Iranian Journal of Numerical Analysis and Optimization
Vol. 15, No. 1, 2025, pp 255–283
https://doi.org/10.22067/ijnao.2024.89176.1490
https://ijnao.um.ac.ir/

Research Article

Parameter-uniform numerical treatment of singularly perturbed parabolic delay differential equations with nonlocal boundary conditions

W.S. Hailu*, and G.F. Duressa

Abstract

This paper focuses on solving singularly perturbed parabolic equations of the convection-diffusion type with a large negative spatial shift and an integral boundary condition. A higher-order uniformly convergent numerical approach is proposed that uses Crank–Nicolson and a hybrid finite difference approximation on a piece-wise uniform Shishkin mesh. Simpson's 1/3 integration rule is used to treat the integral boundary condition. The proposed method has been shown to achieve almost second-order uniform convergence. The computational results derived from the numerical

*Corresponding author

Received 1 August 2024; revised 29 August 2024; accepted 14 September 2024

Wondimagegnehu Simon Hailu
Department of Mathematics, Dilla University, Dilla, Ethiopia. e-mail: wondiye02@gmail.com

Gemechis File Duressa
Department of Mathematics, Jimma University, Jimma, Ethiopia. Department of Mathematics, Arba Minch University, Arba Minch, Ethiopia. e-mail: gammeef@gmail.com

How to cite this article
Hailu, W.S. and Duressa, G.F., Parameter-uniform numerical treatment of singularly perturbed parabolic delay differential equations with nonlocal boundary conditions. *Iran. J. Numer. Anal. Optim.*, 2025; 15(1): 255-283. https://doi.org/10.22067/ijnao.2024.89176.1490

experiment are consistent with the theoretical estimates. Furthermore, the method produces a more accurate result than certain other methods in the literature.

AMS subject classifications (2020): 65M06, 65M12, 65M15, 65M22

Keywords: Singular perturbation; Convection-Diffusion problem; Fitted mesh method; Hybrid finite difference scheme; Integral constraint.

1 Introduction

Many physical phenomena are modeled using delay partial differential equations (DPDEs), which are influenced by both the current state of the system and its history. In order to make the model more realistic, it is occasionally important to take into account the previous states of the system in addition to the one it is in right now. DPDEs are crucial in modeling when the rate of change of a time-dependent phenomenon depends on a preceding state. DPDEs offer more accurate models for processes that exhibit a time lag or an aftereffect than nonDPDEs. Physical problems are modeled using singularly perturbed parabolic delay differential equations (SPPDDEs), whose evolution is influenced by both the system's history and its current state. SPPDDEs are used in numerous scientific fields, including population dynamics, control theory, blood flow models, and others. However, see [4, 13, 20, 6] for more applications and detailed descriptions.

Although SPPDDEs have received less attention in the literature than nondelayed problems such as those presented in [9, 23, 2, 30], many researchers have tried to develop different numerical schemes for solving SPPDDEs. The literature has developed numerical methods for solving SPPDDEs with Dirichlet boundary conditions in great detail; see, for example, [15, 22, 21, 28, 19, 27, 18]. Robin boundary conditions for SPPDDEs are also considered in [7, 8, 24]. Recently, a variety of various parameter-uniform fitted mesh numerical schemes have been suggested for solving a class of SPPDDEs of reaction-diffusion type with an integral boundary condition (IBC) in [5, 14, 29], while fitted operator numerical methods have been developed in [10, 12, 11]. The authors in [25] looked at an SPPDDE of the convection-diffusion type with an IBC for the first time. They developed a parameter-uniform numerical method that provides almost first-order convergence. Very recently, the authors in [17, 16] investigated the SPPDDE of the convection-diffusion type with an IBC. The methods are almost first-order uniformly convergent.

However, due to the presence of boundary layers and multi-scale characters in their solutions, the convergence analysis of higher-order numerical methods for SPPDDEs is difficult. Furthermore, the presence of large delays in SPPDDEs results in an extra interior layer in addition to the boundary layer. This makes convergence analysis more challenging. Traditional numerical techniques require an unnecessarily large number of mesh points to avoid solution oscillations when the perturbation parameter is very small. This is not possible due to computational costs and rounding errors. Scholars employ fitted numerical techniques to overcome these limitations. Higher-order numerical approaches for solving convection-diffusion SPPDDEs with an IBC have made little progress or not gained much attention. As a result, this research intends to provide a higher-order parameter-uniform fitted numerical scheme for solving convection-diffusion SPPDDEs with an IBC as well as a more accurate approximation.

The remaining contents of the article are structured as follows: we outline the governing problem and discuss the continuous solution's features in section 2. The Crank–Nicolson and hybrid difference methods, the semi-discrete maximum principle, and the semi-discrete stability are all covered in section 3. The convergence analysis of the suggested hybrid method is presented in section 4. Section 5 includes numerical examples to support the theoretical estimates and discussion. The conclusion is offered in section 6.

2 The continuous problem

In this article, consider the convection-diffusion SPPDDE with IBC on the entire domain $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 = (0,1] \times (0,T]$ and $\Omega_2 = (1,2) \times (0,T]$. The total boundary is $\partial\Omega = \partial\Omega_l \cup \partial\Omega_b \cup \partial\Omega_r$, where $\partial\Omega_b = \{(x,0) : 0 \leq x \leq 2\}$, $\partial\Omega_l = \{(x,t) : -1 \leq x \leq 0 \text{ and } 0 \leq t \leq T\}$ and

$\partial\Omega_r = \{(2,t):\ 0 \le t \le T\}$. Moreover,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \varepsilon \dfrac{\partial^2 u}{\partial x^2} + A(x)\dfrac{\partial u}{\partial x} + B(x)u(x,t) + C(x)u(x-1,t) = f(x,t), & (x,t) \in \Omega, \\ u(x,0) = \varphi_b(x), & (x,0) \in \partial\Omega_b, \\ u(x,t) = \varphi_l(x,t), & (x,t) \in \partial\Omega_l, \\ \$_r u(x,t) \equiv u(2,t) - \varepsilon \displaystyle\int_0^2 g(x)u(x,t)dx = \varphi_r(x,t), & (x,t) \in \partial\Omega_r, \end{cases} \tag{1}$$

where $0 < \varepsilon \ll 1$ is a perturbation parameter, $A(x)$, $B(x)$, $C(x)$, and $f(x,t)$ are sufficiently smooth functions such that

$$\begin{cases} A(x) \ge A_0 > A_0^* > 0, & B(x) \ge B_0 > 0, \qquad C(x) \le C_0 < 0, \\ A_0^* + B_0 + C_0 > 0, & B(x) + C(x) \ge 2\alpha > 0, \\ u(1^-,t) = u(1^+,t) & u_x(1^-,t) = u_x(1^+,t). \end{cases} \tag{2}$$

Furthermore, we assume that the monotonically nonnegative function $g(x)$ satisfies $\displaystyle\int_0^2 g(x)dx < 1$.

Problem (1) can be written as

$$\$u(x,t) = F(x,t), \quad \text{for all } (x,t) \in \Omega, \tag{3}$$

where

$$\$ = \begin{cases} \$_1 \equiv \left(\dfrac{\partial}{\partial t} - \varepsilon\dfrac{\partial^2}{\partial x^2} + A(x)\dfrac{\partial}{\partial x} + B(x)\right), & \text{on } \ \Omega_1, \\ \$_2 \equiv \left(\dfrac{\partial}{\partial t} - \varepsilon\dfrac{\partial^2}{\partial x^2} + A(x)\dfrac{\partial}{\partial x} + B(x)\right) + C(x)u(x-1,t), & \text{on } \ \Omega_2, \end{cases} \tag{4}$$

$$F(x,t) = \begin{cases} f(x,t) - C(x)\varphi_l(x-1,t), & \text{on } \Omega_1, \\ f(x,t), & \text{on } \Omega_2, \end{cases} \tag{5}$$

subject to

$$\begin{cases} u(x,0) = \varphi_b(x), & \text{on } \ \partial\Omega_b, \\ u(x,t) = \varphi_l(x,t), & \text{on } \ \partial\Omega_l, \\ u(1^-,t) = u(1^+,t), & \dfrac{\partial u}{\partial x}(1^-,t) = \dfrac{\partial u}{\partial x}(1^+,t), \\ \$_r u(x,t) = u(2,t) - \varepsilon \displaystyle\int_0^2 g(x)u(x,t)dx = \varphi_r(x,t), & \text{on } \ \partial\Omega_r. \end{cases} \tag{6}$$

For the existence and uniqueness of a solution for (3)–(6), we assume that the given data and the functions φ_b, φ_l, and φ_r are sufficiently smooth and impose compatibility conditions [1]:

$$\begin{cases} \varphi_b(0,0) = \varphi_l(0,0), \quad \varphi_b(2,0) = \varphi_r(2,0), \\ \dfrac{\partial \varphi_l(0,0)}{\partial t} - \varepsilon \dfrac{\partial^2 \varphi_b(0,0)}{\partial x^2} + A(0)\dfrac{\partial \varphi_b(0,0)}{\partial x} \\ \quad +B(0)\varphi_b(0,0) + C(0)\varphi_l(-1,0) = f(0,0), \\ \dfrac{\partial \varphi_r(2,0)}{\partial t} - \varepsilon \dfrac{\partial^2 \varphi_b(2,0)}{\partial x^2} + A(2)\dfrac{\partial \varphi_b(2,0)}{\partial x} \\ \quad +B(2)\varphi_b(2,0) + C(2)\varphi_b(1,0) = f(2,0). \end{cases} \tag{7}$$

The differential operator $\mathscr{L}$ satisfies the following lemma.

Lemma 1. Suppose that $\Phi(x,t) \in C^{(2,1)}(\overline{\Omega})$ satisfies $\Phi(0,t) \geq 0$, $\Phi(x,0) \geq 0$, $\mathscr{L}_r\Phi(2,t) \geq 0$, $\mathscr{L}_1\Phi(x,t) \geq 0$, $(x,t) \in \Omega_1$, $\mathscr{L}_2\Phi(x,t) \geq 0$, $(x,t) \in \Omega_2$, and $[\Phi_x](1,t) = \Phi_x(1^+,t) - \Phi_x(1^-,t) \leq 0$. Then $\Phi(x,t) \geq 0$ for $(x,t) \in \overline{\Omega}$.

Proof. Refer to [25]. □

When this maximum principle is immediately applied, the following solution bounds are obtained.

Lemma 2. Suppose that $u(x,t)$ is the solution of problem (3)–(6). Then

$$\|u\|_{\infty,\overline{\Omega}} \leq \frac{1}{\alpha}\|F\|_{\infty,\overline{\Omega}} + \|u\|_{\infty,\partial\Omega}.$$

Proof. Refer to [25]. □

Lemma 3. Suppose that $u(x,t)$ is the solution of problem (1). Then

$$\left|\frac{\partial^k u(x,t)}{\partial t^k}\right| \leq C, \qquad \text{for all } (x,t) \in \overline{\Omega} \text{ and } k = 0,1,2, \tag{8}$$

where C is a constant independent of ε.

Proof. See [25]. □

3 Description of the numerical scheme

3.1 Time Semi-discretization

Let $\Omega_t^M = \{t_j = t_0 + jk,\ j = 1, 2, \ldots, M,\ t_0 = 0,\ t_M = T, k = T/M\}$ be mesh points with step size k and M number of mesh elements in the temporal direction. Then, we discretize the time derivative in (3) using the Crank–Nicolson method. We obtained system of singularly perturbed differential equations:

$$\begin{cases} \$^M U^{j+1}(x) = G^{j+1}(x), & j = 0, 1, \ldots, M-1, \quad x \in (0,2), \\ U(x,0) = \varphi_b(x), & 0 \leq x \leq 2, \\ U^{j+1}(x) = \varphi_l(x, t_{j+1}), & -1 \leq x \leq 0, \quad j = 0, 1, \ldots, M-1, \\ \$_r^M U^{j+1}(2) \equiv U^{j+1}(2) & \\ \quad -\varepsilon \int_0^2 g(x) U^{j+1}(x) dx = \varphi_r(x, t_{j+1}), & j = 0, 1, \ldots, M-1, \end{cases} \tag{9}$$

where

$$\$^M U^{j+1}(x) = \begin{cases} \$_1^M U^{j+1}(x) \equiv \left(-\varepsilon \frac{d^2}{dx^2} + A(x)\frac{d}{dx} + E_1(x)\right) U^{j+1}(x), & \text{if } x \in (0,1], \\ \$_2^M U^{j+1}(x) \equiv \left(\$_1^M U^{j+1}(x)\right) + C(x) U^{j+1}(x-1), & \text{if } x \in (1,2), \end{cases}$$

$$G^{j+1}(x) = \begin{cases} G_1^{j+1}(x) = \left(\varepsilon \frac{d^2}{dx^2} - A(x)\frac{d}{dx} - E_2(x)\right) U^j(x) & \\ +f(x,t_j) + f(x,t_{j+1}) & \\ -C(x)\left(\varphi_l^j(x-1) + \varphi_l^{j+1}(x-1)\right), & \text{if } x \in (0,1], \\ G_2^{j+1}(x) = \left(\varepsilon \frac{d^2}{dx^2} - A(x)\frac{d}{dx} - E_2(x)\right) U^j(x) & \\ -C(x) U^j(x-1) + f(x,t_j) + f(x,t_{j+1}), & \text{if } x \in (1,2), \end{cases}$$

where $E_1(x) = B(x) + \frac{2}{k}$ and $E_2(x) = B(x) - \frac{2}{k}$.

The operator $\$^M$ satisfies the following Lemma.

Lemma 4. Let $\Theta(x,t_{j+1})$ be a smooth function satisfies $\Theta(0,t_{j+1}) \geq 0$, $\$_r^M\Theta(2,t_{j+1}) \geq 0$ and $\$^M\Theta(x,t_{j+1}) \geq 0$, for all $x \in (0,2)$. Then $\Theta(x,t_{j+1}) \geq 0$, for all $x \in [0,2]$.

Proof. Take $(\hat{x},t_{j+1}) \in \{(x,t_{j+1}) : x \in (0,2)\}$ such that

$$\Theta(\hat{x},t_{j+1}) = \min_{x\in(0,2)} \Theta(x,t_{j+1}).$$

Then

$$\Theta_x(\hat{x},t_{j+1}) = 0 \quad \text{and} \quad \Theta_{xx}(\hat{x},t_{j+1}) > 0. \tag{10}$$

Assume that $\Theta(\hat{x},t_{j+1}) < 0$. Then

Case i: For $\hat{x} \in (0,1]$, we have

$$\begin{aligned}\$_1^M\Theta(\hat{x},t_{j+1}) &= -\varepsilon\frac{d^2\Theta(\hat{x},t_{j+1})}{dx^2} + A(\hat{x})\frac{d\Theta(\hat{x},t_{j+1})}{dx} + E_1(\hat{x})\Theta(\hat{x},t_{j+1}) \\ &< 0.\end{aligned}$$

Case ii: For $\hat{x} \in (1,2)$, we have

$$\begin{aligned}\$_2^M\Theta(\hat{x},t_{j+1}) &= -\varepsilon\frac{d^2\Theta(\hat{x},t_{j+1})}{dx^2} + A(\hat{x})\frac{d\Theta(\hat{x},t_{j+1})}{dx} + E_1(\hat{x})\Theta(\hat{x},t_{j+1}) \\ &\quad +C(\hat{x})\Theta(\hat{x}-1,t_{j+1}) \\ &\leq -\varepsilon\frac{d^2\Theta(\hat{x},t_{j+1})}{dx^2} + A(\hat{x})\frac{d\Theta(\hat{x},t_{j+1})}{dx} + E_1(\hat{x})\Theta(\hat{x},t_{j+1}) \\ &\quad +C(\hat{x})\Theta(\hat{x},t_{j+1}), \quad \text{since } C(x)<0 \\ &= -\varepsilon\frac{d^2\Theta(\hat{x},t_{j+1})}{dx^2} + \left(B(\hat{x}) + C(\hat{x}) + \frac{2}{k}\right)\Theta(\hat{x},t_{j+1}) \\ &< 0, \qquad (\text{by } (2) \text{ and } (10)).\end{aligned}$$

All cases contradict $\$^M\Theta(x,t_{j+1}) \geq 0$, $0 < x < 2$.

Therefore, $\Theta(x,t_{j+1}) \geq 0$, $0 \leq x \leq 2$. □

Lemma 5. Suppose that $U^{j+1}(x)$ is a semi-discrete solution of (9). Then

$$\begin{aligned}&\left\|U^{j+1}(x)\right\|_{\overline{\Omega}^M} \\ &\leq C\max\left\{\left\|U^{j+1}(x)\right\|_{\partial\Omega_l^M}, \left\|\$_r^M U^{j+1}(2)\right\|_{\partial\Omega_r^M}, \max\left\|\$^M U^{j+1}(x)\right\|\right\}.\end{aligned}$$

Proof. Define the barrier functions as

$$\Im^{\pm}(x,t_{j+1}) = C\mathcal{X} \pm U^{j+1}(x),$$

where $\mathcal{X} = \max\left\{\left\|U^{j+1}(x)\right\|_{\partial\Omega_l^M}, \left\|\$_r^M U^{j+1}(2)\right\|_{\partial\Omega_r^M},\ \max\left\|\$^M U^{j+1}(x)\right\|\right\}$.

It is clear that, $\Im^{\pm}(0,t_{j+1}) \geq 0$ and $\Im^{\pm}(2,t_{j+1}) \geq 0$.

Case i: For $x \in [0,1]$, we have

$$\begin{aligned}
\$_1^M \Im^{\pm}(x,t_{j+1}) &= E_1(x)\mathcal{X} \pm \$_1^M U^{j+1}(x) \\
&\geq \frac{kB(x)+2}{k} \max\left\|\$_1^M U^{j+1}(x)\right\| \pm \$_1^M U^{j+1}(x) \\
&\geq 0.
\end{aligned}$$

Case ii: For $x \in (1,2]$, we have

$$\begin{aligned}
\$_2^M \Im^{\pm}(x,t_{j+1}) &= (C(x)+E_1(x))\mathcal{X} \pm \$_2^M U^{j+1}(s) \\
&\geq \left(C(x) + \frac{kB(x)+2}{k}\right) \max\left\|\$_2^M U^{j+1}(x)\right\| \pm \$_2^M U^{j+1}(x) \\
&\geq \left(B(x) + C(x) + \frac{2}{k}\right) \max\left\|\$_2^M U^{j+1}(x)\right\| \pm \$_2^M U^{j+1}(x) \\
&\geq 0.
\end{aligned}$$

Hence, applying Lemma 4 gives

$$\begin{aligned}
&\left\|U^{j+1}\right\|_{\overline{\Omega}^M} \\
&\leq C \max\left\{\left\|U^{j+1}(x)\right\|_{\partial\Omega_l^M}, \left\|\$_r^M U^{j+1}(2)\right\|_{\partial\Omega_r^M},\ \max\left\|\$^M U^{j+1}(x)\right\|\right\}.
\end{aligned}$$

□

Let $e_{j+1} = u(x,t_{j+1}) - \widetilde{U}^{j+1}(x)$ denote the local truncation error in the $(j+1)$th time step at a point $(x,t_{j+1}) \in \Omega^M$, where $\widetilde{U}^{j+1}(x)$ is the numerical solution of

$$\begin{cases}
\$^M \widetilde{U}^{j+1}(x) = \widetilde{G}^{j+1}(x), \quad x \in (0,2), \\
\widetilde{U}^{j+1}(x) = \varphi_l(x,t_{j+1}), \quad x \in [-1,0], \\
\$_r^M \widetilde{U}^{j+1}(2) = \widetilde{U}^{j+1}(2) - \varepsilon \displaystyle\int_0^2 g(x)\widetilde{U}^{j+1}(x)dx = \varphi_r(2,t_{j+1}),
\end{cases}$$

where

$$\widetilde{G}^{j+1}(x) = \begin{cases} \left(\varepsilon \dfrac{d^2}{dx^2} - A(x)\dfrac{d}{dx} - E_2(x)\right) u(x,t_j) + f(x,t_j) + f(x,t_{j+1}) \\ -C(x)\left(\varphi_l\left(x-1,t_j\right) + \varphi_l\left(x-1,t_{j+1}\right)\right), \qquad \text{if } x \in (0,1], \\ \left(\varepsilon \dfrac{d^2}{dx^2} - A(x)\dfrac{d}{dx} - E_2(x)\right) u(x,t_j) - C(x)u(x-1,t_j) \\ +f(x,t_j) + f(x,t_{j+1}), \qquad \text{if } x \in (1,2), \end{cases}$$

Lemma 6. The $(j+1)$th time step local truncation error $e_{j+1} = u(x,t_{j+1}) - \widetilde{U}^{j+1}(x)$ satisfies

$$\|e_{j+1}\| \leq Ck^3,$$

where C is a constant independent of ε and M.

Proof. Refer to [3] for a detailed proof. □

Lemma 7. Suppose that $U^{j+1}(x)$ is a numerical solution of the semi-discrete problem in (9). Then the global error is

$$\|E_j\|_\infty \leq Ck^2, \qquad \text{for all } j = 1,2,\ldots,M,$$

where C is a constant independent of ε.

Proof. Using the estimate in Lemma 6, we get

$$\begin{aligned} \|E_j\|_\infty &= \left\|\sum_{l=1}^{j} e_l\right\|_\infty \leq \|e_1\|_\infty + \|e_2\|_\infty + \cdots + \|e_j\|_\infty \\ &\leq C_1 jk^3 = C_1(jk)(k^2) \\ &\leq C_1 T(k^2) \qquad \text{because,} \quad jk \leq T. \\ &\leq Ck^2. \end{aligned}$$

□

According to Lemma 7, the semi-discretized scheme is second-order uniformly convergent in time.

The semi-discrete solution U^{j+1} of the semi-discrete problem in (9) can be decomposed as regular $\left(V^{j+1}\right)$ and singular $\left(W^{j+1}\right)$ components and can be written as

$$U^{j+1}(x) = V^{j+1}(x) + W^{j+1}(x),$$

where $V^{j+1}(x)$ satisfies the equation

$$\begin{cases} \$^M V^{j+1}(x) = G^{j+1}(x), & 0 < x < 2, \\ V^{j+1}(x) = \varphi_l(x, t_{j+1}), & -1 \le x \le 0, \\ V^{j+1}(1) = V_0^{j+1}(1), & \$_r^M V^{j+1}(2) = \$_r^M V_0^{j+1}(2), \end{cases}$$

where V_0^{j+1} is solution of the corresponding reduced problem.

Also, $W^{j+1}(x)$ satisfies the homogeneous equation:

$$\begin{cases} \$^M W^{j+1}(x) = 0, & 0 < x < 2, \\ W^{j+1}(x) = 0, & -1 \le x \le 0, \\ \left[W_x^{j+1}\right](1) = \left[V_x^{j+1}\right](1), \\ \$_r^M W^{j+1}(2) = \$_r^M U^{j+1}(2) - \$_r^M V^{j+1}(2). \end{cases}$$

For details of the decomposition of the semi-discrete solution, readers can refer to [26].

The derivatives of regular and singular components of the semi-discrete solution satisfy the following bounds.

Lemma 8. The regular and singular components of the solution $U^{j+1}(x)$ to the semi-discrete problem (9) satisfy the following bounds for $k = 0, 1, 2, 3, 4$:

$$\left|\frac{\mathrm{d}^k V^{j+1}(x)}{\mathrm{d}x^k}\right| \le C\left(1 + \varepsilon^{3-k}\right), \qquad \text{for} \quad \{x : 0 < x < 1\} \cup \{x : 1 < x < 2\},$$

$$\left|\frac{\mathrm{d}^k W_B^{j+1}(x)}{\mathrm{d}x^k}\right| \le C\varepsilon^{-k} \exp\left(\frac{A_0^*(x-2)}{\varepsilon}\right), \qquad \{x : 0 < x < 1\} \cup \{x : 1 < x < 2\},$$

$$\left|\frac{\mathrm{d}^k W_I^{j+1}(x)}{\mathrm{d}x^k}\right| \le C \begin{cases} \varepsilon^{1-k} \exp\left(\frac{A_0^*(x-1)}{\varepsilon}\right), & 0 < x \le 1, \\ \varepsilon^{1-k}, & 1 < x < 2. \end{cases}$$

Proof. Refer to [26]. □

3.2 Spatial discretization

We first construct a piece-wise uniform (Shishkin) mesh and then approximate (9) by applying the hybrid finite difference method. In the x-direction, divide the interval $[0, 2]$ into four sub-intervals because the problem in (9) exhibits a strong boundary layer at $x = 2$, and a weak interior layer at $x = 1$. The sub-interval $[0, 1]$ is subdivided into two sub-intervals $[0, 1-\tau]$ and $[1-\tau, 1]$. Similarly, the interval $[1, 2]$ is subdivided into two sub-intervals $[1, 2-\tau]$ and $[2-\tau, 2]$. Each sub-intervals has $\frac{N}{4}$ mesh elements and τ satisfies

$$\tau = \min\left\{\frac{1}{2}, \tau_0 \varepsilon \ln(N)\right\},$$

where τ_0 and N are positive constant such that $\tau_0 \geq 1/A_0^*$ and the number of mesh elements in the x-direction, respectively. However, assume that $\tau = \tau_0 \varepsilon \ln(N)$ for analysis; otherwise, N^{-1} is exponentially small compared with ε.

Now, the mesh points in the x-direction are defined by

$$x_i = \begin{cases} i h_i, & i = 0, 1, \dots, \frac{N}{4}, \\ 1 - \tau + \left(i - \frac{N}{4}\right) h_i, & i = \frac{N}{4} + 1, \dots, \frac{N}{2}, \\ 1 + \left(i - \frac{N}{2}\right) h_i, & i = \frac{N}{2} + 1, \dots, \frac{3N}{4}, \\ 2 - \tau + \left(i - \frac{3N}{4}\right) h_i, & i = \frac{3N}{4} + 1, \dots, N, \end{cases}$$

where the mesh spacing is given by

$$h_i = \begin{cases} \frac{4\tau}{N}, & i = \frac{N}{4} + 1, \dots, \frac{N}{2}, \frac{3N}{4} + 1, \dots, N, \\ \frac{4(1-\tau)}{N}, & i = 1, \dots, \frac{N}{4}, \frac{N}{2} + 1, \dots, \frac{3N}{4}. \end{cases}$$

Let $H_i = h_i + h_{i+1},\ i = 1, 2, \dots, N-1$. For any mesh function, $Y_i \approx Y(x_i)$ define $Y_{i-\frac{1}{2}} = \frac{Y_i + Y_{i-1}}{2}$ and the following finite difference operators:

$$D^- Y_i = \frac{Y_i - Y_{i-1}}{h_i}, \qquad D^+ Y_i = \frac{Y_{i+1} - Y_i}{h_{i+1}},$$
$$D^0 Y_i = \frac{Y_{i+1} - Y_{i-1}}{H_i}, \qquad \delta^2 Y_i = \frac{2\left(D^+ Y_i - D^- Y_i\right)}{H_i}.$$

For discretization of the problem in (9), the midpoint upwind difference scheme is used in the outer regions $[0, 1-\tau]$ and $[1, 2-\tau]$, and the central difference scheme is used in the interior layer region $[1-\tau, 1]$ and in the boundary layer region $[2-\tau, 2]$. This combination of the two schemes is the proposed hybrid difference scheme which takes the form

$$
\begin{cases}
\$_{md}^{N,M} U_i^{j+1} = R_{i-\frac{1}{2}}^{j+1}, \ \ i = 1, 2, \ldots, \dfrac{N}{4}, \ \dfrac{N}{2} + 1, \ldots, \dfrac{3N}{4}, \\
\$_{ce}^{N,M} U_i^{j+1} = R_i^{j+1}, \ \ i = \dfrac{N}{4} + 1, \ldots, \dfrac{N}{2}, \ \dfrac{3N}{4} + 1, \ldots, N-1,
\end{cases}
\tag{11}
$$

where

$$
\$_{md}^{N,M} U_i^{j+1} = \begin{cases}
\left[-\varepsilon\delta^2 U_i^{j+1} + A_{i-\frac{1}{2}} D^- U_i^{j+1} + E_{1,i-\frac{1}{2}} U_{i-\frac{1}{2}}^{j+1}\right] \\
- \left[\varepsilon\delta^2 U_i^{j} - A_{i-\frac{1}{2}} D^- U_i^{j} - E_{2,i-\frac{1}{2}} U_{i-\frac{1}{2}}^{j}\right], \qquad i = 1, 2, \ldots, \dfrac{N}{4}, \\
\left[-\varepsilon\delta^2 U_i^{j+1} + A_{i-\frac{1}{2}} D^- U_i^{j+1} + E_{1,i-\frac{1}{2}} U_{i-\frac{1}{2}}^{j+1} + C_{i-\frac{1}{2}} U_{i-\frac{N}{2}-\frac{1}{2}}^{j+1}\right] \\
- \left[\varepsilon\delta^2 U_i^{j} - A_{i-\frac{1}{2}} D^- U_i^{j} - E_{2,i-\frac{1}{2}} U_{i-\frac{1}{2}}^{j} - C_{i-\frac{1}{2}} U_{i-\frac{N}{2}-\frac{1}{2}}^{j}\right], \\
\hfill i = \dfrac{N}{2} + 1, \ldots, \dfrac{3N}{4},
\end{cases}
$$

$$
\$_{ce}^{N,M} U_i^{j+1} = \begin{cases}
\left[-\varepsilon\delta^2 U_i^{j+1} + A_i D^0 U_i^{j+1} + E_{1,i} U_i^{j+1}\right] \\
- \left[\varepsilon\delta^2 U_i^{j} - A_i D^0 U_i^{j} - E_{2,i} U_i^{j}\right], \qquad i = \dfrac{N}{4} + 1, \ldots, \dfrac{N}{2}, \\
\left[-\varepsilon\delta^2 U_i^{j+1} + A_i D^0 U_i^{j+1} + E_{1,i} U_i^{j+1} + C_i U_{i-\frac{N}{2}}^{j+1}\right] \\
- \left[\varepsilon\delta^2 U_i^{j} - A_i D^0 U_i^{j} - E_{2,i} U_i^{j} - C_i U_{i-\frac{N}{2}}^{j}\right], \\
\hfill i = \dfrac{3N}{4} + 1, \ldots, N-1.
\end{cases}
$$

$$
R_{i-\frac{1}{2}}^{j+1} = \begin{cases}
f(x_{i-\frac{1}{2}}, t_j) + f(x_{i-\frac{1}{2}}, t_{j+1}) \\
-C_{i-\frac{1}{2}} \left(\varphi_{l,i-\frac{N}{2}-\frac{1}{2}}^{j} + \varphi_{l,i-\frac{N}{2}-\frac{1}{2}}^{j+1}\right), \qquad i = 1, 2, \ldots, \dfrac{N}{4}, \\
f(x_{i-\frac{1}{2}}, t_j) + f(x_{i-\frac{1}{2}}, t_{j+1}), \qquad i = \dfrac{N}{2} + 1, \ldots, \dfrac{3N}{4},
\end{cases}
$$

$$
R_i^{j+1} = \begin{cases}
f(x_i, t_j) + f(x_i, t_{j+1}) \\
-C_i \left(\varphi_{l,i-\frac{N}{2}}^{j} + \varphi_{l,i-\frac{N}{2}}^{j+1}\right), \qquad i = \dfrac{N}{4} + 1, \ldots, \dfrac{N}{2}, \\
f(x_i, t_j) + f(x_i, t_{j+1}), \qquad i = \dfrac{3N}{4} + 1, \ldots, N-1.
\end{cases}
$$

The discrete scheme in (11) can be simplified as

$$
\begin{cases}
\left[\beta_i^- U_{i-1}^{j+1} + \beta_i^0 U_i^{j+1} + \beta_i^+ U_{i+1}^{j+1}\right] - \left[\gamma_i^- U_{i-1}^{j} + \gamma_i^0 U_i^{j} + \gamma_i^+ U_{i+1}^{j}\right] \\
= R_{i-\frac{1}{2}}^{j+1}, \qquad i = 1, 2, \dots, \dfrac{N}{4}, \\
\left[\beta_i^- U_{i-1}^{j+1} + \beta_i^0 U_i^{j+1} + \beta_i^+ U_{i+1}^{j+1}\right] - \left[\gamma_i^- U_{i-1}^{j} + \gamma_i^0 U_i^{j} + \gamma_i^+ U_{i+1}^{j}\right] \\
= R_{i}^{j+1}, \qquad i = \dfrac{N}{4} + 1, \dots, \dfrac{N}{2}, \\
\left[\beta_i^- U_{i-1}^{j+1} + \beta_i^0 U_i^{j+1} + \beta_i^+ U_{i+1}^{j+1} + \dfrac{C_{i-\frac{1}{2}}}{2}\left(U_{i-\frac{N}{2}-1}^{j+1} + U_{i-\frac{N}{2}}^{j+1}\right)\right] \\
- \left[\gamma_i^- U_{i-1}^{j} + \gamma_i^0 U_i^{j} + \gamma_i^+ U_{i+1}^{j} - \dfrac{C_{i-\frac{1}{2}}}{2}\left(U_{i-\frac{N}{2}-1}^{j} + U_{i-\frac{N}{2}}^{j}\right)\right] \\
\quad = R_{i-\frac{1}{2}}^{j+1}, \qquad i = \dfrac{N}{2} + 1, \dots, \dfrac{3N}{4}, \\
\left[\beta_i^- U_{i-1}^{j+1} + \beta_i^0 U_i^{j+1} + \beta_i^+ U_{i+1}^{j+1} + C_i U_{i-\frac{N}{2}}^{j+1}\right] \\
- \left[\gamma_i^- U_{i-1}^{j} + \gamma_i^0 U_i^{j} + \gamma_i^+ U_{i+1}^{j} - C_i U_{i-\frac{N}{2}}^{j}\right] \\
\quad = R_{i}^{j+1}, \qquad i = \dfrac{3N}{4} + 1, \dots, N-1,
\end{cases}
\tag{12}
$$

where for $i = 1, 2, \dots, \dfrac{N}{4},\ \dfrac{N}{2} + 1, \dots, \dfrac{3N}{4}$,

$$
\begin{cases}
\beta_i^- = -\dfrac{2\varepsilon}{H_i h_i} - \dfrac{A_{i-\frac{1}{2}}}{h_i} + \dfrac{B_{i-\frac{1}{2}}}{2} + \dfrac{2}{2k}, \\
\beta_i^0 = \dfrac{2\varepsilon}{H_i h_{i+1}} + \dfrac{2\varepsilon}{H_i h_i} + \dfrac{A_{i-\frac{1}{2}}}{h_i} + \dfrac{B_{i-\frac{1}{2}}}{2} + \dfrac{2}{2k}, \\
\beta_i^+ = -\dfrac{2\varepsilon}{H_i h_{i+1}}, \\
\gamma_i^- = \dfrac{2\varepsilon}{H_i h_i} + \dfrac{A_{i-\frac{1}{2}}}{h_i} - \dfrac{B_{i-\frac{1}{2}}}{2} + \dfrac{2}{2k}, \\
\gamma_i^0 = -\dfrac{2\varepsilon}{H_i h_{i+1}} - \dfrac{2\varepsilon}{H_i h_i} - \dfrac{A_{i-\frac{1}{2}}}{h_i} - \dfrac{B_{i-\frac{1}{2}}}{2} + \dfrac{2}{2k}, \\
\gamma_i^+ = \dfrac{2\varepsilon}{H_i h_{i+1}},
\end{cases}
$$

and for $i = \dfrac{N}{4} + 1, \dots, \dfrac{N}{2},\ \dfrac{3N}{4} + 1, \dots, N-1$,

$$\begin{cases} \beta_i^- = -\dfrac{2\varepsilon}{H_i h_i} - \dfrac{A_i}{H_i}, \\ \beta_i^0 = \dfrac{2\varepsilon}{H_i h_{i+1}} + \dfrac{2\varepsilon}{H_i h_i} + B_i + \dfrac{2}{k}, \\ \beta_i^+ = -\dfrac{2\varepsilon}{H_i h_{i+1}} + \dfrac{A_i}{H_i}, \\ \gamma_i^- = \dfrac{2\varepsilon}{H_i h_i} + \dfrac{A_i}{H_i}, \\ \gamma_i^0 = -\dfrac{2\varepsilon}{H_i h_{i+1}} - \dfrac{2\varepsilon}{H_i h_i} - B_i + \dfrac{2}{k}, \\ \gamma_i^+ = \dfrac{2\varepsilon}{H_i h_{i+1}} - \dfrac{A_i}{H_i}. \end{cases}$$

Hence, the fully discrete problem of the given problem in (1) can be written as

$$\begin{cases} \$_{hyb}^{N,M} U_i^{j+1} = R_i^{j+1}, \qquad i = 1, 2, \dots, N-1,\ j = 0, 1, \dots, M-1, \\ U_i^0 = \varphi_b(x_i), \qquad i = 0, 1, \dots, N, \\ U_i^{j+1} = \varphi_l(x_i, t_{j+1}), \qquad i = -\dfrac{N}{2}, -\dfrac{N}{2}+1, \dots, 0,\ j = 0, 1, \dots, M-1, \\ \$_r^{N,M} U_N^{j+1} = U_N^{j+1} - \varepsilon \displaystyle\sum_{i=1}^{N} \dfrac{g_{i-1}U_{i-1}^{j+1} - 4g_i U_i^{j+1} + g_{i+1}U_{i+1}^{j+1}}{3} h_i \\ = \varphi_r(x_N, t_{j+1}), \qquad j = 0, 1, \dots, M-1, \end{cases} \tag{13}$$

where

$$\$_{hyb}^{N,M} U_i^{j+1} = \begin{cases} \$_{md}^{N,M} U_i^{j+1}, & i = 1, 2, \dots, \dfrac{N}{4},\ \dfrac{N}{2}+1, \dots, \dfrac{3N}{4}, \\ \$_{ce}^{N,M} U_i^{j+1}, & i = \dfrac{N}{4}+1, \dots, \dfrac{N}{2},\ \dfrac{3N}{4}+1, \dots, N-1. \end{cases} \tag{14}$$

As a result, the hybrid difference scheme for solving (1) on $\overline{\Omega}$ is (13)–(14), which provide an $N \times N$ system of algebraic equations.

4 Convergence analysis

In this section, we establish the ε-uniform error estimate of the hybrid difference scheme in (13)–(14).

Lemma 9. Assume that there exists a positive integer N_0 such that for all $N \geq N_0$,

$$\frac{N_0}{\ln N_0} \geq \tau_0 \|A\|_{\overline{\Omega}}, \quad \frac{A_0^* N_0}{2} \geq \left\| B + \frac{1}{k} \right\|, \tag{15}$$

where A and B are the coefficients given in (1).

Let Θ_i^{j+1} be a mesh function satisfies $\Theta_0^{j+1} \geq 0$, $\$_r^{N,M}\Theta_N^{j+1} \geq 0$ and $\$_{hyb}^{N,M}\Theta_i^{j+1} \geq 0$, for $i = 1, 2, \ldots, N-1$. Then $\Theta_i^{j+1} \geq 0$, for $i = 0, 1, 2, \ldots, N$.

Proof. We write the operator $\$_{hyb}^{N,M}$ in a matrix form as

$$\$_{hyb}^{N,M}\Theta_i^{j+1} = \left[y_{i,i-1}\Theta_{i-1}^{j+1} + y_{i,i}\Theta_i^{j+1} + y_{i,i+1}\Theta_{i+1}^{j+1}\right] \tag{16}$$

$$- \left[z_{i,i-1}\Theta_{i-1}^{j} + z_{i,i}\Theta_i^{j} + z_{i,i+1}\Theta_{i+1}^{j}\right], \tag{17}$$

where $Y = (y_{i,j})$ and $Z = (z_{i,j})$ are matrices with entries

$$y_{i,i-1} = \beta_i^-, \quad y_{i,i} = \beta_i^0, \quad y_{i,i+1} = \beta_i^+, \quad \text{and} \quad z_{i,i-1} = \gamma_i^-, \quad z_{i,i} = \gamma_i^0, \quad z_{i,i+1} = \gamma_i^+.$$

Clearly, $Z \geq O$, because $\gamma_i^- \geq 0$, $\gamma_i^0 \geq 0$, $\gamma_i^+ \geq 0$. Using assumptions in (15) to show the coefficient matrix Y is an irreducible M-matrix [26]. Now, use induction to proof the remaining part.

Assume that $\Theta_i^j \geq 0$, $j = 0, 1, \ldots, M-1$.

Then from (17), we obtain

$$Y\Theta_i^{j+1} = Z\Theta_i^j + \$_{hyb}^{N,M}\Theta_i^{j+1}.$$

Since $Z \geq O$, $\Theta_i^j \geq 0$, $Y^{-1} \geq 0$, and $\$_{hyb}^{N,M}\Theta_i^{j+1} \geq 0$ by hypothesis, we have

$$\Theta_i^{j+1} = Y^{-1}\left(Z\Theta_i^j + \$_{hyb}^{N,M}\Theta_i^{j+1}\right) \geq 0, \; for \; i = 1, 2, \ldots, N-1.$$

□

We obtain the following ε-uniform stability bound as an immediate consequence of Lemma 9.

Lemma 10. Let U_i^{j+1} be the numerical solution of the problem in (13) and the assumptions in (15) hold true. Then,

$$\left\|U_i^{j+1}\right\|_{\overline{\Omega}^{N,M}} \leq C \max\left\{\|\varphi_l\|_{\partial\Omega_l^{N,M}}, \; \left|\$_r^{N,M}U_N^{j+1}\right|\right\} + \frac{1}{\alpha}\left\|R_i^{j+1}\right\|_{\overline{\Omega}^{N,M}}, \quad \text{for all} \;\; i = 0, 1, \ldots, N.$$

Proof. Consider the barrier functions

$$\Phi_i^{\pm,j+1} = C\max\left\{\|\varphi_l\|_{\partial\Omega_l^{N,M}},\ \left|\$_r^{N,M}U_N^{j+1}\right|\right\} + \frac{1}{\alpha}\left\|R_i^{j+1}\right\|_{\overline{\Omega}^{N,M}} \pm U_i^{j+1} \quad (18)$$

From (18), we obtain $\Phi_0^{\pm,j+1} \geq 0$, $\$_r^{N,M}\Phi_N^{\pm,j+1} \geq 0$, and $\$_{hyb}^{N,M}\Phi_i^{\pm,j+1} \geq 0$, $i = 1, 2, \ldots, N-1$.

Consequently, the application of the discrete maximum principle (Lemma 9) gives the desired bound. □

Lemma 11. Let $U^{j+1}(x)$ and U_i^{j+1} be solutions of (9) and (13), respectively. Then the local truncation error TE_i at a mesh point (x_i, t_{j+1}) for the discrete problem in (13)–(14) is given by

$$|TE_i| = \begin{cases} \left|\$_{md}^{N,M}U_i^{j+1} - \left(\$^M U^{j+1}\right)(x_i)\right|, \\ \qquad i = 1,\ldots,\frac{N}{4},\ \frac{N}{2}+1,\ldots,\frac{3N}{4}, \\ \left|\$_{ce}^{N,M}U_i^{j+1} - \left(\$^M U^{j+1}\right)(x_i)\right|, \\ \qquad i = \frac{N}{4}+1,\ldots,\frac{N}{2},\ \frac{3N}{4}+1,\ldots,N-1, \end{cases}$$

$$= \begin{cases} C\left[\varepsilon\int_{x_{i-1}}^{x_{i+1}}\left|\frac{\mathrm{d}^3U^{j+1}}{\mathrm{d}\xi^3}(\xi)\right|\mathrm{d}\xi \right. \\ \left. +h_i\int_{x_{i-1}}^{x_i}\left(\left|\frac{\mathrm{d}^3U^{j+1}}{\mathrm{d}\xi^3}(\xi)\right| + \left|\frac{\mathrm{d}^2U^{j+1}}{\mathrm{d}\xi^2}(\xi)\right| + \left|\frac{\mathrm{d}U^{j+1}}{\mathrm{d}\xi}(\xi)\right|\right)\mathrm{d}\xi\right], \\ \qquad i = 1,\ldots,\frac{N}{4},\ \frac{N}{2}+1,\ldots,\frac{3N}{4}, \\ C\left[h_i\int_{x_{i-1}}^{x_{i+1}}\left(\varepsilon\left|\frac{\mathrm{d}^4U^{j+1}}{\mathrm{d}\xi^4}(\xi)\right| + \left|\frac{\mathrm{d}^3U^{j+1}}{\mathrm{d}\xi^3}(\xi)\right|\right)\mathrm{d}\xi\right], \\ \qquad i = \frac{N}{4}+1,\ldots,\frac{N}{2},\ \frac{3N}{4}+1,\ldots,N-1. \end{cases}$$

To obtain parameter-uniform error estimate, we now decompose the discrete solution U_i^{j+1} into regular V_i^{j+1} and W_i^{j+1} components as

$$U_i^{j+1} = V_i^{j+1} + W_i^{j+1}.$$

The regular component V_i^{j+1} is the solution of the nonhomogeneous equation:

$$\begin{cases} \$_{hyb}^{N,M}V_i^{j+1} = R_i^{j+1}, \text{ for all } (x_i, t_{j+1}) \in \Omega^{N,M}, \\ V_0^{j+1} = V^{j+1}(x_0), \\ V_{\frac{N}{2}}^{j+1} = V^{j+1}(1^-),\ V_{\frac{N}{2}+1}^{j+1} = V^{j+1}(1^+), \\ V_N^{j+1} = V^{j+1}(x_N). \end{cases}$$

The singular component W_i^{j+1} satisfies the following homogeneous equation:

$$\begin{cases} \$_{hyb}^{N,M} W_i^{j+1} = 0, \quad \text{for all } i \in \left\{\{1,2,\ldots,N-1\} \,/\left\{\frac{N}{2}\right\}\right\}, \\ W_0^{j+1} = W^{j+1}(x_0), \quad W_N^{j+1} = W^{j+1}(x_N), \\ \$_{hyb}^{N,M}\left(V_{\frac{N}{2}}^{j+1} + W_{\frac{N}{2}}^{j+1}\right) = \$_{hyb}^{N,M}\left(V_{\frac{N}{2}+1}^{j+1} + W_{\frac{N}{2}+1}^{j+1}\right), \\ V_{\frac{N}{2}}^{j+1} + W_{\frac{N}{2}}^{j+1} = V_{\frac{N}{2}+1}^{j+1} + W_{\frac{N}{2}+1}^{j+1}. \end{cases}$$

Now, the error associated with the spatial discretization can be given by

$$e_i^{j+1} = U^{j+1}(x_i) - U_i^{j+1} = \left(V^{j+1}(x_i) - V_i^{j+1}\right) + \left(W^{j+1}(x_i) - W_i^{j+1}\right).$$

Theorem 1. Let $U^{j+1}(x_i)$ be the exact solution of the semi-discrete problem in (9) and let U_i^{j+1} be the discrete solution of the problem in (13) at each mesh point $(x_i, t_{j+1}) \in \overline{\Omega}^{N,M}$. Then under the assumptions in (15), we get

$$\left|U^{j+1}(x_i) - U_i^{j+1}\right| \leq \begin{cases} CN^{-2}, & i = 1,\ldots,\frac{N}{4},\ \frac{N}{2}+1,\ldots,\frac{3N}{4}, N, \\ CN^{-2}\ln^2 N, & i = \frac{N}{4}+1,\ldots,\frac{N}{2},\ \frac{3N}{4}+1,\ldots,N-1. \end{cases}$$

Proof. The proof follows a pattern similar to that shown in [26] for an ordinary differential equation with an IBC for the error at the points $x_i,\ i = 1,2,\ldots,N-1$.

For the error at the point $x_i = x_N$, we have

$$\begin{aligned} e_N^{j+1} &= U^{j+1}(x_N) - U_N^{j+1} \\ &= \varphi_r^{j+1}(x_N) + \varepsilon\int_{x_0}^{x_N} g(x)U^{j+1}(x)dx \\ &\quad - \left(\varphi_r^{j+1}(x_N) + \varepsilon\sum_{i=1}^{N}\frac{g_{i-1}U_{i-1}^{j+1} + 4g_iU_i^{j+1} + g_{i+1}U_{i+1}^{j+1}}{3}h_i\right) \\ &= \varepsilon\int_{x_0}^{x_1} g(x)U^{j+1}(x)dx - \varepsilon\frac{g_0U_0^{j+1} + 4g_1U_1^{j+1} + g_2U_2^{j+1}}{3}h_1 + \cdots \\ &\quad + \varepsilon\int_{x_{N-1}}^{x_N} g(x)U^{j+1}(x)dx - \varepsilon\frac{g_{N-1}U_{N-1}^{j+1} + 4g_NU_N^{j+1} + g_{N+1}U_{N+1}^{j+1}}{3}h_N \\ &= -\varepsilon\frac{1}{90}\left[h_1^4\left(gU^{j+1}\right)^{(iv)}(\xi_1) + h_2^4\left(gU^{j+1}\right)^{(iv)}(\xi_2)\right. \\ &\quad \left. + \cdots + h_N^4\left(gU^{j+1}\right)^{(iv)}(\xi_N)\right], \end{aligned}$$

where $x_{i-1} \leq \xi_i \leq x_i$ for $i = 1, 2, \ldots, N$. Also,

$$\begin{aligned}\left|U^{j+1}(x_N) - U_N^{j+1}\right| &= \Big|C_1\varepsilon\Big(h_1^4\left(gU^{j+1}\right)^{(iv)}(\xi_1) + h_2^4\left(gU^{j+1}\right)^{(iv)}(\xi_2)\\ &\qquad + \cdots + h_N^4\left(gU^{j+1}\right)^{(iv)}(\xi_N)\Big)\Big|\\ &\leq C_1\varepsilon\left(\max\left|\left(gU^{j+1}\right)^{(iv)}\right|\right)\left(h_1^4 + h_2^4 + \cdots + h_N^4\right).\end{aligned}$$

Now, using $\varepsilon \leq CN^{-1}$, $h_i \leq CN^{-1}$, and the result of Lemma 8 in the above inequality, we get

$$\left|U^{j+1}(x_N) - U_N^{j+1}\right| \leq CN^{-2}. \tag{19}$$

□

Theorem 2. Suppose that $u(x_i, t_{j+1})$ is the exact solution of the continuous problem in (1) and that U_i^{j+1} is the numerical solution of the problem in (13) at each mesh point $(x_i, t_{j+1}) \in \overline{\Omega}^{N,M}$. Then under the assumptions in (15), we get

$$\begin{aligned}&\left|u(x_i, t_{j+1}) - U_i^{j+1}\right|\\ &\leq \begin{cases} C\left(k^2 + N^{-2}\right), & i = 1, \ldots, \frac{N}{4}, \ \frac{N}{2} + 1, \ldots, \frac{3N}{4}, N,\\ C\left(k^2 + N^{-2}\ln^2 N\right), & i = \frac{N}{4} + 1, \ldots, \frac{N}{2}, \ \frac{3N}{4} + 1, \ldots, N-1.\end{cases}\end{aligned}$$

Proof. From the triangular inequality, we have

$$\begin{aligned}\left|u(x_i, t_{j+1}) - U_i^{j+1}\right| &= \left|u(x_i, t_{j+1}) - U^{j+1}(x_i) + U^{j+1}(x_i) - U_i^{j+1}\right|\\ &\leq \left|u(x_i, t_{j+1}) - U^{j+1}(x_i)\right| + \left|U^{j+1}(x_i) - U_i^{j+1}\right|.\end{aligned}$$

The designed result is followed from Lemma 7 and Theorem 1. □

5 Numerical illustrations and discussion

To validate the theoretical results obtained by the proposed method, we analyze two model examples involving singularly perturbed parabolic differential equations with IBCs. We use the double mesh principle to determine the maximum point-wise error $\left(E_\varepsilon^{N,M}\right)$ for the considered test examples as:

$$E_{\varepsilon}^{N,M} = \max_{i,j} \left| U_{i,j}^{N,M} - U_{i,j}^{2N,2M} \right|,$$

where $U_{i,j}^{N,M}$ and $U_{i,j}^{2N,2M}$ are the approximate solutions $i = 1,2,\ldots,N-1,\ j = 1,2,\ldots,M-1$ and $i = 1,2,\ldots,2N-1,\ j = 1,2,\ldots,2M-1$, respectively. The parameter-uniform error estimate $\left(E^{N,M}\right)$ is

$$E^{N,M} = \max_{\varepsilon} \left\{ E_{\varepsilon}^{N,M}. \right\}.$$

The formula for parameter-uniform rate of convergence $\left(r^{N,M}\right)$ is

$$r^{N,M} = \log_2 \left(\frac{E^{N,M}}{E^{2N,2M}} \right).$$

Example 1. Consider the SPPDDE:

$$\begin{cases} u_t(x,t) - \varepsilon u_{xx}(x,t) + (2 + x(2-x))u_x(x,t) + 3u(x,t) - u(x-1,t) \\ = 4xe^{-t}t^2, \\ u(x,0) = 0, & x \in [0,2], \\ u(x,t) = 0, & (x,t) \in [-1,0] \times [0,2], \\ u(2,t) = 0 + \dfrac{\varepsilon}{6} \displaystyle\int_0^2 u(x,t)dx, & (x,t) \in \{(2,t), \quad t \in [0,2]\}. \end{cases}$$

Example 2. Consider the SPPDDE:

$$\begin{cases} u_t(x,t) - \varepsilon u_{xx}(x,t) + 3u_x(x,t) + (x+10)u(x,t) - u(x-1,t) \\ = 2(1+x^2)t^2, \\ u(x,0) = 0, & x \in [0,2], \\ u(x,t) = t^2, & (x,t) \in [-1,0] \times [0,2], \\ u(2,t) = 0 + \dfrac{\varepsilon}{6} \displaystyle\int_0^2 x\sin(x)u(x,t)dx, & (x,t) \in \{(2,t), \quad t \in [0,2]\}. \end{cases}$$

The maximum absolute errors and parameter-uniform rate of convergence for Examples 1 and 2 obtained by the proposed hybrid method are presented in Tables 1 and 3, respectively. The tables demonstrate that for any number of mesh points N and M, the maximum absolute error becomes stable and uniform as ε gets closer to zero. This means that the method's convergence is independent of the perturbation parameter used. Our method produces more accurate results and has a better order of convergence than the method developed in [17]. Tables 2 and 4 also compare the rate of convergence of Examples 1 and 2 obtained by the proposed hybrid difference scheme with results from [25]. The rate of convergence obtained with our scheme is observed to be better than that obtained in [25].

Solution profiles when $\varepsilon = 2^{-4}$ for Examples 1 and 2 are shown in Figures 1 and 2, respectively. In Figures 3 and 4, we observe the behavior of the numerical solution, and for very small values of ε, the problem under consideration exhibits a strong boundary layer at $x = 2$. Figures 5 and 6 show the numerical solutions of Examples 1 and 2, respectively, in the neighborhood of $x = 1$. As the perturbation parameter ε approaches zero, it is possible to see an interior layer at $x = 1$. The maximum absolute errors' log-log plots are likewise shown in Figures 7 and 8. It supports the suggested hybrid numerical scheme's theoretical order of convergence. The maximum absolute error reduces with increasing mesh point count, as seen by the negative slope. Since this behavior is unaffected by the value of the perturbation parameter used, the convergence is ε-uniform. This is one of the primary findings that this study promises to prove.

Table 1: Values of $E_\varepsilon^{N,M}$, $E^{N,M}$, and $r^{N,M}$ for Example 1 with different values of ε and $N = M$.

$\downarrow \varepsilon$	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$	$N = 2^{10}$
10^{-2}	1.7153e-02	5.7880e-03	1.9294e-03	6.2634e-04	1.9777e-04	6.0662e-05
10^{-3}	1.6979e-02	5.7452e-03	1.9143e-03	6.2324e-04	1.9722e-04	6.0903e-05
10^{-4}	1.6962e-02	5.7412e-03	1.9128e-03	6.2292e-04	1.9711e-04	6.0858e-05
10^{-5}	1.6960e-02	5.7408e-03	1.9127e-03	6.2289e-04	1.9710e-04	6.0858e-05
10^{-6}	1.6960e-02	5.7407e-03	1.9127e-03	6.2289e-04	1.9710e-04	6.0858e-05
10^{-7}	1.6960e-02	5.7407e-03	1.9127e-03	6.2289e-04	1.9710e-04	6.0858e-05
10^{-8}	1.6960e-02	5.7407e-03	1.9127e-03	6.2289e-04	1.9710e-04	6.0858e-05
10^{-9}	1.6960e-02	5.7407e-03	1.9127e-03	6.2289e-04	1.9710e-04	6.0858e-05
10^{-10}	1.6960e-02	5.7407e-03	1.9127e-03	6.2289e-04	1.9710e-04	6.0858e-05
$E^{N,M}$	**1.7153e-02**	**5.7880e-03**	**1.9294e-03**	**6.2634e-04**	**1.9777e-04**	**6.0903e-05**
$r^{N,M}$	**1.5673**	**1.5849**	**1.6231**	**1.6631**	**1.6992**	**-**
$E^{N,M}$ in [17]	**3.896e-03**	**2.083e-03**	**1.078e-03**	**5.480e-04**	**2.764e-04**	**1.388e-04**
$r^{N,M}$ in [17]	**0.9033**	**0.9510**	**0.9754**	**0.9877**	**0.9938**	**-**

Table 2: Comparison of rate of convergences $\left(r^{N,M}\right)$ of Example 1 with with different values of ε and $N = M$.

$\downarrow \varepsilon$	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$	$N = 2^{10}$
Results of the present scheme						
2^{-1}	0.9785	0.9883	0.9939	0.9962	0.9982	-
2^{-3}	1.7957	2.1799	2.3488	2.0726	0.7899	-
2^{-5}	1.5818	1.5935	1.6535	1.7110	1.8013	-
2^{-7}	1.5663	1.5849	1.6215	1.6612	1.7002	-
2^{-9}	1.5637	1.5854	1.6192	1.6597	1.6949	-
2^{-11}	1.5631	1.5856	1.6188	1.6601	1.6954	-
Results in [25]						
2^{-1}	0.8587	0.9354	0.9666	0.9814	1.4995	1.5612
2^{-3}	0.8213	1.2650	1.3636	1.5621	1.4956	1.5516
2^{-5}	0.8698	0.9499	1.1262	1.3232	1.4918	1.5245
2^{-7}	0.8720	1.1774	1.0161	1.3473	1.4094	1.4126
2^{-9}	0.8483	1.2234	1.1792	1.3062	1.4071	1.4221
2^{-11}	0.7824	1.0547	1.2590	1.1249	1.3026	1.3861

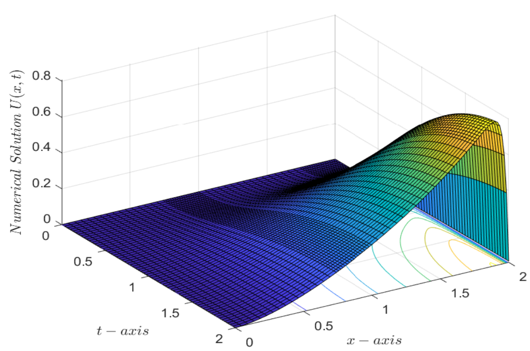

Figure 1: Numerical solution profiles of Example 1 when $\varepsilon = 2^{-4}$ and $N = M = 64$.

Table 3: Values of $E_{\varepsilon}^{N,M}$, $E^{N,M}$ and $r^{N,M}$ for Example 2 with different values of ε and $N = M$.

$\downarrow \varepsilon$	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$	$N = 2^{10}$
10^{-2}	5.3751e-02	1.8327e-02	6.2176e-03	2.0245e-03	6.4245e-04	1.9876e-04
10^{-3}	5.4504e-02	1.8627e-02	6.3046e-03	2.0507e-03	6.5205e-04	2.0152e-04
10^{-4}	5.4582e-02	1.8659e-02	6.3139e-03	2.0535e-03	6.5311e-04	2.0184e-04
10^{-5}	5.4590e-02	1.8662e-02	6.3148e-03	2.0538e-03	6.5322e-04	2.0187e-04
10^{-6}	5.4591e-02	1.8662e-02	6.3149e-03	2.0538e-03	6.5323e-04	2.0188e-04
10^{-7}	5.4591e-02	1.8662e-02	6.3149e-03	2.0538e-03	6.5320e-04	2.0188e-04
10^{-8}	5.4591e-02	1.8662e-02	6.3149e-03	2.0538e-03	6.5320e-04	2.0188e-04
10^{-9}	5.4591e-02	1.8662e-02	6.3149e-03	2.0538e-03	6.5320e-04	2.0188e-04
10^{-10}	5.4591e-02	1.8662e-02	6.3149e-03	2.0538e-03	6.5320e-04	2.0188e-04
$E^{N,M}$	**5.4591e-02**	**1.8662e-02**	**6.3149e-03**	**2.0538e-03**	**6.5323e-04**	**2.0188e-04**
$r^{N,M}$	**1.5486**	**1.5633**	**1.6205**	**1.6527**	**1.6941**	**-**
$E^{N,M}$ in [17]	**5.255e-02**	**2.871e-02**	**1.505e-02**	**7.709e-03**	**3.901e-03**	**1.962e-03**
$r^{N,M}$ in [17]	**0.8720**	**0.9315**	**0.9656**	**0.9827**	**0.9913**	**-**

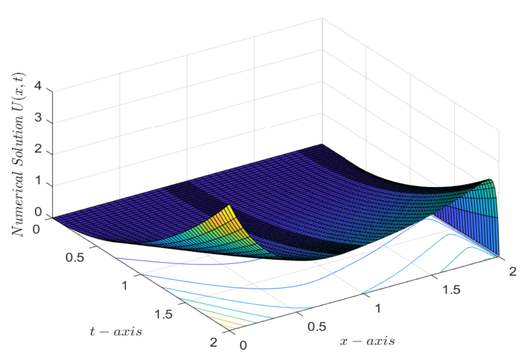

Figure 2: Numerical solution profiles of Example 2 when $\varepsilon = 2^{-4}$ and $N = M = 64$.

Table 4: Comparison of rate of convergences $\left(r^{N,M}\right)$ of Example 2 with different ε and $N = M$.

$\downarrow \varepsilon$	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$	$N = 2^{10}$
Results of the present scheme						
2^{-1}	1.0434	1.0233	1.0111	1.0054	1.0027	-
2^{-3}	1.5902	1.5310	1.6057	1.9093	1.4968	-
2^{-5}	1.5601	1.5535	1.6183	1.6616	1.7025	-
2^{-7}	1.5515	1.5604	1.6191	1.6550	1.6926	-
2^{-9}	1.5494	1.5626	1.6202	1.6533	1.6939	-
2^{-11}	1.5488	1.5631	1.6204	1.6529	1.6941	-
Results in [25]						
2^{-1}	1.5613	1.1790	1.2048	1.5771	1.6907	1.6973
2^{-3}	1.3598	1.2902	1.5089	1.4045	1.5471	1.6603
2^{-5}	1.2941	1.0056	1.2108	1.3531	1.4515	1.5658
2^{-7}	1.0595	1.2923	1.3043	1.1327	1.4157	1.5139
2^{-9}	1.0901	1.3009	1.3411	1.3475	1.4157	1.4652
2^{-11}	0.9815	1.2502	1.1645	1.3379	1.3663	1.4913

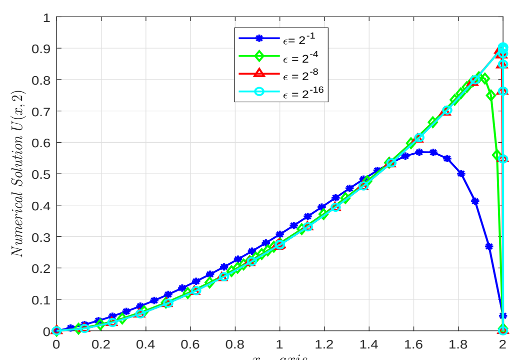

Figure 3: Line plots of Example 1 for $N = M = 32$ at $t = 2$.

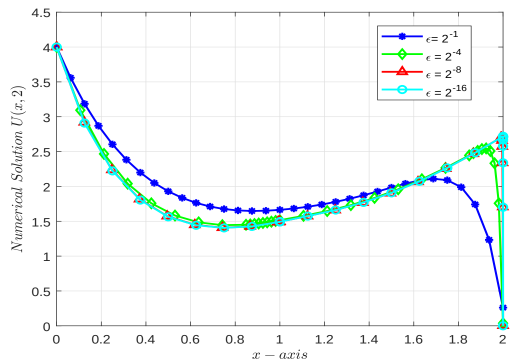

Figure 4: Line plots of Example 2 for $N = M = 32$ at $t = 2$.

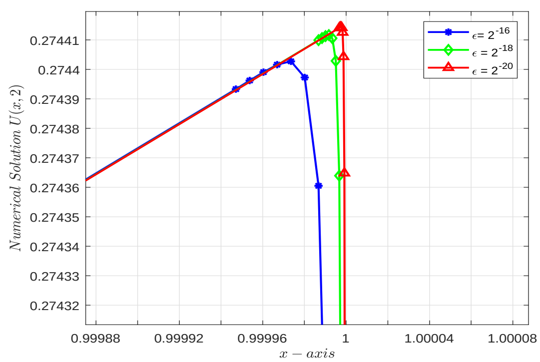

Figure 5: The interior layers for Example 1 at $x = 1$ and $t = 2$.

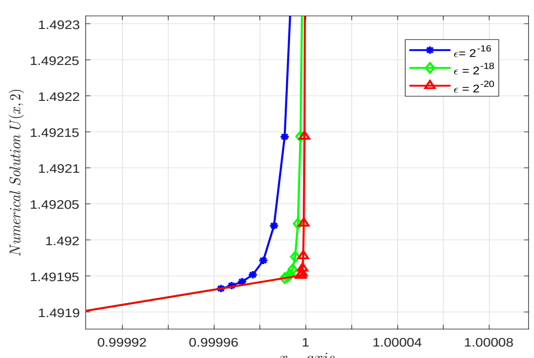

Figure 6: The interior layers for Example 2 at $x = 1$ and $t = 2$.

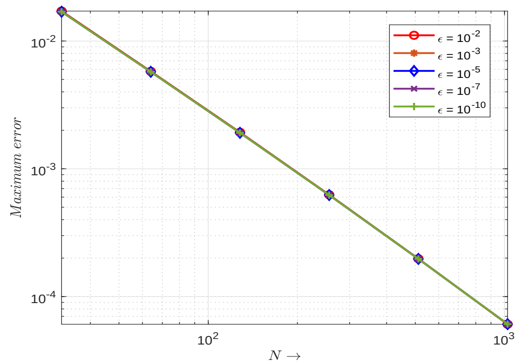

Figure 7: The log-log plot of the maximum absolute errors for Example 1.

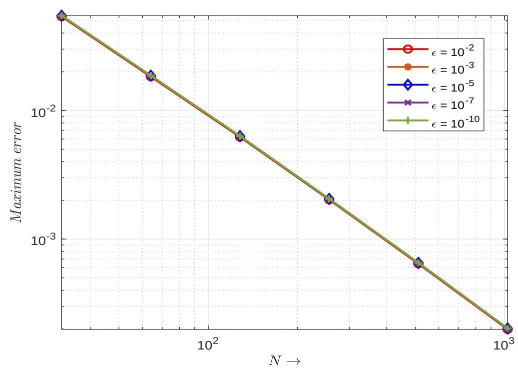

Figure 8: The log-log plot of the maximum absolute errors for Example 2.

6 Conclusions

In this work, a class of SPPDDEs with IBCs and weak interior layers was numerically solved. The Crank–Nicolson difference method for the temporal direction and a hybrid difference scheme composed of a midpoint upwind method outside the layer region and a central difference method in the layer region for the space variable were used to formulate the hybrid numerical method. The convergence analysis of the method revealed that it is of order $\mathcal{O}\left(k^2 + N^{-2}\ln^2 N\right)$, preserving a parameter-uniform convergence. Intensive numerical experimentation was conducted for different values of the perturbation parameter and the number of mesh elements to validate our method. Tables and graphs were used to present the computational findings. Theoretical expectations were satisfied by the convergence obtained in practice. The results showed that the method outperformed certain current numerical methods in the literature.

Acknowledgements

The authors would like to thank the anonymous referees for their useful remarks, which contributed to improving the quality of this article.

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