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Research Article

Explicit collocation algorithm for the nonlinear fractional Duffing equation via third-kind Chebyshev polynomials

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Abstract

Herein, we propose an accurate algorithm to approximate the solution of the nonlinear fractional-order Duffing equation (NFDE). The algorithm is

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based on using shifted Chebyshev polynomials of the third-kind as basis functions and the spectral collocation method as a solver. We study the error analysis of the method in-depth, and we exhibit some numerical test problems to check the applicability of the method. Also, we compare it with other existing techniques to show the superiority of our proposed numerical scheme. Our results show that the method employed provides a useful tool to simulate the solution of the NFDE. The main advantages of the proposed method are that it does not require a huge number of retained modes, simply a few terms, and does not exhaust the machine used to render the codes.

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1 Introduction

Within the fields of mathematics, physics, and engineering, joint efforts have produced a new instrument for characterizing difficult problems: the area of fractional calculus. This field of study allows scientists to accurately handle real-world issues in a variety of domains. Fractional-order differential equations can characterize application issues in physics, biology, mechanics, medicine, astronomy, engineering, and chemistry. Due to the importance of finding solutions to these equations, a number of numerical techniques have been used, including the Laplace transform approach, Chebyshev collocation, domain decomposition technique, differential transformation technique, spectral Legendre method, Tau procedure, variational iteration method, and the wavelet method, which are all numerical techniques used in solving differential equations and related problems. When it comes to breaking down fractional-order problems into systems of algebraic equations, the spectral Tau technique and the operational matrix (OM) method stand out for their simplicity, speed of convergence, and computational ease. Selecting appropriate basis functions is essential in order to use spectral approaches to achieve approximate solutions. Because of their useful features and trigonometric representation, the Chebyshev polynomials of the first kind are one class of functions that have shown to be efficient in this respect. Those who are interested in learning more about Chebyshev polynomials might consult [12]. It is known that Chebyshev polynomials have been used more often in spectral techniques to solve different kinds of differential problems; recent works like [2, 5, 23, 14, 20] attest to this.

A specific class of problems in nonlinear dynamics is the one-dimensional Duffing equation, capturing the behavior of a simple oscillator with cubic nonlinearity. In the context of fractional calculus, the nonlinear fractional-order Duffing equation (NFDE) takes the form:

$$D_t^{\mu}\chi(t) + a\,D_t^{\beta}\chi(t) + b\,\chi(t) + c\,\chi^2(t) + d\,\chi^5(t) + e\,\chi^7(t) = f(t), \tag{1}$$

subject to the simple initial conditions $\chi(0) = \chi_0, \quad \chi'(0) = \chi_1$, where μ is in the range $(1, 2]$ and β in the range $(0, 1]$.

Spectral techniques have expanded to solve fractional-order differential equations in recent years [9, 10, 13, 27, 30, 31, 4, 3, 1], providing a nuanced description of physical phenomena through non-integer derivatives. The inherent characteristics of Chebyshev polynomials provide spectral approaches for such equations, improving numerical efficiency and enabling the reliable approximation of fractional derivatives. Spectral collocation is one of these techniques that stands out for its excellent accuracy and convergence. A very accurate and convergent computational approach for solving nonlinear Volterra integrodifferential equations of high order was presented by the authors in [24]. The spectral collocation approach was used by Thirumalai et al. [21] to solve nonlinear high-order pantograph equations.

As a fundamental spectral approach for solving partial differential equations (PDEs), the Galerkin method is emphasized in [26, 28, 32]. With this approach, the unknown function is expressed as a PDE using a series expansion, and the derivatives of the PDE are then approximated by using the collocation technique [29]. When working with linear or nonlinear PDEs, particularly those that have beginning or boundary conditions, this kind of method is quite helpful.

The NFDE, a well-established nonlinear equation, serves as a potent tool for addressing significant practical phenomena in applied science [19]. This equation gained prominence in the mid-twentieth century, particularly in the study of electronics [18]. Functioning as the simplest oscillator, it exhibits catastrophic increases in amplitude and phase when the frequency of the forcing term varies gradually. The Duffing equation finds wide application in diverse areas, including brain modeling [33], passive islanding detection in inverter-based distributed generation units using Duffing oscillators [22], propagation of electromagnetic pulses in nonlinear media, radar systems, digital communication [7], nonlinear electrical circuits [17], and other fields. Numerous attempts have been made to numerically solve the Duffing equation [15, 11].

The suggested approach presents a novel use of shifted Chebyshev polynomials of the third-kind (SCP3K) in collocation methods, providing an accurate, efficient and fast numerical scheme for spectrally solving the NFDE. It addresses important difficulties in computational complexity and stability, employing few computer resources and greatly improving solution accuracy when compared to existing approaches. To the best of our knowledge, the main contribution and novelty of this paper can be listed in the following points:

- A new theoretical background to the SCP3K is presented.
- Establishing a new OM of fractional derivatives for SCP3K. This matrix is considered an important tool for treating NFDE.
- The study of the error bound is new.

The advantages of the presented approach are as follows:

- By choosing SCP3K as basis functions, a few terms of the retained modes make it possible to produce approximations with excellent precision.
- Less calculation is required to obtain the desired approximate solution.

The following is how this job is organized: In Section 2, the definitions and mathematical equations of SCP3K are presented; OM of fractional order is constructed using SCP3K along with a brief introduction to the tools of fractional calculus. The problem formulation and the proposed numerical solution utilizing derived OM are given in Section 3. For the proposed Chebyshev expansion, Section 4 provides error estimation in L_∞ space. Test cases in Section 5 illustrate the precision and effectiveness of the suggested approach. In conclusion, Section 6 provides some final thoughts.

2 Preliminaries and essential relations

2.1 The fractional derivative in the Caputo sense

Definition 1. [16] The Caputo fractional derivative of order s is defined as

$$D^s u(t) = \frac{1}{\Gamma(m-s)} \frac{d}{dt} \int_0^t (t-y)^{m-s-1} u^{(m)}(y)\, dy, \quad s > 0, \quad t > 0, \tag{2}$$

where $m-1 \leqslant s < m, \quad m \in \mathbb{N}$.

The following properties are satisfied by the operator D^s for $m-1 \leqslant s < m, \quad m \in \mathbb{N}$,

$$D^s c = 0, \ (c \text{is a constant}) \tag{3}$$

$$D^s\, t^r = \begin{cases} 0, & \text{if } r \in \mathbb{N}_0 \quad \textit{and} \quad r < \lceil s \rceil, \\ \frac{r!}{\Gamma(r-s+1)}\, t^{r-s}, & \text{if } r \in \mathbb{N}_0 \quad \textit{and} \quad r \geq \lceil s \rceil, \end{cases} \tag{4}$$

where $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, and the notation $\lceil s \rceil$ denotes the ceiling function.

2.2 A account on the SCP3K

The SCP3K $\mathcal{V}_{S,\ell}(t)$ are special ones of the Jacobi polynomials that can be defined as [25]

$$\mathcal{V}_{S,\ell}(t) = \frac{2^{2\,\ell}}{\binom{2\,\ell}{\ell}}\, P_\ell^{\left(-\frac{1}{2}, \frac{1}{2}\right)}(2\,t-1). \tag{5}$$

The orthogonality relation of $\mathcal{V}_{S,\ell}(t)$ is given by

$$\int_0^1 \mathcal{V}_{S,\ell}(t)\,\mathcal{V}_{S,j}(t)\,\omega(t)dt = \frac{\pi}{2}\,\delta_{\ell,j}, \tag{6}$$

where $\omega(t) = \left(\frac{t}{1-t}\right)^{\frac{1}{2}}$ and $\delta_{\ell,j}$ is the Kronecker delta.

The analytic representation of $\mathcal{V}_{S,\ell}(t)$ can be represented as

$$\mathcal{V}_{S,m}(t) = \sum_{\ell=0}^{m} \frac{(-1)^{\ell}\,(2\,m+1)\,2^{2\,m-2\,\ell}(-\ell+2m)!}{\ell!\,(-2\,\ell+2\,m+1)!}\,t^{m-\ell}, \tag{7}$$

or

$$\mathcal{V}_{S,m}(t) = \sum_{\ell=0}^{m} \frac{2^{2\,\ell}\,(2\,m+1)\,(-1)^{m-\ell}\,(\ell+m)!}{(2\,\ell+1)!\,(-\ell+m)!}\,t^{\ell}. \tag{8}$$

Moreover, the inversion formula of $\mathcal{V}_{S,\ell}(t)$ is

$$t^{r} = \frac{\Gamma(2\,r+2)}{2^{2\,r}} \sum_{\ell=0}^{r} \frac{1}{(-\ell+r)!\,(\ell+r+1)!}\,\mathcal{V}_{S,\ell}(t). \tag{9}$$

Theorem 1. For $0 < \beta < 1$, we have

$$D^{\beta}\mathcal{V}_{S,\ell}(t) = \boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{V}}, \tag{10}$$

where
$\boldsymbol{\mathcal{V}} = [\mathcal{V}_{S,0}(t), \mathcal{V}_{S,1}(t), \dots, \mathcal{V}_{S,M}(t)]^{T}$. Also, $\boldsymbol{\mathcal{A}} = (\mathcal{A}_{\ell,k})$ is an OM of ordinary derivatives of dimension $(M+1)\times(M+1)$, and the entries of this matrix can be written in the form

$$\mathcal{A}_{\ell,k} = \sum_{n=1}^{k} \frac{4^{n}\,(2\,\ell+1)\,(2\,k+1)(-1)^{-n+\ell+k}\,(n+k)!\,\Gamma(n+1)\,\Gamma\left(n-\beta+\frac{3}{2}\right)}{\Gamma(2\,n+2)\,\Gamma(1-n+k)\,\Gamma\left(n-\beta+1\right)} \times {}_{3}\tilde{F}_{2}\left(\left.\begin{matrix}-\ell, \ell+1, -\beta+n+\frac{3}{2}\\ \frac{3}{2}, -\beta+n+2\end{matrix}\right|1\right). \tag{11}$$

Proof. Using Definition 1 jointly with the analytic power form of $\mathcal{V}_{S,n}(t)$, we have

$$D^{\beta}\mathcal{V}_{S,\kappa}(t) = \sum_{n=1}^{\kappa} \frac{4^{n}\,(2\,\kappa+1)\,(-1)^{\kappa-n}\,(\kappa+n)!\,\Gamma(n+1))}{(2\,n+1)!\,(-n+\kappa)!\,\Gamma\left(n-\beta+1\right)}\,t^{n-\beta}. \tag{12}$$

Now, $t^{n-\beta}$ can be approximated as

$$t^{n-\beta} = \sum_{\ell=0}^{M} c_\ell \, \mathcal{V}_{S,\ell}(t), \tag{13}$$

where

$$c_\ell = \frac{2}{\pi} \int_0^1 t^{n-\beta} \, \mathcal{V}_{S,\ell}(t) \, \omega(t) \, dt. \tag{14}$$

Integrating the RHS of the last equation, we have

$$c_\ell = \sum_{s=0}^{\ell} \frac{2\,(2\,\ell+1)\,2^{2\,s}\,(-1)^{\ell-s}\,(\ell+s)!\,\Gamma\left(n+s-\beta+\frac{3}{2}\right)}{\sqrt{\pi}\,(2\,s+1)!\,(\ell-s)!\,(i+s-\beta+1)!}. \tag{15}$$

Equation (15) can be written as

$$c_\ell = (-1)^\ell\,(2\,\ell+1)\,\Gamma\left(n-\beta+\frac{3}{2}\right)\,{}_3\tilde{F}_2\left(\left.\begin{matrix}-\ell, \ell+1, -\beta+n+\frac{3}{2}\\ \frac{3}{2}, -\beta+n+2\end{matrix}\,\right|\,1\right). \tag{16}$$

Inserting (16) into (13) yields

$$\begin{aligned} t^{n-\beta} = &\sum_{\ell=0}^{M} \Bigg((-1)^\ell\,(2\,\ell+1)\,\Gamma\left(n-\beta+\frac{3}{2}\right) \\ &\times {}_3\tilde{F}_2\left(\left.\begin{matrix}-\ell, \ell+1, -\beta+n+\frac{3}{2}\\ \frac{3}{2}, -\beta+n+2\end{matrix}\,\right|\,1\right)\Bigg)\,\mathcal{V}_{S,\ell}(t). \end{aligned} \tag{17}$$

Plugging (17) into (12), we obtain

$$D^\beta \mathcal{V}_{S,\kappa}(t) = \sum_{\ell=0}^{M} \mathcal{A}_{\ell,\kappa}\,\mathcal{V}_{S,\ell}(t) = \boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{V}},$$

where

$$\begin{aligned} \mathcal{A}_{\ell,\kappa} = &\sum_{n=1}^{\kappa} \frac{4^n\,(2\,\ell+1)\,(2\,\kappa+1)(-1)^{-n+\ell+\kappa}\,(n+\kappa)!\,\Gamma(n+1)\,\Gamma\left(n-\beta+\frac{3}{2}\right)}{\Gamma(2\,n+2)\,\Gamma(1-n+\kappa)\,\Gamma\left(n-\beta+1\right)} \\ &\times {}_3\tilde{F}_2\left(\left.\begin{matrix}-\ell, \ell+1, -\beta+n+\frac{3}{2}\\ \frac{3}{2}, -\beta+n+2\end{matrix}\,\right|\,1\right). \end{aligned} \tag{18}$$

Therefore, we conclude that

$$D^\beta \mathcal{V}_{S,\kappa}(t) = \boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{V}},$$

where $\boldsymbol{\mathcal{A}} = (\mathcal{A}_{\ell,\kappa})$ is an OM of derivatives of order $(M+1)\times(M+1)$. □

Theorem 2. For $1 < \mu < 2$, we have

$$D^{\mu}\mathcal{V}_{S,\ell}(t) = \boldsymbol{\mathcal{B}}\,\boldsymbol{\mathcal{V}}, \tag{19}$$

where $\boldsymbol{\mathcal{V}} = [\mathcal{V}_{S,0}(t), \mathcal{V}_{S,1}(t), \dots, \mathcal{V}_{S,M}(t)]^T$. Also, $\boldsymbol{\mathcal{B}} = (\mathcal{B}_{\ell,k})$ is the OM of derivatives with dimension $(M+1)\times(M+1)$, and the entries of this matrix can be written in the form

$$\mathcal{B}_{\ell,k} = \sum_{n=2}^{k} \frac{4^n\,(2\,\ell+1)\,(2\,k+1)(-1)^{-n+\ell+k}\,(n+k)!\,\Gamma(n+1)\,\Gamma\left(n-\mu+\frac{3}{2}\right)}{\Gamma(2\,n+2)\,\Gamma(1-n+k)\,\Gamma\left(n-\mu+1\right)} \times {}_3\tilde{F}_2\left(\begin{matrix} -\ell, \ell+1, -\mu+n+\frac{3}{2} \\ \frac{3}{2}, -\mu+n+2 \end{matrix}\middle|\, 1\right). \tag{20}$$

Proof. The proof of this theorem is similar to the proof of Theorem 1. □

3 Collocation algorithm for the NFDE

In this section, we consider the following NFDE [8]:

$$D_t^{\mu}\chi(t) + a\,D_t^{\beta}\chi(t) + b\,\chi(t) + c\,\chi^2(t) + d\,\chi^5(t) + e\,\chi^7(t) = f(t), \tag{21}$$

$$\mu \in]1,2], \quad \beta \in]0,1], \quad t \in [0,1],$$

subject to the conditions

$$\chi(0) = g_1, \quad \chi'(0) = g_2. \tag{22}$$

The following set $\{\mathcal{V}_{S,i}(t), i : 0,1,2,\dots\}$ forms an orthogonal basis of $L^2_{\omega(t)}(0,1)$. This means that for any given function $\chi(t) \in L^2_{\omega(t)}(0,1)$, one has

$$\chi(t) = \sum_{i=0}^{\infty} \lambda_i\,\mathcal{V}_{S,i}(t), \tag{23}$$

and approximated as

$$\chi(t) \approx \chi^M(t) = \sum_{i=0}^{M} \lambda_i\,\mathcal{V}_{S,i}(t) = \boldsymbol{\lambda}\,\boldsymbol{\mathcal{V}}, \tag{24}$$

where

$$\boldsymbol{\lambda} = [\lambda_0, \lambda_1, \dots, \lambda_M], \tag{25}$$

and

$$\boldsymbol{\mathcal{V}} = [\mathcal{V}_{S,0}(t), \mathcal{V}_{S,1}(t), \dots, \mathcal{V}_{S,M}(t)]^T. \tag{26}$$

By virtue of Theorems 1 and 2, the residual $\mathbf{Res}(t)$ of (21) is given by

$$\begin{aligned}\mathbf{Res}(t) =& D_t^{\mu}\chi^M(t) + a\, D_t^{\beta}\chi^M(t) + b\,\chi^M(t) + c\,\left(\chi^M(t)\right)^2 + d\,\left(\chi^M(t)\right)^5 \\ &+ e\,\left(\chi^M(t)\right)^7 - f(t) \\ =& \boldsymbol{\lambda}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{V}} + a\,\boldsymbol{\lambda}\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{V}} + b\,\boldsymbol{\lambda}\boldsymbol{\mathcal{V}} + c\,(\boldsymbol{\lambda}\boldsymbol{\mathcal{V}})^2 + d\,(\boldsymbol{\lambda}\boldsymbol{\mathcal{V}})^5 + e\,(\boldsymbol{\lambda}\boldsymbol{\mathcal{V}})^7 - f(t).\end{aligned} \tag{27}$$

Now, the application of the standard collocation method yields to the following $(M+1)$ algebraic system of equations in the unknown expansion coefficients b_i

$$\begin{aligned}&\mathbf{Res}(t_i) = 0, \quad i = 1, 2, ..., M-1, \\ &\boldsymbol{\lambda}\bar{\boldsymbol{\mathcal{V}}} = g_1, \quad \boldsymbol{\lambda}\hat{\boldsymbol{\mathcal{V}}} = g_2,\end{aligned} \tag{28}$$

where

$$\begin{aligned}\bar{\boldsymbol{\mathcal{V}}} &= [\mathcal{V}_{S,0}(0), \mathcal{V}_{S,1}(0), ..., \mathcal{V}_{S,M}(0)]^T, \\ \hat{\boldsymbol{\mathcal{V}}} &= [\mathcal{V}'_{S,0}(0), \mathcal{V}'_{S,1}(0), ..., \mathcal{V}'_{S,M}(0)]^T,\end{aligned} \tag{29}$$

and $\{t_i : i = 1, 2, \dots, M\}$ are the first M distinct roots of $\mathcal{V}_{S,M}(t)$. Therefore, the system in (28) can be solved to get λ_i with the aid of the well-known Newton's iterative method.

4 Error analysis

In this section, we study the error analysis of the numerical solution $\chi^M(t)$ to the exact solution $\chi(t)$ of (21) by imitating similar steps as in [29, 6] in L_∞- norm

$$L_\infty[0,1] = \{\chi : \|\chi\|_\infty = \max_{t\in[0,1]} |\chi| < \infty\}.$$

Consider the following space function:

$$L^M[0,1] = \text{span}\{\mathcal{V}_{S,i}(t) : i = 0, 1, \dots, M\}. \tag{30}$$

Assume that $\chi^M(t) \in L^M[0,1]$ is the best approximation of $\chi(t)$; then, by the definition of the best approximation, we have

$$||\chi(t) - \chi^M(t)||_\infty \leq ||\chi(t) - \mathfrak{u}^M(t)||_\infty, \quad \text{for all } \mathfrak{u}^M(t) \in L^M[0,1]. \tag{31}$$

It turns out that the previous inequality is also true if $\mathfrak{u}^M(t)$ denotes the interpolating polynomial for $\chi(t)$ at points t_i, where t_i are the roots of $\mathcal{V}_{S,i}(t)$. Now,

$$\chi(t) - \mathfrak{u}^M(t) = \frac{d^{M+1}\,\chi(s)}{d\,t^{M+1}\,(M+1)!}\prod_{i=0}^{M}(t - t_i), \tag{32}$$

where $s \in [0,1]$, and hence, one has

$$||\chi(t) - \mathfrak{u}^M(t)||_\infty \leq \max_{t\in[0,1]}\left|\frac{d^{M+1}\,\chi(s)}{d\,t^{M+1}}\right| \frac{||\prod_{i=0}^{M}(t - t_i)||_\infty}{(M+1)!}. \tag{33}$$

Since $\chi(t)$ is a smooth function on $[0,1]$, then there exist a constant n such that

$$\max_{t\in[0,1]}\left|\frac{d^{M+1}\,\chi(s)}{d\,t^{M+1}}\right| \leq n. \tag{34}$$

Now, our main aim is to minimize the factor $||\prod_{i=0}^{M}(t - \iota_i)||_\infty$.

Assuming the one-to-one mapping $t = \frac{1}{2}(t+1)$ between the intervals $[-1,1]$ and $[0,1]$ to deduce that

$$\begin{aligned}\min_{t_i\in[0,1]}\max_{t\in[0,1]}\left|\prod_{i=0}^{M}(t - t_i)\right| &= \min_{t_i\in[-1,1]}\max_{t\in[-1,1]}\left|\prod_{i=0}^{M}\frac{1}{2}\,(t - t_i)\right| \\ &= \left(\frac{1}{2}\right)^{M+1}\min_{t_i\in[-1,1]}\max_{t\in[-1,1]}\left|\prod_{i=0}^{M}(t - t_i)\right| \\ &= \left(\frac{1}{2}\right)^{M+1}\min_{t_i\in[-1,1]}\max_{t\in[-1,1]}\left|\frac{\mathcal{V}_{M+1}(t)}{\eta^M}\right|,\end{aligned} \tag{35}$$

where $\eta^M = 2^{M+1}$ is the leading coefficient of Chebyshev polynomials of the third kind $\mathcal{V}_{M+1}(t)$ and t_i are the roots of $\mathcal{V}_{M+1}(t)$.

It is known that

$$\max_{t\in[-1,1]}|\mathcal{V}_{M+1}(t)| = \mathcal{V}_{M+1}(1) = 1. \tag{36}$$

In virtue of inequality (34), equations (35), and (36), we get

$$||\chi(t) - \chi^M(t)||_\infty \leq \frac{n}{2^{2\,(M+1)}\,(M+1)!}. \tag{37}$$

The previous inequality gives us an upper bound estimation of the absolute error (AE). This completes the deducing of our error estimation.

5 Illustrative examples

To demonstrate the effectiveness and validity of the proposed algorithm, the procedure will be illustrated using several numerical examples. Let us define the following L_∞-error:

$$L_\infty = \max_{t\in[0,1]} \left|\chi(t) - \chi^M(t)\right|.$$

Test Problem 1. [8] Consider the following NFDE:

$$D_t^{\mu}\chi(t) + a\,D_t^{\beta}\chi(t) + b\,\chi(t) + c\,\chi^2(t) + d\,\chi^5(t) + e\,\chi^7(t) = \frac{1}{8}e^{-3\,t}\left(3 - 20\,e^{2\,t}\right), \tag{38}$$

subject to the conditions

$$\chi(0) = \frac{1}{2}, \quad \chi'(0) = \frac{-1}{2}, \tag{39}$$

where the exact solution is $\chi(t) = \frac{e^{-t}}{2}$ at $\mu = 2$, $\beta = 1$, $a = 4$, $b = -2$, $c = 3$, $d = 0$, $e = 0$.

Table 1 presents a comparison between our method at $M = 13$ and the method in [8] at $M = 12$, demonstrating the accuracy of our method. Also, Table 2 shows the L_∞-error at $\mu = 2$, $\beta = 1$. Figure 1 shows the AE when $\mu = 2$, $\beta = 1$ at different values of M. Figure 2 illustrates that the approximate solutions have smaller variations for values of μ and β near $\mu = 2$, $\beta = 1$ when $M = 12$. Figure 3 confirms that the method remains stable at $\mu = 2$, $\beta = 1$ for higher values of M, with any indication of numerical instability or divergence in error.

Table 1: Comparison of AE of Example 1 at $\mu = 2$, $\beta = 1$.

t	Method in [8] at $M = 12$	Our method at $M = 13$	Our CPU time
0.1	5.5511×10^{-17}	0	
0.2	0	0	
0.3	5.5511×10^{-17}	0	
0.4	1.1102×10^{-16}	5.55112×10^{-17}	
0.5	0	0	0.952
0.6	5.5511×10^{-17}	$5.55112\times^{-17}$	
0.7	2.7756×10^{-17}	0	
0.8	$8.3267\times^{-17}$	0	
0.9	2.7756×10^{-17}	0	

Table 2: The L_∞-error of Example 1 at $\mu = 2$, $\beta = 1$.

M	3	6	9	12
Error	2.54449×10^{-4}	3.85844×10^{-8}	1.06212×10^{-12}	5.55112×10^{-17}

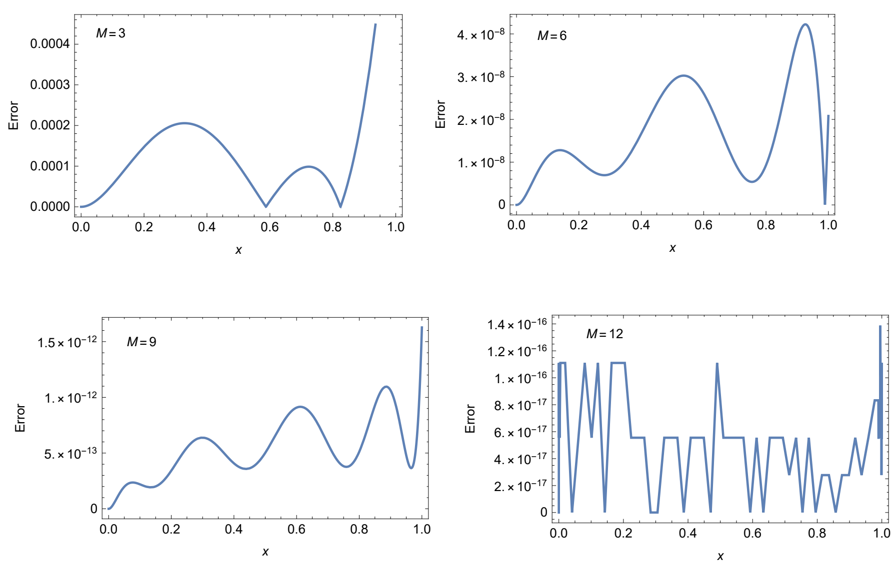

Figure 1: The AE of Example 1 at $\mu = 2$, $\beta = 1$.

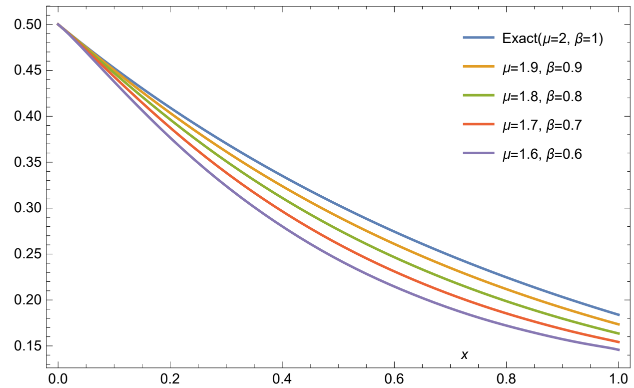

Figure 2: Different solutions of Example 1 at $M = 12$ and different values of μ, β.

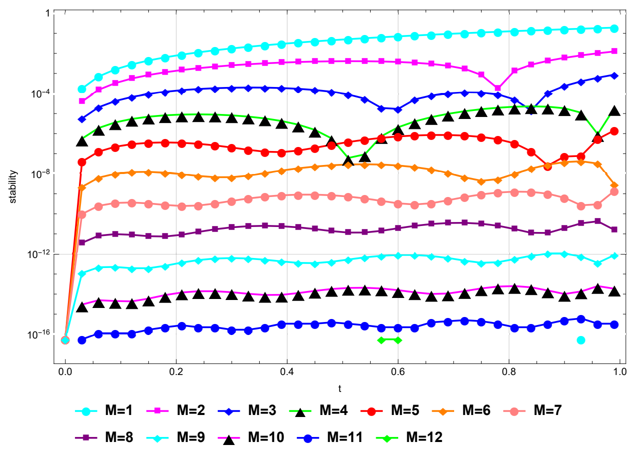

Figure 3: Stability $|\chi^{M+1}(t) - \chi^{M}(t)|$ at $\mu = 2$, $\beta = 1$ for Example 1.

Test Problem 2. [8] Consider the following NFDE:

$$D_t^{\mu}\chi(t)+a\,D_t^{\beta}\chi(t)+b\,\chi(t)+c\,\chi^2(t)+d\,\chi^5(t)+e\,\chi^7(t) = -2\,\sin(t)+\cos^5(t)+8\,\cos^3(t), \tag{40}$$

subject to the conditions

$$\chi(0) = 1, \quad \chi'(0) = 0, \tag{41}$$

where the exact solution is $\chi(t) = \cos(t)$, at $\mu = 2$, $\beta = 1$, $a = 2$, $b = 1$, $c = 8$, $d = 1$, $e = 0$.

Table 3 presents a comparison between our method at $M = 13$ and the method in [8] at $M = 11$, showing the accuracy of our method. Figure 4 shows AE when $\mu = 2$, $\beta = 1$ at different values of M. Figure 5 illustrates that the approximate solutions have smaller variations for values of μ and β near $\mu = 2$, $\beta = 1$ when $M = 12$. Figure 6 confirms that the method remains stable at $\mu = 2$, $\beta = 1$ for higher values of M, with no indication of numerical instability or divergence in error.

Table 3: Comparison of AE of Example 2 at $\mu = 2$, $\beta = 1$.

t	Method in [8] at $M = 11$	Our method at $M = 13$	Our CPU time
0.1	5.9730×10^{-14}	2.22045×10^{-16}	
0.2	1.0669×10^{-13}	0	
0.3	1.1813×10^{-13}	1.11022×10^{-16}	
0.4	9.9032×10^{-14}	1.11022×10^{-16}	
0.5	6.1506×10^{-14}	0	0.891
0.6	1.8208×10^{-14}	0	
0.7	2.0650×10^{-14}	0	
0.8	$4.9738\times^{-14}$	1.11022×10^{-16}	
0.9	6.7835×10^{-14}	1.11022×10^{-16}	

Test Problem 3. Consider the following NFDE:

$$D_t^{\mu}\chi(t) + a\, D_t^{\beta}\chi(t) + b\,\chi(t) + c\,\chi^2(t) + d\,\chi^5(t) + e\,\chi^7(t) = f(t), \tag{42}$$

subject to the conditions

$$\chi(0) = \chi'(0) = 0, \tag{43}$$

where $f(t)$ is chosen such that the exact solution is $\chi(t) = t^{\beta+\mu}$ at $a = 1$, $b = 1$, $c = 1$, $d = 1$, $e = 1$.

Table 4 presents the AE at different values of μ, β when $M = 16$. Figure 7 shows AE when $\mu = 1.8$, $\beta = 0.7$ at different values of M.

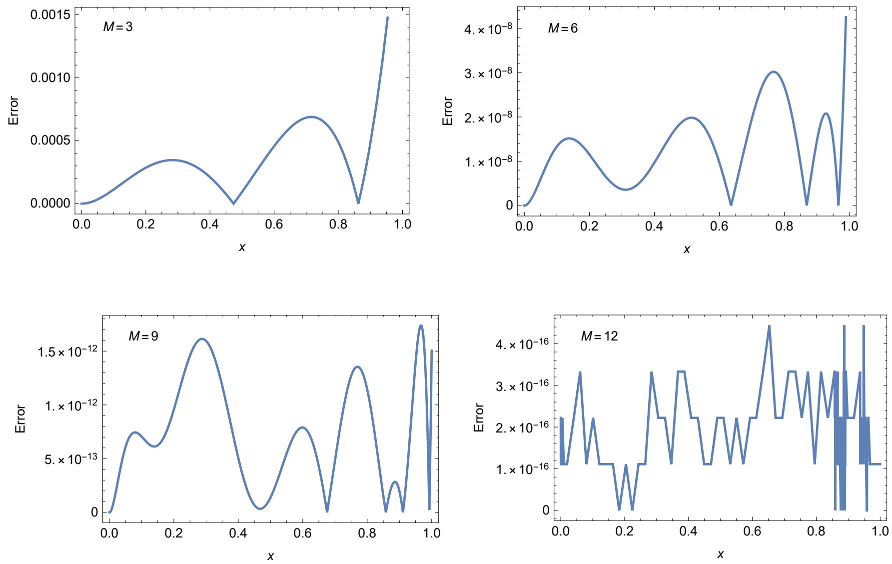

Figure 4: The AE of Example 2 at $\mu = 2$, $\beta = 1$.

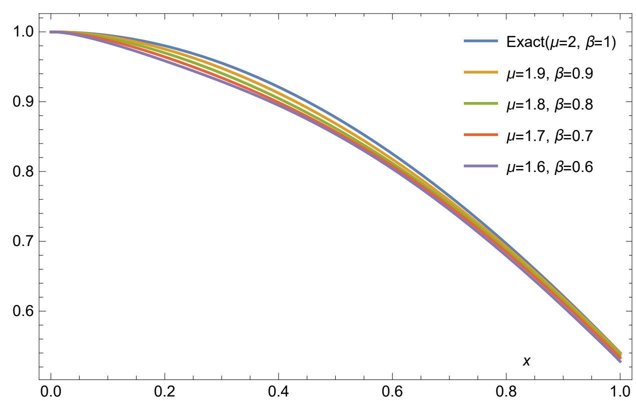

Figure 5: Different solutions of Example 2 at $M = 12$ and different values of μ, β.

6 Concluding Remarks

In this study, we used SCP3K and the spectral collocation method to solve the NFDE. We evaluated the approach with numerical test examples and carried out a thorough error analysis, contrasting it with other approaches that were already in use. Our findings demonstrated that the suggested

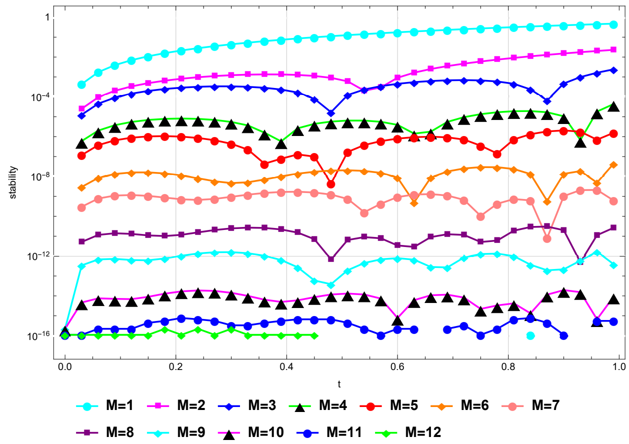

Figure 6: Stability $|\chi^{M+1}(t) - \chi^{M}(t)|$ at $\mu = 2$, $\beta = 1$ for Example 2.

Table 4: The AE of Example 3 at $M = 16$.

t	$\mu = 1.3$, $\beta = 0.3$	CPU time	$\mu = 1.6$, $\beta = 0.6$	CPU time	$\mu = 1.8$, $\beta = 0.7$	CPU time
0.1	1.37101×10^{-4}		9.05091×10^{-6}		5.27826×10^{-6}	
0.2	1.41098×10^{-4}		1.152×10^{-5}		8.17987×10^{-6}	
0.3	7.39605×10^{-5}		1.26987×10^{-5}		1.04948×10^{-5}	
0.4	1.48116×10^{-4}		1.50502×10^{-5}		1.2503×10^{-5}	
0.5	5.9031×10^{-5}	2	1.42085×10^{-5}	1.969	1.35739×10^{-5}	1.609
0.6	1.25943×10^{-4}		1.5650×10^{-5}		1.47554×10^{-5}	
0.7	4.4265×10^{-5}		1.39166×10^{-5}		1.50037×10^{-5}	
0.8	5.48668×10^{-5}		1.40047×10^{-5}		1.53348×10^{-5}	
0.9	1.11037×10^{-4}		1.2731×10^{-5}		1.47999×10^{-5}	

approach, which only needs a small number of terms and little computer power, can successfully simulate the answer.

Conflict of interest

The authors declare that they have no conflict of interest.

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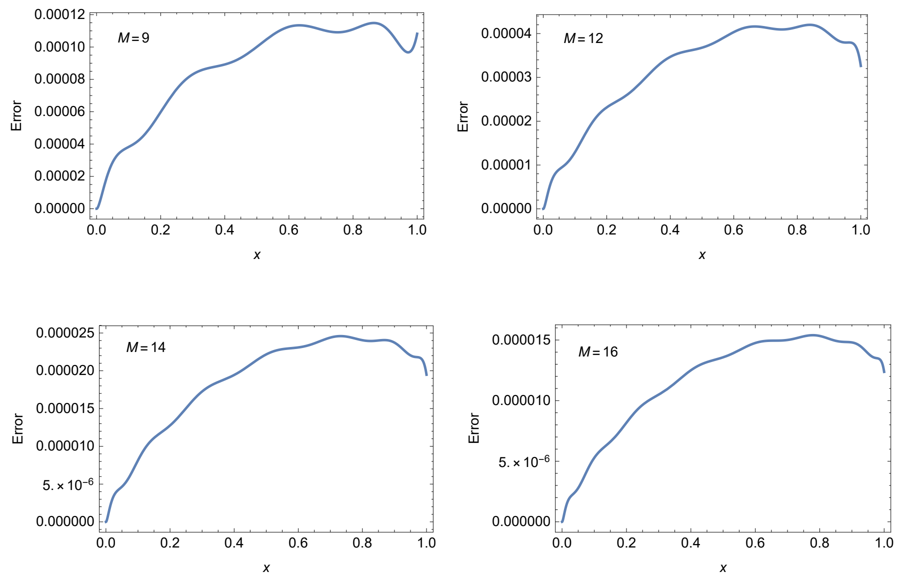

Figure 7: The AE of Example 3 at $\mu = 1.8$, $\beta = 0.7$.

Data availibility

No data is associated with this research.

Author Contributions Statement

YHY conducted the mathematical analysis, developed the methodology, verified the results, wrote the initial draft, and reviewed the final version. AGA contributed to the original manuscript, software development, and methodology. MM and ZYA reviewed and edited the final version of the manuscript.

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