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**Research Article**

# Sequential approximate optimality conditions for a constrained convex vector minimization problem and application to multiobjective fractional programming problem

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**Abstract**

The aim of this paper is to establish sequential necessary and sufficient approximate optimality conditions for a constrained convex vector minimization problem without any constraint qualifications, characterizing the approximate proper and weak efficient solutions. The constraints are described by mappings taking values in different preorder vector spaces. Our approach is based essentially on the sequential approximate subdifferential calculus rule for the sums of a finite family of cone convex mappings. To illustrate our main result, an application to multiobjective fractional programming problem is given. Finally, we present an important subclass of such problems showing the applicability of the obtained conditions.

**AMS subject classifications (2020):** Primary 45D05; Secondary 42C10, 65G99.

**Keywords:** Vector optimization; Approximate Pareto subdifferential of convex mappings; Sequential approximate optimality conditions.

## 1 Introduction

Many multiobjective fractional programming problems arose in different fields of modern research because their algorithmic aspects and abundance of applications, for example, economics, management, medicine, information theory, engineering, and optimal control (see [17, 2, 15, 16, 20] and the references therein).

Generally, for establishing the approximate optimality conditions for convex optimization problems, certain types of constraint qualifications must be imposed. However, we know that the constraint qualifications do not always hold even for finite-dimensional optimization problems and frequently fail for infinite-dimensional optimization problems. In order to eliminate these drawbacks, many authors have paid their attention to characterize optimality conditions for convex optimization problems without any constraint qualifications (see [19, 3, 8, 9, 1]). Thibault [19] derived sequential optimality conditions via the subdifferential calculus for convex functions with cone convex constraints. Boţ, Csetnek, and Wanka [3] obtained sequential characterizations of an optimal solution for composed convex optimization problems. Recently, works have been done in this direction for multiobjective fractional programming problems (see [14, 13, 10, 11, 18]). Moustaid et al. [14, 13] established sequential optimality conditions for a constrained fractional programming problem without any constraint qualifications, characterizing the approximate weak efficient and efficient solution. Kim, Kim, and Lee [10] and Kohli [11] developed sequential optimality conditions for multiobjective fractional programming problems via scalarization and by using the concept of the epigraphs of conjugate functions in terms of approximate subdifferentials at an approximate weak efficient solution.

In this paper, we consider the following constrained vector minimization problem with inequality constraints, which are described by mappings taking values in different preorder vector spaces

$$(P_1) \qquad \begin{cases} \min f(x) \\ x \in C, \\ h_1(x) \in -Z_1^+, \\ \vdots \\ h_p(x) \in -Z_p^+, \end{cases}$$

where $(X, \|\cdot\|_X)$ and $(Z_i, \|\cdot\|_{Z_i})$ $(i = 1, \dots, p)$ are real reflexive Banach spaces and $Y$ is a real Hausdorff topological vector space, $C$, $Y_+ \subset Y$, and $Z_i^+ \subset Z_i$ $(i = 1, \dots, p)$ are convex cones and $f : X \longrightarrow Y \cup \{+\infty_Y\}$ is a proper and $Y_+$-convex and $h_i : X \longrightarrow Z_i \cup \{+\infty_{Z_i}\}$ $(i = 1, \dots, p)$ are proper and $Z_i^+$-convex mappings.

The goal of our paper is to establish, in the absence of constraint qualifications, sequential approximate optimality conditions characterizing properly and weakly approximate efficient solutions for the problem $(P_1)$ in terms of the approximate subdifferentials of the associated functions.

This paper is organized as follows. In Section 2, we present the main notions and give some preliminary results used in what follows. In Section 3, we establish a sequential formula for the approximate subdifferential for the sums of two vector mapping with a sum of $p$ $(p \geq 1)$ vector composites. In Section 4, by the results in Section 3, we study sequential necessary and sufficient approximate optimality conditions for $(P_1)$. Section 5 is devoted to calculus sequential approximate optimality conditions for multiobjective fractional programming problems without any constraint qualification. Finally, we present an example illustrating the main result of this work.

## 2 Notations, definitions, and preliminaries

In this section, we give some basic definitions and results. Let $X$, $Y$, and $Z$ be real Hausdorff topological vector spaces with continuous dual spaces $X^*$, $Y^*$, and $Z^*$ and duality pairing also denoted by $\langle \cdot, \cdot \rangle$. Let $Y_+$ be a convex cone of $Y$ with $\text{int}Y_+ \neq \emptyset$. The subset $l(Y_+) := Y_+ \cap -Y_+$ is the linearity of $Y_+$, when it is null; thus the cone $Y_+$ is said to be pointed. Throughout the paper, we denote by $L(X, Y)$ the set of all continuous linear operators from $X$ into $Y$. In what follows, the convex cone $Y_+$ is not supposed to be a linear subspace so it cannot coincide with its linearity. For any $v_1, v_2 \in Y$, the cone $Y_+$ induces the following ordering relations:

$$v_1 \leq_{Y_+} v_2 \Longleftrightarrow v_2 - v_1 \in Y_+,$$
$$v_1 <_{Y_+} v_2 \Longleftrightarrow v_2 - v_1 \in \text{int}Y_+,$$
$$v_1 \lneq_{Y_+} v_2 \Longleftrightarrow v_2 - v_1 \in Y_+ \backslash l\left(Y_+\right).$$

To space $Y$, we attach an abstract maximal element with respect to "$\leq_{Y_+}$", denoted by $+\infty_Y$ such that $v \leq_{Y_+} +\infty_Y$, for all $v \in Y$ and $v + (+\infty_Y) := (+\infty_Y) + v := +\infty_Y$ for all $v \in Y \cup \{+\infty_Y\}$.

The polar cone $Y_+^*$ and the strict polar cone $(Y_+^*)^\circ$ of $Y_+$ are defined, respectively, as

$$Y_+^* := \{y^* \in Y^* \ : \ y^*(Y_+) \subseteq \mathbb{R}_+\},$$

and

$$(Y_+^*)^\circ := \{y^* \in Y^* \ : \ y^*(Y_+ \backslash l(Y_+)) \subseteq \mathbb{R}_+ \backslash \{0\}\}.$$

Now, let us recall the following definitions.

**Definition 1.** A mapping $F : X \longrightarrow Y \cup \{+\infty_Y\}$ is said to be

- $Y_+$-convex if for any $\beta \in [0, 1]$ and any $u_1, u_2 \in X$

$$F\left(\beta u_1+(1-\beta)u_2\right)\leq_{Y_+}\beta F\left(u_1\right)+(1-\beta)F\left(u_2\right).$$

- proper if its effective domain satisfies

$$\mathrm{dom}F:=\{x\in X: F(x)\in Y\}\neq\emptyset.$$

- $Y_+$-epi-closed if its epigraph

$$\mathrm{epi}F:=\{(x,y)\in X\times Y: F(x)\leq_{Y_+} y\} \text{ is closed.}$$

- star $Y_+$-lower semicontinuous if $y^*\circ F$ is lower semicontinuous for any $y^*\in Y_+^*$.

Let "$\leq_{Z_+}$ " be a partial order on $Z$ induced by a nonempty convex cone $Z_+\subset Z$. We say that a mapping $g: Z\longrightarrow Y\cup\{+\infty_Y\}$ is $(Z_+,Y_+)$-nondecreasing if for any $w_1,w_2\in Z$ we have

$$w_1\leq_{Z_+} w_2\Longrightarrow g(w_1)\leq_{Y_+} g(w_2).$$

If $h: X\longrightarrow Z\cup\{+\infty_Z\}$, then the composed mapping $g\circ h: X\longrightarrow Y\cup\{+\infty_Y\}$ is defined by

$$(g\circ h)(x):=\begin{cases} g(h(x)) & \text{if } x\in \text{ dom } h,\\ +\infty_Y & \text{otherwise.}\end{cases}$$

It is easy to see that if $g$ is $(Z_+,Y_+)$-nondecreasing and $Y_+$-convex and $h$ is $Z_+$-convex, then $g\circ h$ is $Y_+$-convex.

Given $F: X\supseteq S\longrightarrow Y\cup\{+\infty\}$ and $\epsilon\in Y_+$, we consider the following vector minimization problem:

$$(P)\qquad \min_{x\in S} F(x).$$

We recall some $\varepsilon$-efficiency solutions of $(P)$.

**Definition 2.** A point $\bar{x}\in S\cap\mathrm{dom}F$ is said to be

- a weakly $\epsilon$-efficient solution if there does not exist $x\in S$ such that $F(x)<_{Y_+} F(\bar{x})-\epsilon$,

- a properly $\epsilon$-efficient solution if there exists a convex cone $\hat{Y}_+ \subsetneq Y$ that satisfies $Y_+ \backslash l\,(Y_+) \subseteq \operatorname{int} \hat{Y}_+$, there does not exist $x \in X, F(x) \lneq_{\hat{Y}_+} F(\bar{x}) - \epsilon$.

The sets of weakly and properly $\epsilon$-efficient points will be denoted by $E^w_\varepsilon(F,S,Y_+)$ and $E^p_\varepsilon(F,S,Y_+)$, respectively. Let us note that if $E^\sigma_\varepsilon(F,S,Y_+) \neq \emptyset$, then we can see easily $\varepsilon \in D^\sigma$, where

$$D^\sigma = \begin{cases} Y\backslash - \operatorname{int} Y_+ & \text{if } \sigma = w, \\ \\ Y\backslash(-Y_+\backslash l(Y_+)) & \text{if } \sigma = p. \end{cases}$$

The $\varepsilon$-subdifferential of $F$ at $\bar{x} \in \operatorname{dom} F$, may be defined regarding the different concepts of Pareto $\varepsilon$-solutions efficient in the above with respect to $\sigma \in \{w,p\}$ as follows:

$$\partial^\sigma_\varepsilon F(\bar{x}) := \{A \in L(X,Y) : \bar{x} \in E^\sigma_\varepsilon(F-A,S,Y_+)\},$$

that is,

$$\bullet \partial^w_\varepsilon F(\bar{x}) = \{A \in L(X,Y) : \text{ there does not exist } x \in X,$$
$$F(x) - F(\bar{x}) <_{Y_+} A(x-\bar{x}) - \varepsilon\},$$
$$\bullet \partial^p_\varepsilon F(\bar{x}) = \{A \in L(X,Y) : \text{ there exists } \hat{Y}_+ \subsetneq Y \text{ a convex cone that satisfies}$$
$$Y_+ \backslash l\,(Y_+) \subseteq \operatorname{int} \hat{Y}_+,$$
$$\text{there does not exist } x \in X, F(x) - F(\bar{x}) \lneq_{\hat{Y}_+} A(x-\bar{x}) - \varepsilon\}.$$

The conjugate function of any function $F : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is denoted by $F^* : X^* \to \overline{\mathbb{R}}$ and is defined as

$$F^*(x^*) := \sup_{x \in X} \{\langle x^*, x\rangle - F(x)\}.$$

For $\eta \geq 0$, the $\eta$-subdifferential of $F$ at a point $\bar{x} \in \operatorname{dom} F$ is defined by

$$\partial_\eta F(\bar{x}) := \{x^* \in X^* : F(x) - F(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle - \eta, \text{ for all } x \in X\}.$$

It is easy to check that

$$X^* \in \partial_\eta F(\bar{x}) \Longleftrightarrow F^*(x^*) + F(\bar{x}) - \langle x^*, \bar{x}\rangle \leq \eta.$$

By the definition of $F^*$, the so-called Young–Fenchel inequality is

$$F^*(x^*) + F(x) \geq \langle x^*, x\rangle, \quad \text{for all } (x, x^*) \in X \times X^*.$$

If $\varepsilon = 0_Y$, then the set $\partial_0^\sigma F(\bar{x}) := \partial^\sigma F(\bar{x})$ is the Pareto subdifferential (see [7]), for any $\sigma \in \{s, w, p\}$. For simplicity, we consider the following notation:

$$Y_+^\sigma := \begin{cases} Y_+^* \backslash \{0_Y\} \text{ if } \sigma = w, \\ \\ (Y_+^*)^\circ \qquad \text{if } \sigma = p, \end{cases}$$

For any subset $C \subset X$, the vector indicator mapping $\delta_C^v : X \longrightarrow Y \cup \{+\infty_Y\}$ of $C$ is defined by

$$\delta_C^v(x) := \begin{cases} 0, & \text{if } x \in C, \\ +\infty_Y, & \text{otherwise}. \end{cases}$$

When $Y = \mathbb{R}$, the scalar indicator function is denoted by $\delta_C$. The vector indicator mapping $\delta_C^v$ appears to possess properties like the scalar indicator function $\delta_C$. Moreover, we point out the relation between $\delta_C^v$ and $\delta_C$

$$y^* \circ \delta_C^v = \delta_C, \text{ for all } y^* \in Y_+^* \backslash \{0_Y\}.$$

For any $\eta \geq 0$, the $\eta$-normal set to $C$ at $\overline{x} \in C$ is defined as the $\eta$-subdifferential of the indicator function $\delta_C$ at $\overline{x} \in C$, that is,

$$N_\eta(\bar{x}, C) := \partial_\eta \delta_C(\overline{x}) = \{x^* \in X^* : \langle x^*, x - \bar{x}\rangle \leq \eta, \text{ for all } x \in C\}.$$

In what follows, we will need the following theorems. The first one characterizes scalarly the approximate $\sigma$-subdifferential for $\sigma \in \{w, p\}$ and the second one gives a general formula on the sequential approximate subdifferential of the sum of $p$ ($p \geq 2$) proper lower semicontinuous convex functions defined in a reflexive Banach space.

**Theorem 1** ([5])**.** Let $K : X \to Y \cup \{+\infty_Y\}$ and let $\bar{x} \in \text{dom}K$. Then, for $\sigma \in \{p, w\}$,

$$\partial_\varepsilon^\sigma K(\bar{x}) \supseteq \bigcup_{y^* \in Y_+^\sigma} \{A \in L(X, Y) : y^* \circ A \in \partial_{\langle y^*, \varepsilon\rangle}(y^* \circ K)(\bar{x})\}, \quad \text{for all } \varepsilon \in D^\sigma,$$

with equality if $K$ is $Y_+$-convex and $Y_+$ pointed as $\sigma = p$.

Let $(X, \|\cdot\|_X)$ be a reflexive Banach space and let $(X^*, \|\cdot\|_{X^*})$ be its topological dual space. Let $(c_n^*)_{n\in\mathbb{N}}$ be a sequence in $X^*$ (resp., $(c_n)_{n\in\mathbb{N}}$ be a sequence in $X$) and $c^* \in X^*$ (resp., $c \in X$). We write $c_n^* \underset{n\longmapsto+\infty}{\overset{\|\cdot\|_{X^*}}{\longrightarrow}} c^*$ (resp., $c_n \underset{n\longmapsto+\infty}{\overset{\|\cdot\|_X}{\longrightarrow}} c$) if $\|c_n^* - c^*\|_{X^*} \to 0$ when $n \to +\infty$ (resp., $\|c_n - c\|_X \to 0$ when $n \to +\infty$).

**Theorem 2** ([14])**.** Let $X$ be a reflexive Banach space, let $\eta \geq 0$, and let $k_1, \dots, k_p : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be $p$ proper, convex, and lower semicontinuous functions. For any $\bar{e} \in \bigcap_{i=1}^p \mathrm{dom} k_i$, we have $x^* \in \partial_\eta \left(\sum_{i=1}^p k_i\right)(\bar{e})$ if and only if there exist $\eta_i \geq 0$ and $(e_{i,n}, e_{i,n}^*) \in \mathrm{dom}\, k_i \times X^*$ satisfying

$$\begin{cases} \displaystyle\sum_{i=1}^p \eta_i = \eta, \quad e_{i,n} \underset{n\longmapsto+\infty}{\overset{\|\cdot\|_X}{\longrightarrow}} \bar{e}, \\ e_{i,n}^* \in \partial_{\eta_i} k_i\left(e_{i,n}\right), \\ \displaystyle\sum_{i=1}^p e_{i,n}^* \underset{n\longmapsto+\infty}{\overset{\|\cdot\|_{X^*}}{\longrightarrow}} x^*, \\ k_i\left(e_{i,n}\right) - \left\langle e_{i,n}^*, e_{i,n} - \bar{e}\right\rangle \underset{n\mapsto+\infty}{\longrightarrow} k_i(\bar{e}). \end{cases}$$

## 3 Sequential formula for approximate Pareto subdifferential of the convex vector mapping $f_1 + f_2 + \sum_{i=1}^p g_i \circ h_i$ $(p \geq 1)$

In the following, $(X, \|\cdot\|_X)$ and $(Z_1, \|\cdot\|_{Z_1}), \dots, (Z_p, \|\cdot\|_{Z_p})$ are real reflexive Banach spaces, $Y$ is a real Hausdorff topological vector space with continuous dual spaces $(X^*, \|\cdot\|_{X^*})$, $(Z_1^*, \|\cdot\|_{Z_1^*})$,..., $(Z_p^*, \|\cdot\|_{Z_p^*})$ and $Y^*$, duality pairing is also denoted by $\langle\cdot,\cdot\rangle$. We suppose two spaces $Y$ and $Z_i$ are partially ordered by nonempty convex cones $Y_+$ and $Z_i^+$, respectively. Moreover, we use the following norm on $X \times \prod_{k=1}^p Z_k$:

$$\|(x, z_1, \dots, z_p)\|_{X\times Z_1\times\cdots\times Z_p} = \sqrt{\|x\|_X^2 + \textstyle\sum_{i=1}^p \|z_i\|_{Z_i}^2}.$$

Similarly, we define the norm on $X^* \times \prod_{k=1}^p Z_k^*$. Our aim in this section, is to develop the sequential calculus rules for the approximate (weak and proper) Pareto subdifferential of the convex vector mapping $f_1 + f_2 + \sum_{i=1}^p g_i \circ h_i : X \longrightarrow Y \cup \{+\infty_Y\}$, where

- $f_1, f_2 : X \longrightarrow Y \cup \{+\infty_Y\}$ are proper, $Y_+$-convex and star $Y_+$-lower semicontinuous mappings,

- $h_i : X \longrightarrow Z_i \cup \{+\infty_{Z_i}\}$, $i = 1, \dots, p$, are proper, $Z_i^+$-convex and $Z_i^+$-epi-closed mappings,

- $g_i : Z_i \longrightarrow Y \cup \{+\infty_Y\}$, $i = 1, \dots, p$, are proper, $Y_+$-convex, $(Z_i^+, Y_+)$-nondecreasing and star $Y_+$-lower semicontinuous mappings,

- $(\bigcap_{i=1}^p h_i^{-1}(\mathrm{dom} g_i) \cap \mathrm{dom} h_i) \cap \mathrm{dom} f_1 \cap \mathrm{dom} f_2 \neq \emptyset$ and $g_i(+\infty_{Z_i}) = +\infty_Y$, $i = 1, \dots, p$.

The approach that will be used is to reduce the (weak and proper) Pareto $\varepsilon$-subdifferential of $f_1 + f_2 + \sum_{i=1}^p g_i \circ h_i$ to that of the sums of a finite family of proper convex star $Y_+$-lower semicontinuous mappings. For this, let us consider the following auxiliary mappings:

$$\begin{aligned} F_1 : X \times \textstyle\prod_{k=1}^p Z_k &\longrightarrow Y \cup \{+\infty_Y\} \\ (x, z_1, \dots, z_p) &\longrightarrow F_1(x, z_1, \dots, z_p) := f_1(x), \end{aligned}$$

$$\begin{aligned} F_2 : X \times \textstyle\prod_{k=1}^p Z_k &\longrightarrow Y \cup \{+\infty_Y\} \\ (x, z_1, \dots, z_p) &\longrightarrow F_2(x, z_1, \dots, z_p) := f_2(x), \end{aligned}$$

$$\begin{aligned} G_i : X \times \textstyle\prod_{k=1}^p Z_k &\longrightarrow Y \cup \{+\infty_Y\} \qquad (i = 1, \dots, p) \\ (x, z_1, \dots, z_p) &\longrightarrow G_i(x, z_1, \dots, z_p) := g_i(z_i), \end{aligned}$$

$$\begin{aligned} H_i : X \times \textstyle\prod_{k=1}^p Z_k &\longrightarrow Y \cup \{+\infty_Y\} \qquad (i = 1, \dots, p) \\ (x, z_1, \dots, z_p) &\longrightarrow H_i(x, z_1, \dots, z_p) := \delta^v_{\mathrm{epi} h_i}(x, z_i). \end{aligned}$$

The following lemmas will be very helpful in what follows.

**Lemma 1.** Let $\bar{x} \in (\bigcap_{i=1}^p h_i^{-1}(\mathrm{dom} g_i) \cap \mathrm{dom} h_i) \cap \mathrm{dom} f_1 \cap \mathrm{dom} f_2$, let $\sigma \in \{p, w\}$, and let $\varepsilon \in D^\sigma$ with $Y_+$ be pointed as $\sigma = p$. Let $A \in L(X, Y)$ and let $T \in L(X \times \prod_{k=1}^p Z_k, Y)$ be defined by $T(x, z_1, \dots, z_p) := A(x)$, for all $(x, z_1, \dots, z_p) \in X \times \prod_{k=1}^p Z_k$. Then

$$A \in \partial^\sigma_\varepsilon (f_1 + f_2 + \sum_{i=1}^p g_i \circ h_i)(\bar{x})$$

if and only if

$$T \in \partial_{\varepsilon}^{\sigma}(F_1 + F_2 + \sum_{i=1}^{p} G_i + H_i)(\bar{x}, h_1(\bar{x}), \ldots, h_p(\bar{x})).$$

*Proof.* Let us prove the direct implication for the first case $\sigma = w$. Let $A \in \partial_{\varepsilon}^{w}(f_1+f_2+\sum_{i=1}^{p} g_i \circ h_i)(\bar{x})$. We proceed by contradiction, if $T \notin \partial_{\varepsilon}^{w}(F_1+F_2+\sum_{i=1}^{p} G_i+H_i)(\bar{x}, h_1(\bar{x}), \ldots, h_p(\bar{x}))$, then there exists $(x_0, z_{1,0}, \ldots, z_{p,0}) \in X \times \prod_{k=1}^{p} Z_k$ such that

$$(F_1 + F_2 + \sum_{i=1}^{p} G_i + H_i)(x_0, z_{1,0}, \ldots, z_{p,0})$$
$$-(F_1 + F_2 + \textstyle\sum_{i=1}^{p} G_i + H_i)(\bar{x}, h_1(\bar{x}), \ldots, h_p(\bar{x})) - A(x_0 - \bar{x}) + \varepsilon \in -\operatorname{int} Y_+,$$

which implies that

$$x_0 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2, z_{i,0} \in \operatorname{dom} g_i, (x_0, z_{i,0}) \in \operatorname{epi} h_i, \qquad i = 1, \ldots, p, \quad (1)$$

and

$$(f_1(x_0) + f_2(x_0) + \sum_{i=1}^{p} g_i(z_{i,0})) - (f_1 + f_2 + \sum_{i=1}^{p} g_i \circ h_i)(\bar{x}) - A(x_0 - \bar{x}) + \varepsilon \in -\operatorname{int} Y_+. \quad (2)$$

From relations (1), (2) and by using the monotonicity of $g_i$, we obtain

$$(f_1+f_2+\sum_{i=1}^{p} g_i \circ h_i)(x_0) - (f_1+f_2+\sum_{i=1}^{p} g_i \circ h_i)(\bar{x}) - A(x_0-\bar{x}) + \varepsilon \in -\operatorname{int} Y_+ - Y_+,$$

and by using the fact that $-\operatorname{int} Y_+ - Y_+ \subseteq -\operatorname{int} Y_+$, we would obtain

$$(f_1+f_2+\sum_{i=1}^{p} g_i \circ h_i)(x_0) - (f_1+f_2+\sum_{i=1}^{p} g_i \circ h_i)(\bar{x}) - A(x_0-\bar{x}) + \varepsilon \in -\operatorname{int} Y_+,$$

which would contradict $A \in \partial_{\varepsilon}^{w}(f_1 + f_2 + \sum_{i=1}^{p} g_i \circ h_i)(\bar{x})$. To show the case $\sigma = p$, let us consider $A \in \partial_{\varepsilon}^{p}(f_1 + f_2 + \sum_{i=1}^{p} g_i \circ h_i)(\bar{x})$ and $T \notin \partial_{\varepsilon}^{p}(F_1+F_2+\sum_{i=1}^{p} G_i+H_i)(\bar{x}, h_1(\bar{x}), \ldots, h_p(\bar{x}))$. Hence, there exists a convex cone $\hat{Y}_+ \subsetneq Y$ such that $Y_+ \setminus \{0_Y\} \subseteq \operatorname{int} \hat{Y}_+$ and $(x_0, z_{i,0}) \in ((\operatorname{dom} f_1 \cap \operatorname{dom} f_2) \times \operatorname{dom} g_i) \cap \operatorname{epi} h_i$ $(i = 1, \ldots, p)$ satisfying

$$(F_1 + F_2 + \sum_{i=1}^{p} G_i + H_i)(x_0, z_{1,0}, \ldots, z_{p,0})$$

$$-(F_1+F_2+\sum_{i=1}^{p}G_i+H_i)(\bar{x},h_1(\bar{x}),\dots,h_p(\bar{x}))-A(x_0-\bar{x})+\varepsilon\in-\hat{Y}_+\backslash l(\hat{Y}_+),$$

Reasoning in the same way as above for $\sigma=w$, we obtain

$$(f_1+f_2+\sum_{i=1}^{p}g_i\circ h_i)(x_0)-(f_1+f_2+\sum_{i=1}^{p}g_i\circ h_i)(\bar{x})-A(x_0-\bar{x})+\varepsilon\in-\hat{Y}_+\backslash l(\hat{Y}_+)-Y_+.$$

Observe that $-\hat{Y}_+\backslash l(\hat{Y}_+)-Y_+\subseteq-\hat{Y}_+\backslash l(\hat{Y}_+)$. Indeed, let $v:=v_1+v_2$ with $v_1\in-\hat{Y}_+\backslash l(\hat{Y}_+)$ and $v_2\in-Y_+$. If $v_2:=0_Y$, then $v\in-\hat{Y}_+\backslash l(\hat{Y}_+)$. Otherwise, $v_2\in-Y_+\backslash\{0_Y\}$ and since $-Y_+\backslash\{0_Y\}\subseteq-\operatorname{int}\hat{Y}_+\subseteq-\hat{Y}_+\backslash l(\hat{Y}_+)$, we get $v\in-\hat{Y}_+\backslash l(\hat{Y}_+)-\hat{Y}_+\backslash l(\hat{Y}_+)\subseteq-\hat{Y}_+\backslash l(\hat{Y}_+)$, and hence we have

$$(f_1+f_2+\sum_{i=1}^{p}g_i\circ h_i)(x_0)-(f_1+f_2+\sum_{i=1}^{p}g_i\circ h_i)(\bar{x})-A(x_0-\bar{x})+\varepsilon\in-\hat{Y}_+\backslash l(\hat{Y}_+),$$

obtaining again a contradiction with $A\in\partial^p_\varepsilon(f_1+f_2+\sum_{i=1}^p g_i\circ h_i)(\bar{x})$.

Conversely, it is obvious by contradiction, too. □

**Lemma 2.** Let $(\bar{x},\bar{z_1},\dots,\bar{z_p})\in\operatorname{dom}F_1\cap\operatorname{dom}F_2\cap\operatorname{dom}G_i\cap\operatorname{dom}H_i$ and $\alpha_1$,$\alpha_2$,$\eta_i$,$\beta_i\geq0$ $(i=1,\dots,p)$. Then for all $y^*\in Y_+^\sigma$, we have

1)
$$(x^*,z_1^*,\dots,z_p^*)\in\partial_{\beta_i}(y^*\circ H_i)(\bar{x},\bar{z_1},\dots,\bar{z_p})$$
$$\Longleftrightarrow\begin{cases}\text{there exist}\beta_{i,1}\geq0,\beta_{i,2}\geq0\text{ with }\beta_{i,1}+\beta_{i,2}=\beta_i,\\ x^*\in\partial_{\beta_{i,1}}(-z_i^*\circ h_i)(\bar{x}),\\ -z_i^*\in(Z_i^+)^*,\ \langle z_i^*,h_i(\bar{x})-\bar{z_i}\rangle\leq\beta_{i,2},\\ z_j^*=0,\ j\in\{1,\dots,p\}\backslash\{i\}.\end{cases}$$

2) $$(x^*,z_1^*,\dots,z_p^*)\in\partial_{\alpha_1}(y^*\circ F_1)(\bar{x},\bar{z_1},\dots,\bar{z_p})\Longleftrightarrow\begin{cases}x^*\in\partial_{\alpha_1}(y^*\circ f_1)(\bar{x}),\\ z_j^*=0,\ j\in\{1,\dots,p\}.\end{cases}$$

$$3)\ (x^*, z_1^*, \ldots, z_p^*) \in \partial_{\alpha_2}(y^* \circ F_2)(\bar{x}, \bar{z}_1, \ldots, \bar{z}_p) \Longleftrightarrow \begin{cases} x^* \in \partial_{\alpha_2}(y^* \circ f_2)(\bar{x}), \\ \\ z_j^* = 0,\ j \in \{1, \ldots, p\}. \end{cases}$$

$$4)\ (x^*, z_1^*, \ldots, z_p^*) \in \partial_{\eta_i}(y^* \circ G_i)(\bar{x}, \bar{z}_1, \ldots, \bar{z}_p) \Longleftrightarrow \begin{cases} z_i^* \in \partial_{\eta_i}(y^* \circ g_i)(\bar{z}_i), \\ \\ x^* = 0,\ z_j^* = 0, \\ j \in \{1, \ldots, p\} \backslash \{i\}. \end{cases}$$

*Proof.* 1) Let $(x, z_1, \ldots, z_p) \in \mathrm{dom} H_i$, let $\beta_i \geq 0$, $i = 1, \ldots, p$, and let $\left(x^*, z_1^*, \ldots, z_p^*\right) \in X^* \times \prod_{k=1}^{p} Z_k^*$. We can see that

$$H_i^*(x^*, z_1^*, \ldots, z_p^*) = \delta^*_{Z_i^*}(z_i^*) + (-z_i^* \circ h_i)^*(x^*) + \sum_{\substack{k=1 \\ k \neq i}}^{p} \delta_{\{0\}}(z_k^*).$$

Thus,

$$\left(x^*, z_1^*, \ldots, z_p^*\right) \in \partial_{\beta_i} H_i(x, z_1, \ldots, z_p)$$

if and only if

$$H_i^*\left(x^*, z_1^*, \ldots, z_p^*\right) + H_i(x, , z_1, \ldots, z_p) - \langle x^*, x\rangle - \sum_{\substack{k=1 \\ k \neq i}}^{p} \langle z_k^*, z_k\rangle - \langle z_i^*, z_i\rangle \leq \beta_i,$$

that is,

$$z_k^* = 0, \qquad k \in \{1, \ldots, p\} \backslash \{i\},$$

and

$$(-z_i^* \circ h_i)^*(x^*) + \delta^*_{Z_i^+}(z_i^*) + \delta_{\mathrm{epi} h_i}(\bar{x}, \bar{z}_i) - \langle x^*, \bar{x}\rangle - \langle z_i^*, \bar{z}_i\rangle \leq \beta_i, \tag{3}$$

by taking $w_i := \bar{z}_i - h_i(\bar{x}) \in Z_i^+$, we may rewrite (3) as follows:

$$[(-z_i^* \circ h_i)^*(x^*) + (-z_i^* \circ h_i)(\bar{x}) - \langle x^*, \bar{x}\rangle] + [\delta^*_{Z_i^+}(z_i^*) + \delta_{Z_i^+}(w_i) - \langle z_i^*, w_i\rangle] \leq \beta_i.$$

By using the Young–Fenchel inequality, it follows that

$$\begin{cases} (-z_i^* \circ h_i)^*(x^*) + (-z_i^* \circ h_i)(\bar{x}) - \langle x^*, \bar{x}\rangle \geq 0, \\ \\ \delta^*_{Z_i^+}(z_i^*) + \delta_{Z_i^+}(w_i) - \langle z_i^*, w_i\rangle \geq 0, \end{cases}$$

and hence, there exist some $\beta_{i,1} \geq 0$ and $\beta_{i,2} \geq 0$ satisfying $\beta_i = \beta_{i,1} + \beta_{i,2}$ and

$$\begin{cases} (-z_i^* \circ h_i)^*(x^*) + (-z_i^* \circ h_i)(\bar{x}) - \langle x^*, \bar{x} \rangle \leq \beta_{i,1}, \\ \\ \delta^*_{Z_i^+}(z_i^*) + \delta_{Z_i^+}(w_i) - \langle z_i^*, w_i \rangle \leq \beta_{i,2}, \end{cases}$$

that is,

$$\begin{cases} x^* \in \partial_{\beta_{i,1}}(-z_i^* \circ h_i)(\bar{x}), \\ \\ z_i^* \in \partial_{\beta_{i,2}} \delta_{Z_i^+}(w_i) = N_{\beta_{i,2}}(\bar{z}_i - h_i(\bar{x}), Z_i^+). \end{cases}$$

Since $Z_1^+, \ldots, Z_p^+$ are all convex cones, we get easily

$$z_i^* \in N_{\beta_{i,2}}(\bar{z}_i - h_i(\bar{x}), Z_i^+) \Longleftrightarrow \begin{cases} -z_i^* \in (Z_i^+)^*, \\ \\ \langle z_i^*, h_i(\bar{x}) - \bar{z}_i \rangle \leq \beta_{i,2}. \end{cases}$$

The proof of (2), (3), and (4) is similar to (1). □

Now, we present our main result in this section.

**Theorem 3.** Let $f_1, f_2 : X \longrightarrow Y \cup \{+\infty_Y\}$ be two proper, $Y_+$-convex and star $Y_+$-lower semicontinuous mappings, let $h_i : X \longrightarrow Z_i \cup \{+\infty_{Z_i}\}$, $i = 1, \ldots, p$, be $p$ proper, $Z_i^+$-convex and $Z_i^+$-epi-closed mappings, and let $g_i : Z_i \longrightarrow Y \cup \{+\infty_Y\}$, $i = 1, \ldots, p$, be $p$ proper, $Y_+$-convex, $(Z_i^+, Y_+)$-nondecreasing and star $Y_+$-lower semicontinuous mappings. Let $\bar{x} \in (\bigcap_{i=1}^p h_i^{-1}(\mathrm{dom} g_i) \cap \mathrm{dom} h_i) \cap \mathrm{dom} f_1 \cap \mathrm{dom} f_2$, $\varepsilon \in D^\sigma$ and $\sigma \in \{p, w\}$. Suppose that $Y_+$ is pointed as $\sigma = p$. Then, $A \in \partial_\varepsilon^\sigma(f_1 + f_2 + \sum_{i=1}^p g_i \circ h_i)(\bar{x})$ if and only if there exist $y^* \in Y_+^\sigma$, $x_n \in \mathrm{dom} f_1$, $r_n \in \mathrm{dom} f_2$, $(x_{i,n}, z_{i,n}) \in \mathrm{epi} h_i$, $v_{i,n} \in \mathrm{dom} g_i$, $x_n^*, r_n^*, x_{i,n}^* \in X^*$, $z_{i,n}^*, v_{i,n}^* \in Z_i^*$, and $\alpha_1, \alpha_2, \lambda_i, \gamma_i, \eta_i \geq 0$ satisfying

$$\begin{cases} \alpha_1 + \alpha_2 + \displaystyle\sum_{i=1}^p \lambda_i + \gamma_i + \eta_i = \langle y^*, \varepsilon \rangle, \\ \begin{cases} x_n \overset{\|\cdot\|_X}{\underset{n \longmapsto +\infty}{\longrightarrow}} \bar{x}, \quad r_n \overset{\|\cdot\|_X}{\underset{n \longmapsto +\infty}{\longrightarrow}} \bar{x}, \quad x_{i,n} \overset{\|\cdot\|_X}{\underset{n \longmapsto +\infty}{\longrightarrow}} \bar{x}, \\ z_{i,n} \overset{\|\cdot\|_{Z_i}}{\underset{n \longmapsto +\infty}{\longrightarrow}} h_i(\bar{x}), \quad v_{i,n} \overset{\|\cdot\|_{Z_i}}{\underset{n \longmapsto +\infty}{\longrightarrow}} h_i(\bar{x}), \end{cases} \\ \\ x_n^* \in \partial_{\alpha_1}(y^* \circ f_1)(x_n), r_n^* \in \partial_{\alpha_2}(y^* \circ f_2)(x_n), v_{i,n}^* \in \partial_{\eta_i}(y^* \circ g_i)(v_{i,n}), \\ \\ x_{i,n}^* \in \partial_{\lambda_i}(-z_{i,n}^* \circ h_i)(x_{i,n}), -z_{i,n}^* \in (Z_i^+)^*, \langle z_{i,n}^*, h_i(x_{i,n}) - z_{i,n} \rangle \leq \gamma_i, \end{cases}$$

and

$$\begin{cases} x_n^* + r_n^* + \sum_{i=1}^{p} x_{i,n}^* \quad \xrightarrow[n\longmapsto+\infty]{\|\cdot\|_{X^*}} y^* \circ A \text{ and } z_{i,n}^* + v_{i,n}^* \quad \xrightarrow[n\longmapsto+\infty]{\|\cdot\|_{Z_i^*}} 0, \\ (y^* \circ f_1)(x_n) - \langle x_n^*, x_n - \bar{x}\rangle \quad \xrightarrow[n\mapsto+\infty]{} (y^* \circ f_1)(\bar{x}), \\ (y^* \circ f_2)(r_n) - \langle r_n^*, r_n - \bar{x}\rangle \quad \xrightarrow[n\mapsto+\infty]{} (y^* \circ f_2)(\bar{x}), \\ \langle x_{i,n}^*, x_{i,n} - \bar{x}\rangle + \langle z_{i,n}^*, z_{i,n} - h_i(\bar{x})\rangle \quad \xrightarrow[n\mapsto+\infty]{} 0, \\ (y^* \circ g_i)(v_{i,n}) - \langle v_{i,n}^*, v_{i,n} - h_i(\bar{x})\rangle \quad \xrightarrow[n\mapsto+\infty]{} (y^* \circ g_i)(h_i(\bar{x})). \end{cases}$$

*Proof.* Let $A \in \partial_\varepsilon^\sigma(f_1 + f_2 + \sum_{i=1}^p g_i \circ h_i)(\bar{x})$. By applying Lemma 1, we have

$$A \in \partial_\varepsilon^\sigma(f_1 + f_2 + \sum_{i=1}^{p} g_i \circ h_i)(\bar{x})$$
$$\Longleftrightarrow T \in \partial_\varepsilon^\sigma(F_1 + F_2 + \sum_{i=1}^{p} G_i + H_i)(\bar{x}, h_1(\bar{x}), \ldots, h_p(\bar{x})).$$

By virtue of scalarization Theorem 1, there exists $y^* \in Y_+^\sigma$ such that

$$y^* \circ T \in \partial_{\langle y^*,\varepsilon\rangle}(y^* \circ F_1 + y^* \circ F_2 + \sum_{i=1}^{p} y^* \circ G_i + y^* \circ H_i)(\bar{x}, h_1(\bar{x}), \ldots, h_p(\bar{x})),$$

which is equivalent to

$$(y^*\circ A, 0, 0, \ldots, 0) \in \partial_{\langle y^*,\varepsilon\rangle}(y^*\circ F_1 + y^*\circ F_2 + \sum_{i=1}^{p} y^*\circ G_i + y^*\circ H_i)(\bar{x}, h_1(\bar{x}), \ldots, h_p(\bar{x})).$$

The functions $y^* \circ G_i$, $y^* \circ H_i$ $(i = 1, \ldots, p,)$ $y^* \circ F_1$ and $y^* \circ F_2$ are all lower semicontinuous, convex, and proper on $X \times \prod_{k=1}^p Z_k$.

The condition $\bar{x} \in (\bigcap_{i=1}^p h_i^{-1}(\mathrm{dom} g_i) \cap \mathrm{dom} h_i) \cap \mathrm{dom} f_1 \cap \mathrm{dom} f_2$ is equivalent to $(\bar{x}, h_1(\bar{x}), \ldots, h_p(\bar{x})) \in (\bigcap_{i=1}^p \mathrm{dom} G_i \cap \mathrm{dom} H_i) \cap \mathrm{dom} F_1 \cap \mathrm{dom} F_2 = (\bigcap_{i=1}^p \mathrm{dom}(y^* \circ G_i) \cap \mathrm{dom}(y^* \circ H_i)) \cap \mathrm{dom}(y^* \circ F_1) \cap \mathrm{dom}(y^* \circ F_2)$. Hence, $y^* \circ G_i$, $y^* \circ H_i$, $i = 1, \ldots, p$, $y^* \circ F_1$ and $y^* \circ F_2$ satisfy all the hypotheses of Theorem 2, and therefore there exist $(x_n, w_{1,n}, \ldots, w_{p,n}) \in \mathrm{dom}(y^* \circ F_1)$,

$(r_n,t_{1,n},\ldots,t_{p,n}) \in \operatorname{dom}(y^*\circ F_2)$ , $(x_{i,n},w_{1,i,n},\ldots,w_{p,i,n}) \in \operatorname{dom}(y^*\circ H_i)$, $(c_{i,n},u_{1,i,n},\ldots,u_{p,i,n}) \in \operatorname{dom}(y^*\circ G_i)$, $(x_n^*,w_{1,n}^*,\ldots,w_{p,n}^*)$, $(r_n^*,t_{1,n}^*,\ldots,t_{p,n}^*)$ $(x_{i,n}^*,w_{1,i,n}^*,\ldots,w_{p,i,n}^*)$, $(c_{i,n}^*,u_{1,i,n}^*,\ldots,u_{p,i,n}^*) \in X^*\times\prod_{k=1}^p Z_k^*$ and $\alpha_1$,$\alpha_2$,$\beta_i$,$\eta_i \geq 0$, $i=1,\ldots,p$, satisfying

$$\alpha_1+\alpha_2+\sum_{i=1}^{p}\beta_i+\eta_i=\langle y^*,\varepsilon\rangle, \tag{4}$$

$$\begin{cases} (x_n,w_{1,n},\ldots,w_{p,n}) \overset{\|\cdot\|_{X\times Z_1\times\cdots\times Z_p}}{\underset{n\longmapsto+\infty}{\longrightarrow}} (\bar{x},h_1(\bar{x}),\ldots,h_p(\bar{x})),\\ (r_n,t_{1,n},\ldots,t_{p,n}) \overset{\|\cdot\|_{X\times Z_1\times\cdots\times Z_p}}{\underset{n\longmapsto+\infty}{\longrightarrow}} (\bar{x},h_1(\bar{x}),\ldots,h_p(\bar{x})),\\ (c_{i,n},u_{1,i,n},\ldots,u_{p,i,n}) \overset{\|\cdot\|_{X\times Z_1\times\cdots\times Z_p}}{\underset{n\longmapsto+\infty}{\longrightarrow}} (\bar{x},h_1(\bar{x}),\ldots,h_p(\bar{x})),\\ (x_{i,n},w_{1,i,n},\ldots,w_{p,i,n}) \overset{\|\cdot\|_{X\times Z_1\times\cdots\times Z_p}}{\underset{n\longmapsto+\infty}{\longrightarrow}} (\bar{x},h_1(\bar{x}),\ldots,h_p(\bar{x})), \end{cases} \tag{5}$$

$$\begin{cases} (x_n^*,w_{1,n}^*,\ldots,w_{p,n}^*) \in \partial_{\alpha_1}(y^*\circ F_1)(x_n,w_{1,n},\ldots,w_{p,n}),\\ (r_n^*,t_{1,n}^*,\ldots,t_{p,n}^*) \in \partial_{\alpha_2}(y^*\circ F_2)(r_n,t_{1,n},\ldots,t_{p,n}),\\ (c_{i,n}^*,u_{1,i,n}^*,\ldots,u_{p,i,n}^*) \in \partial_{\eta_i}(y^*\circ G_i)(c_{i,n},u_{1,i,n},\ldots,u_{p,i,n}),\\ (x_{i,n}^*,w_{1,i,n}^*,\ldots,w_{p,i,n}^*) \in \partial_{\beta_i}(y^*\circ H_i)(x_{i,n},w_{1,i,n},\ldots,w_{p,i,n}), \end{cases} \tag{6}$$

$$\begin{cases} (y^*\circ F_1)(x_n,w_{1,n},\ldots,w_{p,n})-\langle(x_n^*,w_{1,n}^*,\ldots,w_{p,n}^*),(x_n,w_{1,n},\ldots,w_{p,n})\\ \qquad -(\bar{x},h_1(\bar{x}),\ldots,h_p(\bar{x}))\rangle \underset{n\mapsto+\infty}{\longrightarrow} (y^*\circ F_1)(\bar{x},h_1(\bar{x}),\ldots,h_p(\bar{x})),\\ \\ (y^*\circ F_2)(r_n,t_{1,n},\ldots,t_{p,n})-\langle(r_n^*,t_{1,n}^*,\ldots,t_{p,n}^*),(r_n,t_{1,n},\ldots,t_{p,n})\\ \qquad -(\bar{x},h_1(\bar{x}),\ldots,h_p(\bar{x}))\rangle \underset{n\mapsto+\infty}{\longrightarrow} (y^*\circ F_2)(\bar{x},h_1(\bar{x}),\ldots,h_p(\bar{x})),\\ \\ (y^*\circ G_i)(c_{i,n},u_{1,i,n},\ldots,u_{p,i,n})-\langle(c_{i,n}^*,u_{1,i,n}^*,\ldots,u_{p,i,n}^*),(c_{i,n},u_{1,i,n},\ldots,u_{p,i,n})\\ \qquad -(\bar{x},h_1(\bar{x}),\ldots,h_p(\bar{x}))\rangle \underset{n\mapsto+\infty}{\longrightarrow} (y^*\circ G_i)(\bar{x},h_1(\bar{x}),\ldots,h_p(\bar{x})),\\ \\ (y^*\circ H_i)(x_{i,n},w_{1,i,n},\ldots,w_{p,i,n})-\langle(x_{i,n}^*,w_{1,i,n}^*,\ldots,w_{p,i,n}^*),(x_{i,n},w_{1,i,n},\ldots,w_{p,i,n})\\ \qquad -(\bar{x},h_1(\bar{x}),\ldots,h_p(\bar{x}))\rangle \underset{n\mapsto+\infty}{\longrightarrow} (y^*\circ H_i)(\bar{x},h_1(\bar{x}),\ldots,h_p(\bar{x})), \end{cases} \tag{7}$$

$$(x_n^* + r_n^* + \sum_{i=1}^{p} x_{i,n}^* + c_{i,n}^*, w_{1,n}^* + t_{1,n}^* + \sum_{i=1}^{p} w_{1,i,n}^* + u_{1,i,n}^*, \dots, w_{p,n}^* + t_{p,n}^* + \sum_{i=1}^{p} w_{p,i,n}^* + u_{p,i,n}^*) \xrightarrow[n\longmapsto+\infty]{\|\cdot\|_{x^*\times z_1^*\times\cdots\times z_p^*}} (y^* \circ A, 0, \dots, 0). \tag{8}$$

Note that (5) becomes equivalent to

$$\begin{cases} x_n \xrightarrow[n\longmapsto+\infty]{\|\cdot\|_X} \bar{x}, \quad r_n \xrightarrow[n\longmapsto+\infty]{\|\cdot\|_X} \bar{x}, \quad x_{i,n} \xrightarrow[n\longmapsto+\infty]{\|\cdot\|_X} \bar{x}, \quad c_{i,n} \xrightarrow[n\longmapsto+\infty]{\|\cdot\|_X} \bar{x}, \\ w_{j,n} \xrightarrow[n\longmapsto+\infty]{\|\cdot\|_{Z_j}} h_j(\bar{x}), \quad t_{j,n} \xrightarrow[n\longmapsto+\infty]{\|\cdot\|_{Z_j}} h_j(\bar{x}) \qquad (j = 1, \dots, p), \\ w_{j,i,n} \xrightarrow[n\longmapsto+\infty]{\|\cdot\|_{Z_j}} h_j(\bar{x}), \ u_{j,i,n} \xrightarrow[n\longmapsto+\infty]{\|\cdot\|_X} h_j(\bar{x}) \qquad (j = 1, \dots, p). \end{cases}$$

According to Lemma 2, (6) becomes equivalent to

$$\begin{cases} (x_n^*, w_{1,n}^*, \dots, w_{p,n}^*) \in \partial_{\alpha_1}(y^* \circ F_1)(x_n, w_{1,n}, \dots, w_{p,n}) \\ \qquad \Longleftrightarrow \begin{cases} x_n^* \in \partial_{\alpha_1}(y^* \circ f_1)(x_n), \\ w_{j,n}^* = 0, \ j \in \{1, \dots, p\}. \end{cases} \\ (r_n^*, t_{1,n}^*, \dots, t_{p,n}^*) \in \partial_{\alpha_2}(y^* \circ F_2)(r_n, t_{1,n}, \dots, t_{p,n}) \\ \qquad \Longleftrightarrow \begin{cases} r_n^* \in \partial_{\alpha_2}(y^* \circ f_2)(r_n), \\ t_{j,n}^* = 0, \ j \in \{1, \dots, p\}. \end{cases} \\ (c_{i,n}^*, u_{1,i,n}^*, \dots, u_{p,i,n}^*) \in \partial_{\eta_i}(y^* \circ G_i)(c_{i,n}, u_{1,i,n}, \dots, u_{p,i,n}) \\ \qquad \Longleftrightarrow \begin{cases} u_{i,i,n}^* \in \partial_{\eta_i}(y^* \circ g_i)(u_{i,i,n}), \\ c_{i,n}^* = 0, \ u_{j,i,n} = 0, \ j \in \{1, \dots, p\} \backslash \{i\}. \end{cases} \\ (x_{i,n}^*, w_{1,i,n}^*, \dots, w_{p,i,n}^*) \in \partial_{\beta_i}(y^* \circ H_i)(x_{i,n}, w_{1,i,n}, \dots, w_{p,i,n}) \\ \qquad \Longleftrightarrow \begin{cases} \text{there exist} \beta_{i,1} \geq 0, \beta_{i,2} \geq 0 \text{ with } \beta_{i,1} + \beta_{i,2} = \beta_i, \\ x_{i,n}^* \in \partial_{\beta_{i,1}}(-w_{i,i,n}^* \circ h_i)(x_{i,n}), \\ -w_{i,i,n}^* \in (Z_i^+)^*, \langle w_{i,i,n}^*, h_i(x_{i,n}) - w_{i,i,n} \rangle \leq \beta_{i,2}, \\ w_{j,i,n}^* = 0, \ j \in \{1, \dots, p\} \backslash \{i\}. \end{cases} \end{cases}$$

By putting $\lambda_i := \beta_{i,1}$, $\gamma_i := \beta_{i,2}$, $i = 1, \ldots, p$, and using (4), we get $\alpha_1 + \alpha_2 + \sum_{i=1}^{p} \lambda_i + \gamma_i + \eta_i = \langle y^*, \varepsilon\rangle$. As $(x_{i,n}, w_{i,i,n}) \in \text{epi} h_i$ and $(\bar{x}, h_i(\bar{x})) \in \text{epi} h_i$, then obviously the relation (7) is equivalent to

$$\begin{cases} (y^* \circ f_1)(x_n) - \langle x_n^*, x_n - \bar{x}\rangle \underset{n\mapsto+\infty}{\longrightarrow} (y^* \circ f_1)(\bar{x}), \\ \\ (y^* \circ f_2)(x_n) - \langle x_n^*, x_n - \bar{x}\rangle \underset{n\mapsto+\infty}{\longrightarrow} (y^* \circ f_2)(\bar{x}), \\ \\ (y^* \circ g_i)(u_{i,i,n}) - \langle u_{i,i,n}^*, u_{i,i,n} - h_i(\bar{x})\rangle \underset{n\mapsto+\infty}{\longrightarrow} (y^* \circ g_i)(h_i(\bar{x})), \\ \\ \langle x_{i,n}^*, x_{i,n} - \bar{x}\rangle + \langle w_{i,i,n}^*, w_{i,i,n} - h_i(\bar{x})\rangle \underset{n\mapsto+\infty}{\longrightarrow} 0. \end{cases}$$

W can write (8) equivalently as

$$x_n^* + r_n^* + \sum_{i=1}^{p} x_{i,n}^* \overset{\|\cdot\|_{x^*}}{\underset{n\longmapsto+\infty}{\longrightarrow}} y^* \circ A \text{ and } z_{i,i,n}^* + v_{i,i,n}^* \overset{\|\cdot\|_{Z_i^*}}{\underset{n\longmapsto+\infty}{\longrightarrow}} 0.$$

Since $w_{j,n}$, $t_{j,n}$, $j = 1, \ldots, p$, $c_{i,n}$, $u_{j,i,n}$, $j \in \{1, \ldots, p\}\backslash\{i\}$, $i = 1, \ldots, p$, and $w_{j,i,n}$, $j \in \{1, \ldots, p\}\backslash\{i\}$, $i = 1, \ldots, p$, are superfluous, we put $z_{i,n} := w_{i,i,n}$, $v_{i,n} = u_{i,i,n}$, $z_{i,n}^* := w_{i,i,n}^*$ and $v_{i,n}^* := u_{i,i,n}^*$, $i = 1, \ldots, p$, which completes the proof. □

**Remark 1.** Let us note that if $\varepsilon = 0_Y$ in the above theorem, then $\alpha_1 = \alpha_2 = \lambda_i = \gamma_i = \eta_i = 0$ $(i = 1, \ldots, p)$.

## 4 Approximate optimality conditions for a constrained vector minimization problem

In this section, our main objective is to establish sequential approximate efficiency optimality conditions for the convex problem $(P_1)$. In fact, by introducing the vector indicator mappings $\delta_C^v$ and $\delta_{-Z_i^+}^v$, $i = 1, \ldots, p$, the problem $(P_1)$ becomes equivalent to the unconstrained vector minimization problem

$$(Q_1) \quad \min_{x\in X} \left( f + \delta_C^v + \sum_{i=1}^{p} \delta_{-Z_i^+}^v \circ h_i \right)(x).$$

**Theorem 4.** Let $f : X \longrightarrow Y \cup \{+\infty_Y\}$ be a proper, $Y_+$-convex and star $Y_+$-lower semicontinuous mapping, and let $h_i : X \longrightarrow Z_i \cup \{+\infty_{Z_i}\}$, $i = 1, \dots, p$, be $p$ proper, $Z_i^+$-convex and $Z_i^+$-epi-closed mappings. Let $\bar{x} \in \bigcap_{i=1}^{p} h_i^{-1}(-Z_i^+) \cap C$, $\varepsilon \in D^\sigma$ and let $\sigma \in \{w, p\}$. Suppose that $Y_+$ is pointed as $\sigma = p$ and that $C, Z_1^+, \dots, Z_p^+$ are all closed convex cones. Then, $\bar{x}$ is a $\varepsilon$-$\sigma$-efficient solution of $(P_1)$ if and only if there exist $y^* \in Y_+^{\sigma}$, $x_n \in \mathrm{dom} f$, $r_n \in C$, $(x_{i,n}, z_{i,n}) \in \mathrm{epi} h_i$, $v_{i,n} \in -Z_i^+$,

$x_n^*, r_n^*, x_{i,n}^* \in X^*$, $z_{i,n}^*, v_{i,n}^* \in Z_i^*$ and $\alpha_1, \alpha_2, \lambda_i, \gamma_i, \eta_i \geq 0$, $i = 1, \dots, p$, satisfying

$$\begin{cases} \alpha_1 + \alpha_2 + \sum_{i=1}^{p} \lambda_i + \gamma_i + \eta_i = \langle y^*, \varepsilon \rangle, \\ \begin{cases} x_n \xrightarrow[n \longmapsto +\infty]{\|\cdot\|_X} \bar{x}, \quad r_n \xrightarrow[n \longmapsto +\infty]{\|\cdot\|_X} \bar{x}, \quad x_{i,n} \xrightarrow[n \longmapsto +\infty]{\|\cdot\|_X} \bar{x}, \\ z_{i,n} \xrightarrow[n \longmapsto +\infty]{\|\cdot\|_{Z_i}} h_i(\bar{x}), \quad v_{i,n} \xrightarrow[n \longmapsto +\infty]{\|\cdot\|_{Z_i}} h_i(\bar{x}), \end{cases} \\ x_n^* \in \partial_{\alpha_1}(y^* \circ f)(x_n), r_n^* \in N_{\alpha_2}(r_n, C), x_{i,n}^* \in \partial_{\lambda_i}(-z_{i,n}^* \circ h_i)(x_{i,n}), \\ v_{i,n}^* \in (Z_i^+)^* \text{ and } \langle v_{i,n}^*, -v_{i,n} \rangle \leq \eta_i, \\ -z_{i,n}^* \in (Z_i^+)^*, \langle z_{i,n}^*, h_i(x_{i,n}) - z_{i,n} \rangle \leq \gamma_i, \end{cases}$$

and

$$\begin{cases} x_n^* + \sum_{i=1}^{p} x_{i,n}^* \xrightarrow[n \longmapsto +\infty]{\|\cdot\|_{x^*}} 0 \text{ and } z_{i,n}^* + v_{i,n}^* \xrightarrow[n \longmapsto +\infty]{\|\cdot\|_{Z_i^*}} 0, \\ (y^* \circ f)(x_n) - \langle x_n^*, x_n - \bar{x} \rangle \xrightarrow[n \mapsto +\infty]{} (y^* \circ f)(\bar{x}), \\ \langle x_{i,n}^*, x_{i,n} - \bar{x} \rangle + \langle z_{i,n}^*, z_{i,n} - h_i(\bar{x}) \rangle \xrightarrow[n \mapsto +\infty]{} 0, \\ \langle r_n^*, r_n - \bar{x} \rangle \xrightarrow[n \mapsto +\infty]{} 0, \langle v_{i,n}^*, v_{i,n} - h_i(\bar{x}) \rangle \xrightarrow[n \mapsto +\infty]{} 0. \end{cases}$$

*Proof.* Since the problem $(P_1)$ is equivalent to the unconstrained problem $(Q_1)$, we have $\bar{x}$ is a $\varepsilon$-$\sigma$-efficient solution of $(P_1)$ if and only if

$$0 \in \partial_{\varepsilon}^{\sigma}(f + \delta_C^v + \textstyle\sum_{i=1}^{p} \delta_{-Z_i^+}^v \circ h_i)(\bar{x}).$$

The vector indicator mappings $\delta^v_C$ and $\delta^v_{-Z_1^+}$ $(i = 1, \ldots, p)$ are $Y_+$-convex and star $Y_+$-lower semicontinuous. Let us recall that $\delta^v_{-Z_i^+}$ $(i = 1, \ldots, p)$ are $(Z_i^+, Y_+)$-nondecreasing (see [7]) and since the mappings $f_1 = f$, $f_2 = \delta^v_C$, $g_i = \delta^v_{-Z_i^+}$ and $h_i$ satisfy together all the assumptions of Theorem 3, therefore there exist $y^* \in Y_+^\sigma$, $x_n \in \mathrm{dom} f$, $r_n \in C$, $(x_{i,n}, z_{i,n}) \in \mathrm{epi} h_i$, $v_{i,n} \in -Z_i^+$, $x_n^*, r_n^*, x_{i,n}^* \in X^*$, $z_{i,n}^*, v_{i,n}^* \in Z_i^*$, and $\alpha_1, \alpha_2, \lambda_i, \gamma_i, \eta_i \geq 0$, $i = 1, \ldots, p$, satisfying

$$\begin{cases} \alpha_1 + \alpha_2 + \sum\limits_{i=1}^{p} \lambda_i + \gamma_i + \eta_i = \langle y^*, \varepsilon \rangle, \\ \begin{cases} x_n \underset{n \longmapsto +\infty}{\overset{\|\cdot\|_X}{\longrightarrow}} \bar{x}, \quad r_n \underset{n \longmapsto +\infty}{\overset{\|\cdot\|_X}{\longrightarrow}} \bar{x}, \quad x_{i,n} \underset{n \longmapsto +\infty}{\overset{\|\cdot\|_X}{\longrightarrow}} \bar{x}, \\ z_{i,n} \underset{n \longmapsto +\infty}{\overset{\|\cdot\|_{Z_i}}{\longrightarrow}} h_i(\bar{x}), \quad v_{i,n} \underset{n \longmapsto +\infty}{\overset{\|\cdot\|_{Z_i}}{\longrightarrow}} h_i(\bar{x}), \end{cases} \\ x_n^* \in \partial_{\alpha_1}(y^* \circ f)(x_n), r_n^* \in N_{\alpha_2}(r_n, C), v_{i,n}^* \in N_{\eta_i}(v_{i,n}, \\ -Z_i^+), x_{i,n}^* \in \partial_{\lambda_i}(-z_{i,n}^* \circ h_i)(x_{i,n}) - z_{i,n}^* \in (Z_i^+)^*, \langle z_{i,n}^*, h_i(x_{i,n}) - z_{i,n} \rangle \leq \gamma_i, \end{cases}$$

and

$$\begin{cases} x_n^* + \sum\limits_{i=1}^{p} x_{i,n}^* \underset{n \longmapsto +\infty}{\overset{\|\cdot\|_{x^*}}{\longrightarrow}} 0 \text{ and } z_{i,n}^* + v_{i,n}^* \underset{n \longmapsto +\infty}{\overset{\|\cdot\|_{Z_i^*}}{\longrightarrow}} 0, \\ (y^* \circ f)(x_n) - \langle x_n^*, x_n - \bar{x} \rangle \underset{n \mapsto +\infty}{\longrightarrow} (y^* \circ f)(\bar{x}), \\ \langle x_{i,n}^*, x_{i,n} - \bar{x} \rangle + \langle z_{i,n}^*, z_{i,n} - h_i(\bar{x}) \rangle \underset{n \mapsto +\infty}{\longrightarrow} 0, \\ \langle r_n^*, r_n - \bar{x} \rangle \underset{n \mapsto +\infty}{\longrightarrow} 0, \langle v_{i,n}^*, v_{i,n} - h_i(\bar{x}) \rangle \underset{n \mapsto +\infty}{\longrightarrow} 0. \end{cases}$$

It is easy to see that the condition $v_{i,n}^* \in N_{\eta_i}(v_{i,n}, -Z_i^+)$ is equivalent to $v_{i,n}^* \in (Z_i^+)^*$ and $\langle v_{i,n}^*, -v_{i,n} \rangle \leq \eta_i$. Now the proof is complete. $\square$

## 5 Applications to a constrained multiobjective fractional programming problems

In this section, by applying the previous results we present, without any constraint qualification, sequential weak approximate efficiency optimality conditions for the following multiobjective fractional programming problem

$$(P_2)\qquad \begin{cases} \min\left\{\dfrac{f_1(x)}{g_1(x)},\ldots,\dfrac{f_s(x)}{g_s(x)}\right\} \\ x\in C, \\ h_1(x)\in -Z_1^+, \\ \vdots \\ h_p(x)\in -Z_p^+, \end{cases}$$

where $f_j$, -$g_j$: $X\longrightarrow\mathbb{R}$ $(j=1,\ldots,s)$ are convex and lower semicontinuous functions and $h_i: X\longrightarrow Z_i\cup\{+\infty_{Z_i}\}$ $(i=1,\ldots,p)$ are proper, $Z_i^+$-convex and $Z_i^+$-epi-closed mappings. Also, $C$ and $Z_i^+$ $(i=1,\ldots,p)$ are nonempty closed convex. Moreover, we suppose that $f_j(x)\geq 0$ and $g_j(x)>0$ $(j=1,\ldots,s)$ for any $x\in\bigcap_{i=1}^p h_i^{-1}(-Z_i^+)\cap C$. The finite-dimensional space $Y:=\mathbb{R}^s$ is equipped with its natural order induced by the positive cone $Y:=\mathbb{R}^s_+=\{(y_1,\ldots,y_s)\in\mathbb{R}^s,\ y_j\geq 0,\ \text{for all } j=1,\ldots,s\}$. The following notations will be used in what follows:

$$\begin{aligned} &\varepsilon:=(\varepsilon_1,\ldots,\varepsilon_s)\in\mathbb{R}^s_+,\\ &\overline{\varepsilon}:=(\varepsilon_1 g_1(\bar{x}),\ldots,\varepsilon_s g_s(\bar{x})),\\ &\nu_j:=\tfrac{f_j(\bar{x})}{g_j(\bar{x})}-\varepsilon_j\geq 0 \text{ and } \nu=(\nu_1,\ldots,\nu_s)\in\mathbb{R}^s_+. \end{aligned}$$

Now, we recall the definition of weakly $\varepsilon$-efficient solution of $(P_2)$, which can be found in [10].

**Definition 3.** A point $\bar{x}\in\bigcap_{i=1}^p h_i^{-1}(-Z_i^+)\cap C$ is said to be weakly $\varepsilon$-efficient solution of $(P_2)$ if there does not exist $x\in\bigcap_{i=1}^p h_i^{-1}(-Z_i^+)\cap C$ such that

$$\frac{f_j(x)}{g_j(x)}<\frac{f_j(\bar{x})}{g_j(\bar{x})}-\varepsilon_j \qquad (\text{ for all } j=1,\ldots,s).$$

By using parametric approach of Dinkelbach [4], we can transform the problem $(P_2)$ into a vector convex nonfractional programming problem defined as follows:

$$(P_\nu) \qquad \begin{cases} \min F(x) \\ x \in C, \\ h_1(x) \in -Z_1^+, \\ \vdots \\ h_p(x) \in -Z_p^+, \end{cases}$$

where $F : X \longrightarrow \mathbb{R}^s$ is defined for any $x \in X$ by

$$F(x) := (f_1(x) - \nu_1 g_1(x), \ldots, f_s(x) - \nu_s g_s(x)).$$

**Lemma 3.** ([10]) A point $\bar{x} \in \bigcap_{i=1}^p h_i^{-1}(-Z_i^+) \cap C$ is a weakly $\varepsilon$-efficient solution of $(P_2)$ if and only if $\bar{x}$ is a weakly $\bar{\varepsilon}$-efficient solution of $(P_\nu)$.

**Theorem 5.** Let $\bar{x} \in \bigcap_{i=1}^p h_i^{-1}(-Z_i^+) \cap C$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_s) \in \mathbb{R}_+^s$ and $\nu_j = \frac{f_j(\bar{x})}{g_j(\bar{x})} - \varepsilon_j \geq 0$, $j = 1, \ldots, s$. Suppose that $C, Z_1^+, \ldots, Z_p^+$ are all closed convex cones. Then $\bar{x}$ is a weakly $\varepsilon$-efficient solution of $(P_2)$ if and only if there exist $y^* = (y_1, \ldots, y_s) \in \mathbb{R}_+^s \setminus \{0\}$, $x_n \in X$, $r_n \in C$, $(x_{i,n}, z_{i,n}) \in \text{epi} h_i$, $v_{i,n} \in -Z_i^+$,

$x_n^*, r_n^*, x_{i,n}^* \in X^*$, $z_{i,n}^*, v_{i,n}^* \in Z_i^*$ and $\alpha_1, \alpha_2, \lambda_i, \gamma_i, \eta_i \geq 0$, $i = 1, \ldots, p$, satisfying

$$\begin{cases} \alpha_1 + \alpha_2 + \sum_{i=1}^p \lambda_i + \gamma_i + \eta_i = \sum_{i=1}^s y_i \varepsilon_i, \\ \begin{cases} x_n \xrightarrow[n\longmapsto+\infty]{\|\cdot\|_X} \bar{x}, \quad r_n \xrightarrow[n\longmapsto+\infty]{\|\cdot\|_X} \bar{x}, \quad x_{i,n} \xrightarrow[n\longmapsto+\infty]{\|\cdot\|_X} \bar{x}, \\ z_{i,n} \xrightarrow[n\longmapsto+\infty]{\|\cdot\|_{Z_i}} h_i(\bar{x}), \quad v_{i,n} \xrightarrow[n\longmapsto+\infty]{\|\cdot\|_{Z_i}} h_i(\bar{x}), \end{cases} \\ x_n^* \in \partial_{\alpha_1}\left(\sum_{j=1}^s y_j\big(f_j + \nu_j(-g_j)\big)\right)(x_n), r_n^* \in N_{\alpha_2}(r_n, C), \\ x_{i,n}^* \in \partial_{\lambda_i}(-z_{i,n}^* \circ h_i)(x_{i,n}), v_{i,n}^* \in (Z_i^+)^* \text{ and } \langle v_{i,n}^*, -v_{i,n}\rangle \leq \eta_i, \\ -z_{i,n}^* \in (Z_i^+)^*, \langle z_{i,n}^*, h_i(x_{i,n}) - z_{i,n}\rangle \leq \gamma_i, \end{cases}$$

and

$$\begin{cases} x_n^* + r_n^* + \sum_{i=1}^{p} x_{i,n}^* \overset{\|\cdot\|_{x^*}}{\underset{n\longmapsto+\infty}{\longrightarrow}} 0 \text{ and } z_{i,n}^* + v_{i,n}^* \overset{\|\cdot\|_{Z_i^*}}{\underset{n\longmapsto+\infty}{\longrightarrow}} 0, \\ \left(\sum_{j=1}^{s} y_j\big(f_j + \nu_j(-g_j)\big)\right)(x_n) - \langle x_n^*, x_n - \bar{x}\rangle \\ \qquad\qquad \underset{n\mapsto+\infty}{\longrightarrow} \left(\sum_{j=1}^{s} y_j\big(f_j + \nu_j(-g_j)\big)\right)(\bar{x}), \\ \langle x_{i,n}^*, x_{i,n} - \bar{x}\rangle + \langle z_{i,n}^*, z_{i,n} - h_i(\bar{x})\rangle \underset{n\mapsto+\infty}{\longrightarrow} 0, \\ \langle r_n^*, r_n - \bar{x}\rangle \underset{n\mapsto+\infty}{\longrightarrow} 0, \langle v_{i,n}^*, v_{i,n} - h_i(\bar{x})\rangle \underset{n\mapsto+\infty}{\longrightarrow} 0. \end{cases}$$

*Proof.* According to Lemma 3, $\bar{x}$ is a weakly $\varepsilon$-efficient solution of $(P_2)$ if and only if $\bar{x}$ is a weakly $\bar{\varepsilon}$-efficient solution of $(P_v)$. The mappings $f := F$ and $h_i$ satisfy together all the assumptions of Theorem 4, then it follows that there exist $y^* = (y_1, \ldots, y_s) \in (\mathbb{R}_+^s)^* \setminus \{0\} = \mathbb{R}_+^s \setminus \{0\}$, $x_n \in \mathrm{dom}F = X$, $r_n \in C$, $(x_{i,n}, z_{i,n}) \in \mathrm{epi}h_i$, $v_{i,n} \in -Z_i^+$,

$x_n^*, r_n^*, x_{i,n}^* \in X^*$, $z_{i,n}^*, v_{i,n}^* \in Z_i^*$ and $\alpha_1, \alpha_2, \lambda_i, \gamma_i, \eta_i \geq 0$, $i = 1, \ldots, p$, satisfying

$$\begin{cases} \alpha_1 + \alpha_2 + \sum_{i=1}^{p} \lambda_i + \gamma_i + \eta_i = \sum_{i=1}^{s} y_i \varepsilon_i, \\ \begin{cases} x_n \overset{\|\cdot\|_X}{\underset{n\longmapsto+\infty}{\longrightarrow}} \bar{x}, \quad r_n \overset{\|\cdot\|_X}{\underset{n\longmapsto+\infty}{\longrightarrow}} \bar{x}, \quad x_{i,n} \overset{\|\cdot\|_X}{\underset{n\longmapsto+\infty}{\longrightarrow}} \bar{x}, \\ z_{i,n} \overset{\|\cdot\|_{Z_i}}{\underset{n\longmapsto+\infty}{\longrightarrow}} h_i(\bar{x}), \quad v_{i,n} \overset{\|\cdot\|_{Z_i}}{\underset{n\longmapsto+\infty}{\longrightarrow}} h_i(\bar{x}), \end{cases} \\ x_n^* \in \partial_{\alpha_1}\left(\sum_{j=1}^{s} y_j\big(f_j + \nu_j(-g_j)\big)\right)(x_n), r_n^* \in N_{\alpha_2}(r_n, C), \\ x_{i,n}^* \in \partial_{\lambda_i}(-z_{i,n}^* \circ h_i)(x_{i,n}), v_{i,n}^* \in (Z_i^+)^* \text{ and } \langle v_{i,n}^*, -v_{i,n}\rangle \leq \eta_i, \\ -z_{i,n}^* \in (Z_i^+)^*, \langle z_{i,n}^*, h_i(x_{i,n}) - z_{i,n}\rangle \leq \gamma_i, \end{cases}$$

and

$$\begin{cases} x_n^* + r_n^* + \sum_{i=1}^{p} x_{i,n}^* \xrightarrow[n\longmapsto+\infty]{\|\cdot\|_{x^*}} 0 \text{ and } z_{i,n}^* + v_{i,n}^* \xrightarrow[n\longmapsto+\infty]{\|\cdot\|_{Z_i^*}} 0, \\ \left(\sum_{j=1}^{s} y_j\big(f_j + \nu_j(-g_j)\big)\right)(x_n) - \langle x_n^*, x_n - \bar{x}\rangle \\ \qquad\qquad \underset{n\mapsto+\infty}{\longrightarrow} \left(\sum_{j=1}^{s} y_j\big(f_j + \nu_j(-g_j)\big)\right)(\bar{x}), \\ \langle x_{i,n}^*, x_{i,n} - \bar{x}\rangle + \langle z_{i,n}^*, z_{i,n} - h_i(\bar{x})\rangle \underset{n\mapsto+\infty}{\longrightarrow} 0, \\ \langle r_n^*, r_n - \bar{x}\rangle \underset{n\mapsto+\infty}{\longrightarrow} 0, \langle v_{i,n}^*, v_{i,n} - h_i(\bar{x})\rangle \underset{n\mapsto+\infty}{\longrightarrow} 0. \end{cases}$$

□

In what follows, we present an important sub-class of such problems showing the applicability of the suggested conditions of Theorem 5. This is the class of multiobjective linear fractional programming problems, which have significant application in different real life areas such as production planning, financial sector, health care and all engineering fields. This type of problems is modeled as follows (see [12]):

$$(P_3) \qquad \begin{cases} \min\{K_1(x), \dots, K_s(x)\} \\ x \in \mathbb{R}_+^n, \\ \mathcal{A}x - b \in -\mathbb{R}_+^p, \end{cases}$$

where $\mathcal{A}$ is a matrix $(p \times n)$, $b \in \mathbb{R}^p$, $K_j(x) = \frac{f_j(x)}{f_j(x)} = \frac{a_j^t x + b_j}{c_j^t x + d_j}$ $(a_j, c_j, x \in \mathbb{R}^n, b_j, d_j \in \mathbb{R}, j = 1, \dots, s)$, and $t$ denotes the transpose operation. Note that $\mathcal{S} = \{x \in \mathbb{R}_+^n, \mathcal{A}x - b \in -\mathbb{R}_+^p, b \in \mathbb{R}^p\}$ is the feasible set in decision space. We assume that for each feasible solution $x$, $f_j(x) \geq 0$ and $g_j(x) > 0$ $(j = 1, \dots, s)$.

By taking $C := \mathbb{R}_+^n$ and $h(x) := \mathcal{A}x - b$, we observe that all assumptions of Theorem 5 are satisfied. Then we deduce the following result.

**Theorem 6.** Let $\bar{x} \in \mathcal{S}$, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_s) \in \mathbb{R}_+^s$ and let $\nu_j = \frac{a_j^t \bar{x} + b_j}{c_j^t \bar{x} + d_j} - \varepsilon_j \geq 0$, $j = 1, \dots, s$. Then $\bar{x}$ is a weakly $\varepsilon$-efficient solution of $(P_3)$ if and only if there exist $y^* = (y_1, \ldots, y_s) \in \mathbb{R}^s_+ \setminus \{0\}$, $x_n \in \mathbb{R}^n$, $r_n \in \mathbb{R}^n_+$, $(w_n, z_n) \in \operatorname{epi}(\mathcal{A}(.)+b)$, $v_n \in -\mathbb{R}^p_+$,

$x_n^*, r_n^*, w_n^* \in \mathbb{R}^n$, $z_n^*, v_n^* \in \mathbb{R}^p$ and $\alpha_1, \alpha_2, \lambda, \gamma, \eta \geq 0$ satisfying

$$\begin{cases} \alpha_1 + \alpha_2 + \lambda + \gamma + \eta = \sum_{i=1}^{s} y_i \varepsilon_i, \\ \begin{cases} x_n \underset{n \longmapsto +\infty}{\overset{\|\cdot\|_{\mathbb{R}^n}}{\longrightarrow}} \bar{x}, \quad r_n \underset{n \longmapsto +\infty}{\overset{\|\cdot\|_{\mathbb{R}^n}}{\longrightarrow}} \bar{x}, \\ w_n \underset{n \longmapsto +\infty}{\overset{\|\cdot\|_{\mathbb{R}^n}}{\longrightarrow}} \bar{x}, \quad z_n \underset{n \longmapsto +\infty}{\overset{\|\cdot\|_{\mathbb{R}^p}}{\longrightarrow}} \mathcal{A}\bar{x} + b, \end{cases} \\ x_n^* \in \partial_{\alpha_1}\left(\sum_{j=1}^{s} y_j\big(f_j + \nu_j(-g_j)\big)\right)(x_n), r_n^* \in N_{\alpha_2}(r_n, \mathbb{R}^n_+), \\ w_n^* \in \partial_\lambda(-z_n^* \circ (\mathcal{A}(.) + b))(w_n), v_n^* \in \mathbb{R}^p_+ \text{ and } \langle v_n^*, -v_n\rangle \leq \eta, \\ -z_n^* \in \mathbb{R}^p_+, \langle z_n^*, \mathcal{A}w_n + b - z_n\rangle \leq \gamma, \end{cases}$$

and

$$\begin{cases} x_n^* + r_n^* + w_n^* \underset{n \longmapsto +\infty}{\overset{\|\cdot\|_{\mathbb{R}^n}}{\longrightarrow}} 0 \text{ and } z_n^* + v_n^* \underset{n \longmapsto +\infty}{\overset{\|\cdot\|_{\mathbb{R}^p}}{\longrightarrow}} 0, \\ \left(\sum_{j=1}^{s} y_j\big(f_j + \nu_j(-g_j)\big)\right)(x_n) - \langle x_n^*, x_n - \bar{x}\rangle \\ \qquad\qquad \underset{n \mapsto +\infty}{\longrightarrow} \left(\sum_{j=1}^{s} y_j\big(f_j + \nu_j(-g_j)\big)\right)(\bar{x}), \\ \langle w_n^*, w_n - \bar{x}\rangle + \langle z_n^*, z_n - \mathcal{A}\bar{x} + b\rangle \underset{n \mapsto +\infty}{\longrightarrow} 0, \\ \langle r_n^*, r_n - \bar{x}\rangle \underset{n \mapsto +\infty}{\longrightarrow} 0, \langle v_n^*, v_n - \mathcal{A}\bar{x} + b\rangle \underset{n \mapsto +\infty}{\longrightarrow} 0. \end{cases}$$

Next, we close this section by presenting an example showing that the Moreau–Rockafellar and Attouch–Brézis constraint qualifications (see [6]) fail and also the sequential $\varepsilon$-optimality conditions of Theorem 5 hold.

**Example 1.** Let us consider the following multiobjective fractional programming problem:

$$(Q)\qquad \begin{cases} \min f(x,y), \\ (x,y)\in C, \\ \sqrt{x^2+y^2}-y\leq 0, \\ (-x,-x)\leq_{\mathbb{R}_2^+}(0,0), \end{cases}$$

where $f:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ is defined by

$$f(x,y):=\left(\frac{x+1}{2y},\frac{-x}{2(y+1)}\right) \text{ and } C:=\{0\}\times[1,2].$$

Let $f_1(x,y):=\frac{x+1}{2}$, $f_2(x,y):=\frac{-x}{2}$, $g_1(x,y):=y$, $g_2(x,y):=y+1$, $h_1(x,y):=\sqrt{x^2+y^2}-y$ and $h_2(x,y):=(-x,-x)$. Let $\varepsilon:=(\varepsilon_1,\varepsilon_2)=(\frac{1}{2},0)$, $(\overline{x},\overline{y}):=(0,1)$ be a feasible point, $\nu_1=\frac{f_1(\overline{x},\overline{y})}{g_1(\overline{x},\overline{y})}-\varepsilon_1=0$, $\nu_2=\frac{f_2(\overline{x},\overline{y})}{g_2(\overline{x},\overline{y})}-\varepsilon_2=0$ and $\overline{\varepsilon}=(\varepsilon_1g_1(\overline{x},\overline{y}),\varepsilon_2g_2(\overline{x},\overline{y}))=(\frac{1}{2},0)$. It is easy to check that $(\overline{x},\overline{y})$ is a $\varepsilon$-weakly efficient solution of $(Q)$. Indeed, putt $h:=(h_1,h_2)$. Then we observe that $h(C)\cap(-\text{int}\mathbb{R}^3_+)=\emptyset$ and that $\mathbb{R}_+[\mathbb{R}^3_++h(C)]=\mathbb{R}^3_+$, which is not a subspace of $\mathbb{R}^3_+$. Hence, Moreau–Rockafellar and Attouch–Brézis qualification conditions are not satisfied.

For each $n\in\mathbb{N}$, we take $\alpha_1=\frac{1}{2}$, $\alpha_2=0$, $\lambda_i=\gamma_i=\eta_i=0$ $(i=1,2)$, $x_n=(0,1)$, $x_{i,n}=(0,1)$ $(i=1,2)$, $r_n=(0,1)$, $z_{1,n}=v_{1,n}=0$, $z_{2,n}=v_{2,n}=(0,0)$ . Let $x_n^*=(0,0)$, $x_{i,n}^*=(0,0)$ $(i=1,2)$, $r_n^*=(0,0)$, $z_{1,n}^*=0\in-\mathbb{R}_+$, $v_{1,n}^*=0\in\mathbb{R}_+$, $z_{2,n}^*=(0,0)\in-\mathbb{R}^2_+$, $v_{2,n}^*=(0,0)\in\mathbb{R}^2_+$, and $(y_1,y_2)=(1,1)\in\mathbb{R}^2_+\setminus\{(0,0)\}$. We have

$$\begin{cases} \alpha_1+\alpha_2+\sum_{i=1}^{2}\lambda_i+\gamma_i+\eta_i=\frac{1}{2}=\sum_{i=1}^{2}y_i\varepsilon_ig_i(0,1), \\ x_n \underset{n\mapsto+\infty}{\longrightarrow}(0,1),\quad r_n \underset{n\mapsto+\infty}{\longrightarrow}(0,1),\quad x_{i,n} \underset{n\mapsto+\infty}{\longrightarrow}(0,1), \\ z_{1,n} \underset{n\mapsto+\infty}{\longrightarrow} h_1(0,1)=0,\quad v_{1,n} \underset{n\mapsto+\infty}{\longrightarrow} h_1(0,1)=0, \\ z_{2,n} \underset{n\mapsto+\infty}{\longrightarrow} h_2(0,1)=(0,0),\quad v_{2,n} \underset{n\mapsto+\infty}{\longrightarrow} h_2(0,1)=(0,0). \end{cases}$$

It is immediate to see that $x_n^*\in\partial_{\alpha_1}\left(\sum_{i=1}^{2}y_i\big(f_i+\nu_i(-g_i)\big)\right)(x_n)=\{(0,0)\}$. Moreover, $r_n^*\in N_{\alpha_2}(r_n,C)$, $\langle v_{i,n}^*,-v_{i,n}\rangle=0\leq\eta_i=0$ $(i=1,2)$, $x_{i,n}^*\in\partial_{\lambda_i}(-z_{i,n}^*\circ h_i)(x_{i,n})=\{(0,0)\}$ $(i=1,2)$, $\langle z_{i,n}^*,h_i(x_{i,n})-z_{i,n}\rangle=0\leq\gamma_i=0$ $(i=1,2)$ and

$$\begin{cases} x_n^* + r_n^* + \sum_{i=1}^{2} x_{i,n}^* \underset{n \mapsto +\infty}{\longrightarrow} (0,0) \text{ and } z_{i,n}^* + v_{i,n}^* \underset{n \mapsto +\infty}{\longrightarrow} (0,0), \\ \left( \sum_{i=1}^{2} y_i \big( f_i + \nu_i(-g_i) \big) \right)(x_n) - \langle x_n^*, x_n - (0,0) \rangle = \frac{1}{2} \\ \qquad \underset{n \mapsto +\infty}{\longrightarrow} \left( \sum_{i=1}^{2} y_i \big( f_i + \nu_i(-g_i) \big) \right)(0,1) = \frac{1}{2}, \\ \langle x_{i,n}^*, x_{i,n} - (0,1) \rangle + \langle z_{i,n}^*, z_{i,n} - h_i(0,1) \rangle = 0 \underset{n \mapsto +\infty}{\longrightarrow} 0, \\ \langle r_n^*, r_n - (0,1) \rangle \underset{n \mapsto +\infty}{\longrightarrow} 0, \langle v_{i,n}^*, v_{i,n} - h_i(0,1) \rangle = 0 \underset{n \mapsto +\infty}{\longrightarrow} 0. \end{cases}$$

Then, by Theorem 5 the point $(\overline{x}, \overline{y})$ is a weakly approximate solution for the problem $(Q)$.

## 6 Conclusion

It is well known that the constraint qualifications are required to obtain approximate optimality conditions but sometimes these constraint qualifications become very difficult to compute. In this work, we focused to establish sequential approximate optimality conditions without any constraint qualification for a constrained convex vector minimization problem $(P_1)$ via scalarization process in terms of the approximate subdifferentials of the associated functions. As an application, we derive sequential weakly approximate optimality conditions to a constrained multiobjective fractional programming problem $(P_2)$.

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