sliding mode control (ISMC). The synchronization controller and parameter identification technique are designed based on Lyapunov stability theory, and the theoretical findings are validated through computer simulations. The primary contributions of this work include:

1. A groundbreaking four-dimensional fractional-order hyperchaotic system is introduced from the $3-D$ Rikitake dynamo system.

2. Rigorous analysis confirms the hyperchaotic nature of the proposed system, highlighting its complex and unpredictable behavior.

3. Developing of an ISMC strategy for synchronizing two identical instances of the novel system.

4. Lyapunov theory and numerical simulations to validate the effectiveness of the synchronization approach.

The rest of the manuscript is arranged as follows: Section 2 presents some preliminary concepts for fractional calculus. Section 3 introduces the construction of four dimensional fractional-order system, it delves into a detailed analysis of its fundamental dynamical property. Section 4 develops an integral sliding mode controller for synchronizing two identical hyperchaotic systems. The feasibility and effectiveness of both control schemes are verified by numerical simulations.

2 Basic definitions for fractional-order systems

In this section, some basic definitions related to fractional differentiation are presented.

Definition 1. In the field of fractional calculus, a vital function is the Euler gamma function Γ, which is defined as follows

$$\Gamma(a)=\int_{0}^{+\infty} t^{a-1}e^{-t}dt. \tag{1}$$

Definition 2. Fractional calculus extends the concepts of integration and differentiation to non-integer orders. It uses a fundamental operator, denoted

${}_aD_t^\alpha$, which defines fractional integration and differentiation. The integro-differential operator ${}_aD_t^\alpha$ is defined by

$$
{}_aD_t^\alpha = \begin{cases} \dfrac{d^\alpha}{(dt)^\alpha}, & \alpha > 0, \\ 1, & \alpha = 0, \\ \int_a^t (d\tau)^\alpha, & \alpha < 0, \end{cases} \tag{2}
$$

where a and t are the bounds of the operator and $t \in \mathbb{R}$.

Several definitions exist regarding fractional derivative, but the Caputo definition in (3) is used the most in engineering applications, since this definition incorporates initial values for a function f and its standard derivatives, that is, initial values that are physically appealing in the traditional way. Also, a modified version of Adams–Bashforth–Moulton method [12] will be employed for numerical simulation of the Caputo fractional-order equations.

Definition 3 (Caputo fractional derivative [28])**.** The Caputo fractional derivative of order α is defined as follows:

$$
D^\alpha f(t) = I^{n-\alpha} f^n(t), \tag{3}
$$

where, $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, I^α is the αth-order Riemann–Liouville fractional integration given by

$$
I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\xi)^{\alpha-1} f(\xi) d\xi. \tag{4}
$$

Theorem 1. Consider the following commensurate fractional-order dynamics system:

$$
D^\alpha x(t) = f(x(t)),
$$

where D^α is a Caputo differential operator, $0 < \alpha < 1$, and $x \in \mathbb{R}^n$ is the state vector of system and f is continuous vector function of the system. The equilibrium points of this system are calculated by solving the equation $f(x) = 0$. These points are locally asymptotically stable if all eigenvalues λ_i of the Jacobian matrix $J = \partial f / \partial x$ satisfy

$$
|arg(\lambda_i)| > \alpha \frac{\pi}{2}.
$$

3 Model description of the novel fractional-order hyperchaotic system

As reported in the literature, the $3-D$ Rikitake dynamo system [41] is described as follows

$$\begin{cases} \dot{x_1} = -ax_1 + x_2x_3, \\ \dot{x_2} = -ax_2 + x_1(x_3 - b), \\ \dot{x_3} = 1 - x_1x_2, \end{cases} \tag{5}$$

where x_1, x_2, and x_3 are the state variables, a and b are positive parameters and represent the resistive dissipation and the difference in the angular velocities of the two disks, respectively.

An extensive study established chaos in this system has been described by Cook and Roberts [10].

By adding a state feedback control to the previous system and replacing standard derivatives by fractional-order derivatives, we get the following system

$$\begin{cases} D^\alpha x_1 = -ax_1 + x_2x_3 - cx_4, \\ D^\alpha x_2 = -ax_2 + x_1(x_3 - b), \\ D^\alpha x_3 = d - x_1x_2, \\ D^\alpha x_4 = x_1, \end{cases} \tag{6}$$

where x_1, x_2, x_3, and x_4 are the state variables of the system, a, b, c, and $d \neq 0$ are real constants, $\alpha \in (0, 1]$ is the fractional-order derivative, and D^α is the Caputo differential operator.

In the following, some basic properties of the novel system (6) are numerically analyzed, including: Dissipativity, equilibrium points, Poincaré map, hyperchaotic attractors, Lyapunov exponents, and Kaplan-Yorke dimension.

3.1 Dissipativity

We use the divergence of the vector field. The system is dissipative if the sum of the coefficients of the fractional derivatives is negative. The divergence of the system (6) is calculated as

$$\nabla f = \frac{\partial D^{\alpha} x_1}{\partial x_1} + \frac{\partial D^{\alpha} x_2}{\partial x_2} + \frac{\partial D^{\alpha} x_3}{\partial x_3} + \frac{\partial D^{\alpha} x_4}{\partial x_4}.$$

Substituting the partial derivatives

$$\nabla f = -2a,$$

since $a > 0$, the sum of the coefficients of the fractional derivatives is negative, that is, $\nabla f < 0$. Therefore, the system is dissipative. In this case, all the trajectories of the system tend to an attractor, when $t \to +\infty$.

3.2 Equilibrium points

In order to find the equilibrium points of the system (6), we solve the four following equations

$$\begin{cases} -ax_1 + x_2 x_3 - cx_4 = 0, \\ -ax_2 + x_1(x_3 - b) = 0, \\ d - x_1 x_2 = 0, \\ x_1 = 0. \end{cases} \tag{7}$$

Since $d \neq 0$, the last two equations in the system (7) contradict each other. This means that the proposed system (6) has no equilibrium points. Thus, the system lacks the characteristic bifurcation phenomena (e.g., pitchfork, Hopf) typically observed in hyperchaotic systems. Additionally, the absence of equilibrium points precludes the existence of a stable sink state.

3.3 Poincaré map

To elucidate the complexity and characteristic features of the novel system, we employ the Poincaré map. This technique provides valuable insights into the underlying behavior.

Figures 1 and 2 show the Poincaré map of the system (6) in $2-D$ plan and $3-D$ space, respectively.

The system's Poincaré map (Figure 2) reveals intricate structures, suggesting hyperchaotic behavior.

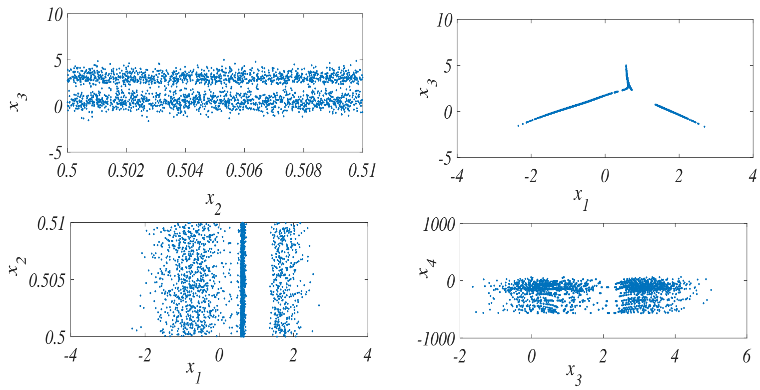

Figure 1: Poincaré map in $2-D$ plan.

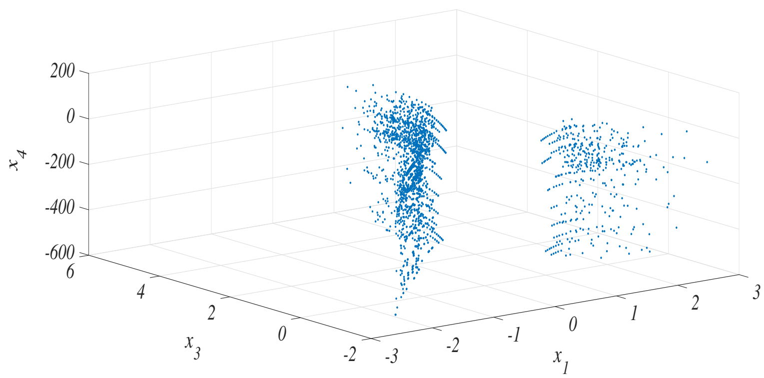

Figure 2: Poincaré map in $3-D$ space.

3.4 Hyperchaotic attractors

It is interesting that our system can generate hyperchaotic signals although there is no equilibrium points. The system (6) exhibits hyperchaotic behavior, as shown in Figure 3, where $\alpha = 0.98$, the system's parameters are as follows

$$a = b = d = 1 \quad and \quad c = 0,$$

and the initial values of the hyperchaotic system (6) are selected as

$$x_1(0) = x_2(0) = x_3(0) = -0.1 \quad and \quad x_4(0) = 0.2. \tag{8}$$

We used the MATLAB/Simulink environment to develop a comprehensive

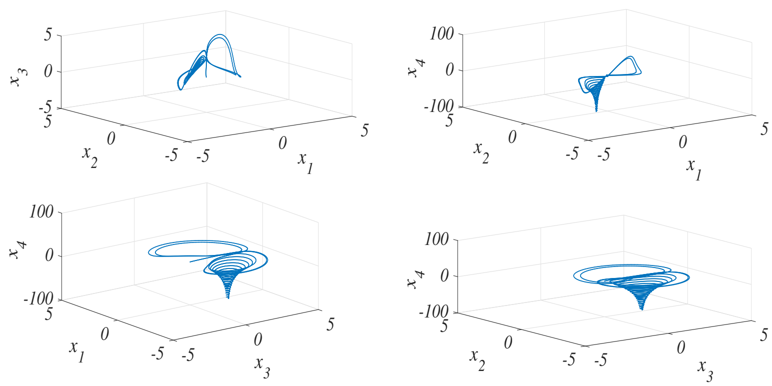

Figure 3: Hyperchaotic attractors of system (6), when $\alpha = 0.98$.

simulation framework for the system described by (6). To achieve accurate modeling of fractional-order dynamics, we utilized the capabilities of the FOMCON toolkit in Simulink. This toolbox provides robust blocks for implementing fractional integration of various orders. The detailed block diagram of the simulation setup is presented in Figure 4. The temporal evolution

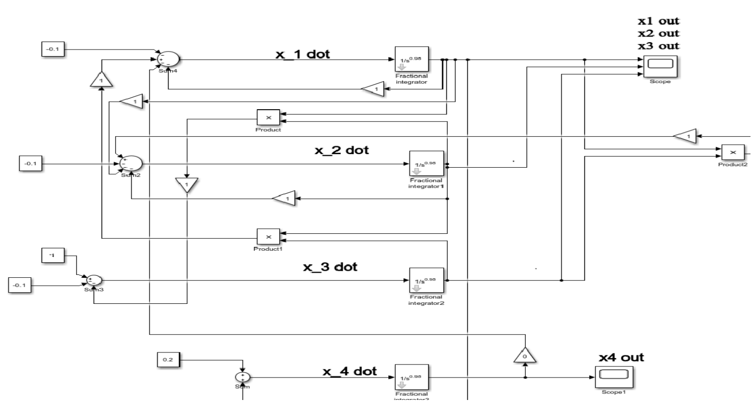

Figure 4: The block diagram of the new hyperchaotic system (6) in MATLAB Simulink.

of the state variables x_1, x_2, x_3, *and* x_4 are illustrated in Figures 5 and 6, respectively.

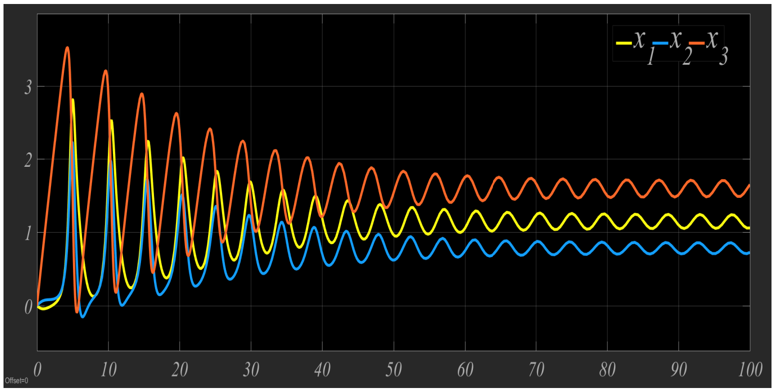

Figure 5: Temporal evolution of x_1, x_2, x_3.

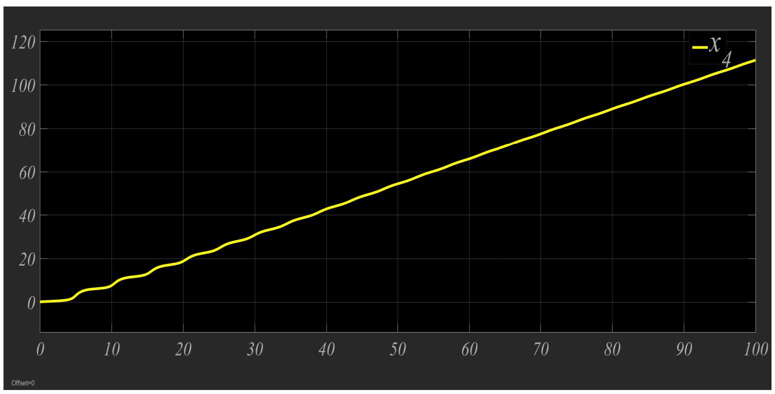

Figure 6: Temporal evolution of x_4.

In order to provide definitive confirmation of the hyperchaotic behavior exhibited by the proposed system, a computation of the Lyapunov exponents will be undertaken.

3.5 Lyapunov exponents

In this section, we assume that the parameters a, b, and d remain constant and c is varied in $[0, 5]$. By using MATLAB code for Lyapunov exponents of fractional-order systems [11], the Lyapunov exponents spectrum of system (6) with $a = 1, b = 1$, and $d = 1$ is represented in Figure 7.

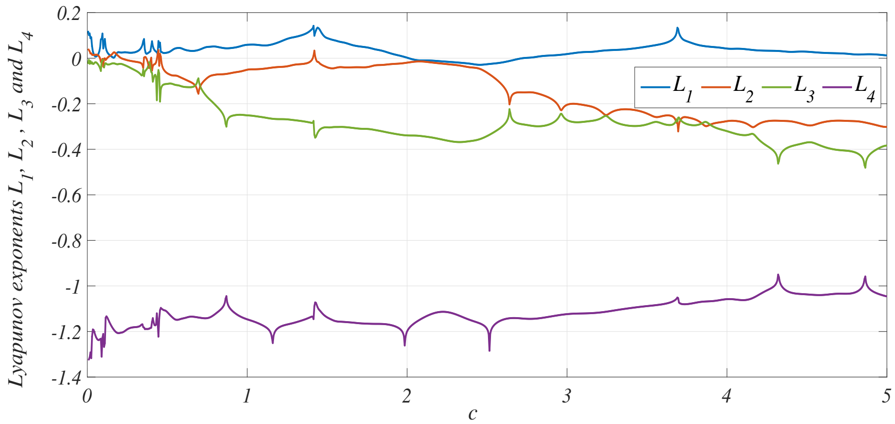

Figure 7: The dynamics of the Lyapunov exponents of the system (6).

Particularly, when the parameter systems are taken as

$$a = b = d = 1 \quad and \quad c = 0.03,$$

the Lyapunov exponents of the proposed system are obtained as

$$L_1 = 0.1208, \; L_2 = 0.0202, \; L_3 = 0, \; \text{and} \; L_4 = -1.2740.$$

The Kaplan–Yorke dimension of the system is calculated as

$$D_{KY} = 3 + \frac{L_1 + L_2 + L_3}{|L_4|} = 3.1107.$$

Thus, the system is hyperchaotic.

4 Sliding mode controller design and synchronization analysis

To assess the synchronization capabilities of the proposed hyperchaotic system, SMC is employed in order to achieve complete synchronization between the states of the master-slave systems.

4.1 General method of synchronization

Consider the fractional-order dynamic system defined by

$$D^{\alpha}x(t) = f(x(t)), \tag{9}$$

where D^{α} is Caputo differential operator, $0 < \alpha < 1$, $x \in \mathbb{R}^n$ is the state vector of system, and f is continuous vector function. We consider the system (9) as a drive system. The controlled response system is given by

$$D^{\alpha}y(t) = f(y(t)) + u. \tag{10}$$

Define the synchronization error as $e(t) = y(t) - x(t)$. The main purpose is to choose a suitable controller u, so that the response system accurately tracks the drive system's trajectory, such that

$$\lim_{t\to\infty} ||e(t)|| = 0. \tag{11}$$

4.2 Sliding mode controller design

The drive system is described by

$$\begin{cases} D^{\alpha}x_1 = -ax_1 + x_2x_3 - cx_4, \\ D^{\alpha}x_2 = -ax_2 + x_1(x_3 - b), \\ D^{\alpha}x_3 = d - x_1x_2, \\ D^{\alpha}x_4 = x_1. \end{cases} \tag{12}$$

The controlled response system is given by

$$\begin{cases} D^{\alpha}y_1 = -ay_1 + y_2y_3 - cy_4 + u_1, \\ D^{\alpha}y_2 = -ay_2 + y_1(y_3 - b) + u_2, \\ D^{\alpha}y_3 = d - y_1y_2 + u_3, \\ D^{\alpha}y_4 = y_1 + u_4, \end{cases} \tag{13}$$

where u_1, u_2, u_3, u_4 are the integral sliding mode controllers to be designed.

Define the synchronization errors as $e_i(t) = y_i(t) - x_i(t),\ i = 1, 2, 3, 4$. It is easy to see that the error dynamics system can be obtained as follows

$$\begin{cases} D^{\alpha}e_1 = -ae_1 + y_2y_3 - x_2x_3 - ce_4 + u_1, \\ D^{\alpha}e_2 = -ae_2 + y_1y_3 - x_1x_3 - be_1 + u_2, \\ D^{\alpha}e_3 = -y_1y_2 + x_1x_2 + u_3, \\ D^{\alpha}e_4 = e_1 + u_4. \end{cases} \tag{14}$$

In order to achieve the stability of system (14), four sliding surfaces s_1, s_2, s_3, and s_4 are introduced as

$$s_i = (D^{\alpha} + \lambda_i)\int_0^t e_i(\tau)d\tau, \qquad i = 1, 2, 3, 4. \tag{15}$$

A simple calculation yields

$$\begin{cases} \dot{s_1} = D^{\alpha}e_1(t) + \lambda_1 e_1(t), \\ \dot{s_2} = D^{\alpha}e_2(t) + \lambda_2 e_2(t), \\ \dot{s_3} = D^{\alpha}e_3(t) + \lambda_3 e_3(t), \\ \dot{s_4} = D^{\alpha}e_4(t) + \lambda_4 e_4(t), \end{cases} \tag{16}$$

where $\lambda_i > 0$ for $i = 1, 2, 3, 4$.
To design SMC, the reaching law [33] is chosen as

$$\begin{cases} \dot{s_1} = -\beta_1 sgn(s_1) - K_1 s_1, \\ \dot{s_2} = -\beta_2 sgn(s_2) - K_2 s_2, \\ \dot{s_3} = -\beta_3 sgn(s_3) - K_3 s_3, \\ \dot{s_4} = -\beta_4 sgn(s_4) - K_4 s_4, \end{cases} \tag{17}$$

where $\beta_i > 0$, $K_i > 0$, for $i = 1, 2, 3, 4$, and $sgn(\cdot)$ represents sign function, that is

$$sgn(s_i) = \begin{cases} 1, & s_i > 0, \\ 0, & s_i = 0, \\ -1, & s_i < 0. \end{cases}$$

By comparing (16) and (17), we deduce

$$\begin{cases} D^{\alpha}e_1(t) + \lambda_1 e_1(t) = -\beta_1 sgn(s_1) - K_1 s_1, \\ D^{\alpha}e_2(t) + \lambda_1 e_2(t) = -\beta_2 sgn(s_2) - K_2 s_2, \\ D^{\alpha}e_3(t) + \lambda_1 e_3(t) = -\beta_3 sgn(s_3) - K_3 s_3, \\ D^{\alpha}e_4(t) + \lambda_1 e_4(t) = -\beta_4 sgn(s_4) - K_4 s_4. \end{cases} \tag{18}$$

Combining (14) and (18), we get

$$\begin{cases} -ae_1 + y_2y_3 - x_2x_3 - ce_4 + u_1 + \lambda_1 e_1(t) = -\beta_1 sgn(s_1) - K_1 s_1, \\ -ae_2 + y_1y_3 - x_1x_3 - be_1 + u_2 + \lambda_2 e_2(t) = -\beta_2 sgn(s_2) - K_2 s_2, \\ -y_1y_2 + x_1x_2 + u_3 + \lambda_3 e_3(t) = -\beta_3 sgn(s_3) - K_3 s_3, \\ e_1 + u_4 + \lambda_4 e_4(t) = -\beta_4 sgn(s_4) - K_4 s_4. \end{cases} \tag{19}$$

In order to synchronize the drive system (12) with the response system (13), the control input can be obtained as

$$\begin{cases} u_1 = (a - \lambda_1)e_1 + ce_4 - y_2y_3 + x_2x_3 - \beta_1 sgn(s_1) - K_1 s_1, \\ u_2 = -b^* e_1 + (a - \lambda_2)e_2 - y_1y_3 + x_1x_3 - \beta_2 sgn(s_2) - K_2 s_2, \\ u_3 = -\lambda_3 e_3 + y_1y_2 - x_1x_2 - \beta_3 sgn(s_3) - K_3 s_3, \\ u_4 = -e_1 - \lambda_4 e_4 - \beta_4 sgn(s_4) - K_4 s_4. \end{cases} \tag{20}$$

4.3 Synchronization analysis

To enhance the practical applicability of the proposed system, an ISMC method is designed to achieve complete synchronization of the system. In addition, numerical simulations are performed to demonstrate and verify the theoretical analysis.

Theorem 2. The systems (12) and (13) are globally and asymptotically synchronized under the integral sliding mode controller (20), where $\alpha_i, \beta_i, K_i,\ (i = 1, 2, 3, 4)$ are constant positives.

Proof. We establish this theorem using Lyapunov stability theory. First, we consider the quadratic and positive definite Lyapunov function defined by

$$V(s_1, s_2, s_3, s_4) = \frac{1}{2}(s_1^2 + s_2^2 + s_3^2 + s_4^2).$$

Differentiating V with respect to the time t, we obtain

$$\dot{V} = s_1\dot{s_1} + s_2\dot{s_2} + s_3\dot{s_3} + s_4\dot{s_4}. \tag{21}$$

Substituting from (17) to (21), we get

$$\dot{V} = \sum_{i=1}^{4} [-\beta_{|} s_i| - K_i s_i^2)], \tag{22}$$

which is a negative definite function on $\mathbb{R}^4$.

Thus, by the Lyapunov stability theory, we conclude that $s_i \to 0$ as $t \to +\infty$, for $i = 1, 2, 3, 4$. Hence, it follows that $e_i \to 0$, as $t \to +\infty$, for $i = 1, 2, 3, 4$. □

4.4 Numerical simulations

For MATLAB simulations, we take the parameter values for the drive and response systems as in the case of hyperchaotic attractors

$$a = 1, b = 1, c = 0, \; and \; d = 1.$$

We take the sliding constants as

$$\lambda_i = 0.2, \; \beta_i = 0.2, \; and \; K_i = 3, \qquad for \; all \; i = 1, 2, 3, 4.$$

We set the initial conditions of the drive and the response systems, respec-

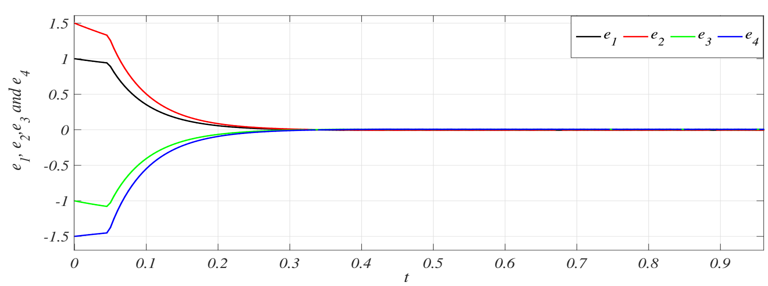

Figure 8: Time history of the synchronization errors.

tively, as

$$(x_1(0), x_2(0), x_3(0), x_4(0)) = (0.5, -0.5, 0.5, 1)$$

and

$$(y_1(0), y_2(0), y_3(0), y_4(0)) = (1.5, 1, -0.5, -0.5).$$

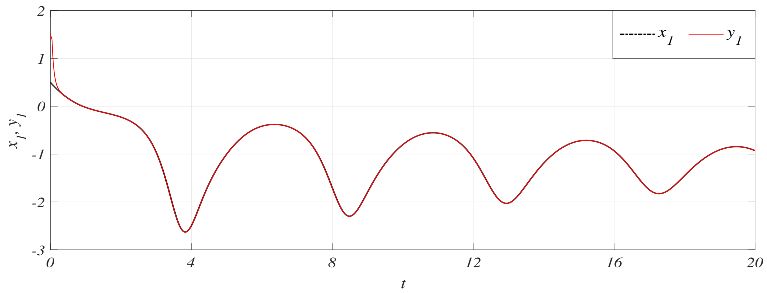

Figure 9: Complete synchronization between signals x_1 and y_1.

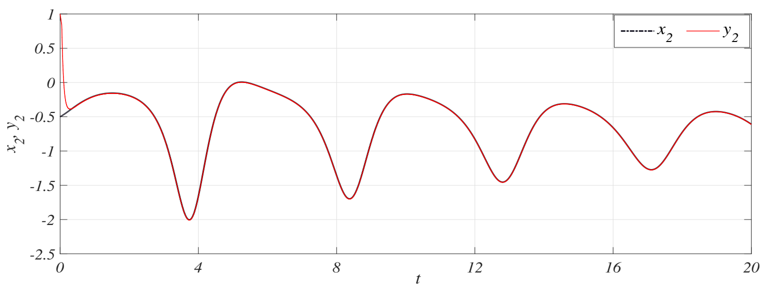

Figure 10: Complete synchronization between signals x_2 and y_2.

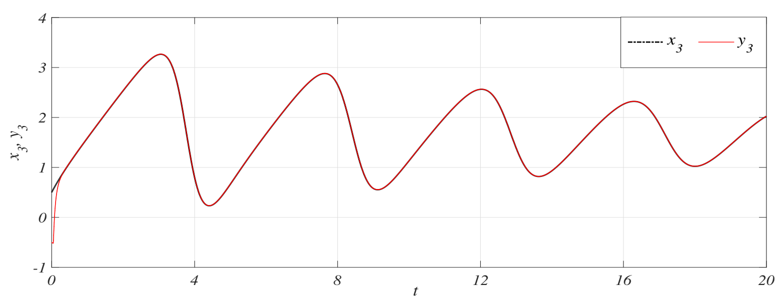

Figure 11: Complete synchronization between signals x_3 and y_3.

Figure 8 shows time history of the synchronization errors. Furthermore, Figures 9, 10, 11, and 12 show the complete synchronization between the states x_1 and y_1, x_2 and y_2, x_3 and y_3, x_4 and y_4, respectively .

From these figures, it is straightforward to see the complete synchronization between the drive and response systems (12) and (13) are achieved.

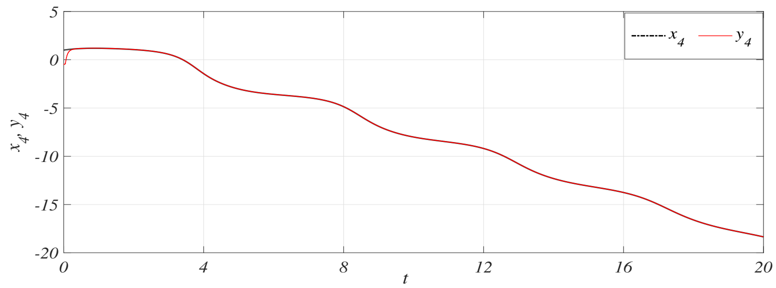

Figure 12: Complete synchronization between signals x_4 and y_4.

5 Conclusion

There is great interest in the modeling of hyperchaotic systems devoid of equilibrium points within the field of chaos literature. This research introduced a novel fractional-order system exhibiting complex dynamics, thereby expanding the understanding of chaotic behavior beyond traditional equilibrium-based frameworks. The comprehensive analysis conducted in this study, including the identification of hyperchaotic attractors and the computation of Lyapunov exponents, has yielded valuable insights into the intricate dynamics of the system.

Furthermore, the creation and application of an ISMC strategy for synchronization have illustrated the potential for controlling and coordinating these complex systems. The rigorous validation through MATLAB simulations has confirmed the effectiveness of the proposed approach, even in the face of the inherent challenges posed by the system's sensitivity to initial conditions.

The findings of this research have significant implications for a number of applications in fields such as secure communications, signal processing, and control systems. The capacity to generate and regulate complex, erratic dynamics can be harnessed for operations that necessitate randomness, noise, or chaotic behavior. Moreover, the demonstrated synchronization capabilities provide a foundation for the construction of complex networks and the coordination of multiple chaotic systems.

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