Iranian Journal of Numerical Analysis and Optimization
Vol. 13, No. 4, 2023, pp 578–603
https://doi.org/10.22067/ijnao.2023.79288.1192
https://ijnao.um.ac.ir/

How to cite this article
Research Article

A generalized form of the parametric spline methods of degree $(2k + 1)$ for solving a variety of two-point boundary value problems

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Abstract

In this paper, a high order accuracy method is developed for finding the approximate solution of two-point boundary value problems. The present approach is based on a special algorithm, taken from Pascal's triangle, for obtaining a generalized form of the parametric splines of degree $(2k + 1)$, $k = 1, 2, \dots,$ which has a lower computational cost and gives the better approximation. Some appropriate band matrices are used to obtain a matrix form for this algorithm.

The approximate solution converges to the exact solution of order $O(h^{4k})$, where k is a quantity related to the degree of parametric splines and the number of matrix bands that are applied in this paper. Some examples are given to illustrate the applicability of the method, and we compare the computed results with other existing known methods. It is observed that our approach produced better results.

AMS subject classifications (2020): Primary 45D05, Secondary 42C10, 65G99.

Keywords: Boundary value problems; Parametric spline; Band matrices; Pascal's triangle.

1 Introduction

Spline function is a piecewise polynomial satisfying certain conditions of the continuity of the function and its derivatives. In other words, a spline function

Received 20 October 2022; revised 25 March 2023; accepted 27 March 2023

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$S(x)$ of degree d is defined in a region $[a,b]$ such that there exists a mesh $\Delta = \{a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b\}$ with $h_i = x_i - x_{i-1}$ for $i = 1, 2, \ldots, n$. This function satisfies the following conditions:

(i) In each subinterval $[x_i, x_{i+1}]$, $i = 0, 1, \ldots, n-1$, $S(x)$ is a polynomial of degree d.

(ii) $S(x)$ and its first $(d-1)$ derivatives are continuous on $[a,b]$.

The spline's theory and application were thoroughly discussed by Ahlberg Nilson, and Walsh [1] and Greville [9]. So far, different types of spline methods, such as approximating, interpolating, and curve fitting functions, have been developed and used to solve a wide variety of differential equations; see, for example, [2, 3, 14, 15, 17, 20, 22, 26, 25, 24, 31] and references therein. One type of splines, considered in this paper, is the parametric splines developed to address some shortcomings of ordinary spline methods. These splines, depending on a parameter $\tau > 0$, are defined through the solution of a differential equation in each subinterval. The arbitrary constants are chosen to satisfy certain smoothness conditions at the joints. These splines reduce to polynomial splines (ordinary splines) as $\tau \to 0$. The exact form depends upon the manner in which the parameter is introduced. Therefore, different types of parametric splines with distinct convergence orders can be generated. Although, these methods obtained significant results, due to the lengthy calculations, no attempt was made to extend parametric splines of higher degrees. Note that using the word "degree", in this paper, for parametric splines is only for numbering and ordering them. It does not have the common meaning that is used for polynomials.

For the first time, this paper presents a general form of parametric splines with the degree $(2k+1)$, $k = 1, 2, \ldots$, which has a lower computational cost and a higher-order of convergence than the usual methods using parametric splines. Before going into details about the method, it seems necessary to review some of the fundamental properties and definitions of these parametric splines in the following subsection.

1.1 Parametric spline methods with the degree $(2k+1)$

For simplicity, it is assumed that the subintervals are of equal length, so $h = h_i = h_{i+1}$. The interval $[a,b]$ is divided into n equal subintervals using knots x_i and the partition $\Delta = \{a = x_0, x_1, \ldots, x_n = b\}$, where $x_i = x_0 + ih$, $h = \frac{b-a}{n}$ and n is a positive integer. The parametric spline function $S(x)$, with the degree $(2k+1)$, $k = 1, 2, \ldots$, is obtained in the subinterval $[x_{i-1}, x_i]$ by solving the following differential equation and determining the constants of integration:

$$S^{(2k)}(x) + \tau^2 S^{(2k-2)}(x) = (S^{(2k)}(x_i) + \tau^2 S^{(2k-2)}(x_i))(\frac{x - x_{i-1}}{h})$$

$$+(S^{(2k)}(x_{i-1})+\tau^2 S^{(2k-2)}(x_{i-1}))(\frac{x_i-x}{h}).$$

This function of class $C^{2k}[a,b]$ depends on a parameter τ and reduces to an ordinary spline function with the degree $(2k+1)$, as $\tau\to 0$. The continuity of its derivatives at the grid points, that is, $S_{i-1}^{(\nu)}(x_i)=S_i^{(\nu)}(x_i)$, $\nu=1,3,\dots,2k-1$, yields spline relations. Note that S_i is the spline function in the subintervals $[x_i,x_{i+1}]$. Using algebraic manipulation on these relations, a differential relation, called "consistency relation", is obtained in terms of u and its derivatives at knots. In the parametric spline methods, the approximate solution of a given boundary value problem (BVP) is determined by solving the system defined by this consistency relation.

Now, to further explain how parametric splines are used to solve equations, we consider a simple second order BVP as follows:

$$\begin{cases} u''(x)=f(x)+g(x)u(x), & x\in[a,b],\\ u(a)=\lambda, \\ u(b)=\gamma, \end{cases} \tag{1}$$

where λ and γ are finite real constants and the functions $f(x)$ and $g(x)$ are continuous on $[a,b]$. Such problems arise in the theory that describes the deflection of plates and a number of other scientific applications [10].

The consistency relation associated with (1) in spline methods, is in terms of u_i and u_i''. Note that $u_i=u(x_i)$ and $u_i''=u''(x_i)$. A system of linear algebraic equations is generated by substituting discretized (1) in the mentioned consistency relation. Finally, by solving this system, the approximate solution of (1) is obtained. One can observe that, for $k>1$, the number of equations in this system is less than the number of unknowns; see, for example, [2, 3, 4, 8, 7, 11, 13, 14, 16, 18, 20, 19, 21, 26, 25, 24, 30, 31] To obtain the unique solution of the system, more equations, called "end conditions or boundary formulas", are needed.

When k is a large number, two problems are encountered. First, the number of additional equations that must be defined to complete the aforementioned system increases. Second, as the value of k increases, so does the number of relations resulting from the derivative continuities of splines, and consequently, the combination of them becomes more difficult.

In this paper, we present a method that allows us to obtain a general form for the consistency relations of parametric spline methods of degree $(2k+1)$ without going through a lengthy and complex calculation process. For large values of k, it does not face the drawbacks mentioned above, because, in the proposed method, we do not need to obtain the spline function and use its continuity properties and derivatives directly. This means that we do not solve any differential equation. In fact, by providing a general pattern, both the consistency relation and the required additional equations are obtained without a need to generate many spline relations. Moreover, we transform the desired algorithm into a matrix form by defining proper band matrices, which gives us more insight into the method and facilitates convergence study. Furthermore, the convergence order of splines is improved by this general formulation.

We apply our method to (1), which is in terms of u and u''. However, the method can be applied to more complex models of (1), such as the nonlinear form or the system of these equations. It will be demonstrated in the numerical results section. Our method along with Newton–Raphson method is used to solve Bratu's problem (Example (2)) as a nonlinear equation. Also, our method is applied to solve problems such as Perturbed (Example (3)) and two-dimensional problems in the calculus of variations (Example (4)).

It should be noted that, if an equation other than (1) is considered, then the execution process of the method, such as generating a consistency relation and additional equations, will be changed. Because they are produced and defined according to the type of equation. In addition, while this paper focuses on parametric splines with the degree $(2k+1)$, the extending of our method can be probed for other types of splines as well, that is, nonpolynomials, ordinary splines, and parametric splines with the degree of $(2k)$, $k = 2, 3, 4, \dots$.

The outline of this paper is as follows. In section 2, a comprehensive description of the method is given. To demonstrate the efficiency and superiority of the presented method, we solve examples of linear, nonlinear, perturb, and system of two-point BVPs and compare the obtained results with the other quoted methods in section 3. Finally, some important concluding remarks are given in section 4.

2 Derivation of the method

In this section, we describe our method in detail. As mentioned previously, in the common form of the spline method, we need to generate a consistency relation proportional to the type of BVP that we are going to solve numerically. This can be time-consuming and even complicated due to the numerous calculations required, such as developing the spline function criterion, determining its coefficients, and computing successive derivatives. We provide a generalized form for the consistency relation of all parametric spline methods of degree $(2k+1)$, while solving (1), without the need for long calculations. To produce this relation, we find a specific pattern and then convert it to a matrix form by using only the properties of band matrices, especially, the following widely used a matrix C, which is $(n-1)\times(n-1)$-dimensional and evident in the majority of spline-based papers (see [8, 7, 13, 14, 18, 20, 19, 21, 31]):

$$(C)_{i,j} = \begin{cases} 2, & i = j, \\ -1, & |i-j| = 1, \\ 0, & otherwise. \end{cases} \tag{2}$$

To shed light on the above-mentioned issues, we begin subsection 2.1 by evaluating some samples of the consistency relations associated with the spline methods previously used by researchers and then identifying a general pattern for our desired consistency relation. In subsection 2.2, we define its matrix form. Subsections 2.3 and 2.4 are also dedicated to solving (1) and developing boundary formulas according to the contents of the previous two subsections.

2.1 The consistency relation

There are two types of coefficients in the consistency relations of spline methods: the coefficients of u and its derivatives. By studying the spline papers (see the references on page 3), we find that just the coefficients of u follow a certain pattern. The coefficients of u are of two kinds: known and unknown. The first type of coefficients exists in the consistency relations of splines with the degree $(2k+1)$ in solving the BVPs of order $(2k)$, while the second ones can be seen in the consistency relations of the same splines in solving the BVPs of order $2, 4, 6, \dots, 2k-2$, for $k = 2, 3, 4, \dots$ (i.e., for $k = 2$, we have spline with the degree 5 in solving BVP of order 2, for $k = 3$, we have spline with the degree 7 in solving BVP of order 2 and 4, for $k = 4$, we have spline with the degree 9 in solving BVP of order 2,4 and 6, etc.) We first establish a pattern for known coefficients and then extend this to unknown ones.

In the following, we highlight them in the sample format. For convenience, we consider the coefficients of u to be on the left side of the consistency relation and assume that the coefficients of derivatives of u be on the right side.

Sample 1 (k=1): *The left side of the consistency relation of spline method with the degree 3 in solving a BVP of order 2*:

For $i = 1, 2, \dots, n-1$:

$$\mathbf{1}u_{i-1} - \mathbf{2}u_i + \mathbf{1}u_{i+1} = \cdots .$$

One can see this relation in [15, 29].

Sample 2 (k=2): *The left side of the consistency relation of spline method with the degree 5 in solving a BVP of order 4*:

For $i = 2, 3, \dots, n-2$,

$$\mathbf{1}u_{i-2} - \mathbf{4}u_{i-1} + \mathbf{6}u_i - \mathbf{4}u_{i+1} + \mathbf{1}u_{i+2} = \cdots .$$

One can see this relation in [18].

Sample 3 (k=3): *The left side of the consistency relation of spline method with the degree 7 in solving a BVP of order 6*:

For $i = 3, 4, \dots, n-3$,

$$\mathbf{1}u_{i-3}-\mathbf{6}u_{i-2}+\mathbf{15}u_{i-1}-\mathbf{20}u_i+\mathbf{15}u_{i+1}-\mathbf{6}u_{i+2}+\mathbf{1}u_{i+3}=\cdots.$$

One can see this relation in [3, 11, 26].

Sample 4 (k=4): *The left side of the consistency relation of spline method with the degree 9 in solving a BVP of order 8*:

For $i=4,5,\dots,n-4$,

$$\begin{aligned}\mathbf{1}u_{i-4}-\mathbf{8}u_{i-3}+\mathbf{28}u_{i-2}-\mathbf{56}u_{i-1}+\mathbf{70}u_i\\ -\mathbf{56}u_{i+1}+\mathbf{28}u_{i+2}-\mathbf{8}u_{i+3}+\mathbf{1}u_{i+4}=\cdots.\end{aligned}$$

One can see this relation in references [2, 19].

Sample 5 (k=5): *The left side of the consistency relation of spline method with the degree 11 in solving a BVP of order 10*:

For $i=5,6,\dots,n-5$,

$$\begin{aligned}\mathbf{1}u_{i-5}-\mathbf{10}u_{i-4}+\mathbf{45}u_{i-3}-\mathbf{120}u_{i-2}+\mathbf{210}u_{i-1}-\mathbf{252}u_i\\ +\mathbf{210}u_{i+1}-\mathbf{120}u_{i+2}+\mathbf{45}u_{i+3}-\mathbf{10}u_{i+4}+\mathbf{1}u_{i+5}=\cdots.\end{aligned}$$

One can see this relation in [20, 24].

Sample 6 (k=6): *The left side of the consistency relation of spline method with the degree 13 in solving a BVP of order 12*:

For $i=6,7,\dots,n-6$,

$$\begin{aligned}\mathbf{1}u_{i-6}-\mathbf{12}u_{i-5}+\mathbf{66}u_{i-4}-\mathbf{220}u_{i-3}+\mathbf{495}u_{i-2}-\mathbf{792}u_{i-1}+\mathbf{924}u_i\\ -\mathbf{792}u_{i+1}+\mathbf{495}u_{i+2}-\mathbf{220}u_{i+3}+\mathbf{66}u_{i+4}-\mathbf{12}u_{i+5}+\mathbf{1}u_{i+6}=\cdots.\end{aligned}$$

One can see this relation in [25].

By considering the above coefficients, we find that, regardless of their sign, they are the same as the binomial coefficients or the entries in the rows of Pascal's triangle:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

	1	11	55	165	330	462	462	330	165	55	11	1
1	**12**	**66**	**220**	**495**	**792**	**924**	**792**	**495**	**220**	**66**	**12**	**1**

...

The correlation between the above-known coefficients of u and Pascal's triangle motivates us to find a similar correlation for the unknown ones that we will deal with in this paper.

After studying the references such as [8, 14, 15, 16, 21, 22, 29, 30, 31] (Previous studies have only investigated 3rd-, 5th-, 7th-, and 9th-degree spline methods. Higher degree splines have not been used yet), we find out to consider the initial form for the consistency relations of spline methods with the degree $(2k+1)$, $k = 2, 3, 4, \dots$ in solving a BVP of order two as follows:

$$\begin{aligned} &*(u_{i-k} + u_{i+k}) + *(u_{i-k+1} + u_{i+k-1}) + \cdots + *(u_{i-1} + u_{i+1}) \\ &+ * \, u_i = -h^2 \left(\beta_0 u_i'' + \beta_1(u_{i-1}'' + u_{i+1}'') + \cdots + \beta_k(u_{i-k}'' + u_{i+k}'')\right), \end{aligned}$$

where β_j's are the coefficients which will be determined numerically during the process of the method. We have displayed the vacancy of the unknown coefficients of u with $*$. We intend to find a pattern for them, inspired by Pascal's triangle. For this purpose, we first consider the parameters as $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_{k-1}$, for each k, and then we implement Pascal's algorithm for them. It should be mentioned that the numerical value of these coefficients for each k is independent of the values for other k, so it is preferable to write β_j's and α_j's with the exponent (k) as $\beta_j^{(k)}$ and $\alpha_j^{(k)}$. However, to reduce the complexity of the text, the exponent (k) could be removed from the coefficients without disturbing the whole. Indeed, the number of coefficients, which is indicated by an index in them, is affected by k.

Hence, we have the following Pascal's algorithm for $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_{k-1}$:

$$\begin{aligned} &\alpha_{k-1}, \alpha_{k-2}, \dots, \alpha_2, \alpha_1, \alpha_0, \alpha_1, \alpha_2, \dots, \alpha_{k-2}, \alpha_{k-1} \\ &\alpha_{k-1}, \alpha_{k-1} + \alpha_{k-2}, \dots, \alpha_1 + \alpha_0, \alpha_0 + \alpha_1, \dots, \alpha_{k-2} + \alpha_{k-1}, \alpha_{k-1} \\ &\alpha_{k-1}, 2\alpha_{k-1} + \alpha_{k-2}, \dots, \alpha_2 + 2\alpha_1 + \alpha_0, 2(\alpha_1 + \alpha_0), \alpha_2 + 2\alpha_1 + \alpha_0, \dots, 2\alpha_{k-1} + \alpha_{k-2}, \alpha_{k-1} \\ &\cdots . \end{aligned}$$

According to the consistency relations of the mentioned references, the third row of the above triangle, regardless of the signs, is the same as the coefficients of u in the consistency relation of the spline method of degree $(2k+1)$ for solving a BVP of second order as (1). We will demonstrate that the next rows of this triangle are the coefficients of u in the consistency relations of the spline methods of degree $(2k+1)$ for solving BVPs of higher orders (bigger than 2) in future research.

Thus, the following relation with the sign $(-1)^{q+p}$ for each phrase $\alpha_p u_{i\pm q}$, can be defined as the desired consistency relation for $k = 2, 3, 4, \dots$:

$$-\alpha_{k-1}(u_{i-k}+u_{i+k})+(2\alpha_{k-1}-\alpha_{k-2})(u_{i-k+1}+u_{i+k-1})$$
$$+(-\alpha_{k-1}+2\alpha_{k-2}-\alpha_{k-3})(u_{i-k+2}+u_{i+k-2})+\cdots+(2\alpha_0-2\alpha_1)u_i$$
$$=-h^2\left(\beta_0 u_i''+\sum_{j=1}^{k}\beta_j(u_{i-j}''+u_{i+j}'')\right),\quad i=k,k+1,\ldots,n-k. \qquad (3)$$

In the following, to verify the correctness of (3), we compare it to the consistency relations developed in related papers. A quick review shows that although the appearance of the coefficients in the consistency relations of available references slightly differs from what we propose, they are identical in content. In fact, the other authors have defined these coefficients in terms of parameter τ:

Equation (3) for $k = 2$, is the consistency relation of parametric spline with degree 5 (quintic spline):

$$-\alpha_1(u_{i-2}+u_{i+2})+(2\alpha_1-\alpha_0)(u_{i-1}+u_{i+1})+(-2\alpha_1+2\alpha_0)u_i$$
$$=-h^2\left[\beta_2(u_{i-2}''+u_{i+2}'')+\beta_1(u_{i-1}''+u_{i+1}'')+\beta_0 u_i''\right],\quad i=2,3,\ldots,n-2.$$

One can compare it to the consistency relations in [16, 21, 30, 31].

Equation (3) for $k = 3$, is the consistency relation of parametric spline with degree 7 (septic spline):

$$-\alpha_2(u_{i-3}+u_{i+3})+(2\alpha_2-\alpha_1)(u_{i-2}+u_{i+2})$$
$$+(-\alpha_0+2\alpha_1-\alpha_2)(u_{i-1}+u_{i+1})+(-2\alpha_1+2\alpha_0)u_i$$
$$=-h^2[\beta_3(u_{i-3}''+u_{i+3}'')+\beta_2(u_{i-2}''+u_{i+2}'')+\beta_1(u_{i-1}''+u_{i+1}'')+\beta_0 u_i''],$$
$$i=3,4,\ldots,n-3.$$

One can compare it to the consistency relations in [14].

Equation (3) for $k = 4$, is the consistency relation of parametric spline with degree 9 (nonic spline):

$$-\alpha_3(u_{i-4}+u_{i+4})+(2\alpha_3-\alpha_2)(u_{i-3}+u_{i+3})+(-\alpha_3+2\alpha_2-\alpha_1)(u_{i-2}+u_{i+2})$$
$$+(-\alpha_2+2\alpha_1-\alpha_0)(u_{i-1}+u_{i+1})+(-2\alpha_1+2\alpha_0)u_i$$
$$=-h^2[\beta_4(u_{i-4}''+u_{i+4}'')+\beta_3(u_{i-3}''+u_{i+3}'')+\beta_2(u_{i-2}''+u_{i+2}''),$$
$$+\beta_1(u_{i-1}''+u_{i+1}'')+\beta_0 u_i''] \qquad i=4,5,\ldots,n-4.$$

One can compare it to the consistency relations in [8, 7].

2.2 The matrix form

To demonstrate the accuracy of the above cases, namely, the validity of our claim about the correlation between the consistency relation of the parametric spline of degree $(2k+1)$ and Pascal's algorithm, we need to obtain a matrix form for (3), dependent on k. For this purpose, we use a matrix C and the following band matrices, which are $(n-1)\times(n-1)$-dimensional and play a major role in our method:

$$(A)_{i,j} = \begin{cases} \alpha_0, & i=j, \\ \alpha_1, & |i-j|=1, \\ \vdots & \\ \alpha_{k-1}, & |i-j|=k-1, \\ 0, & otherwise, \end{cases} \tag{4}$$

$$(B)_{i,j} = \begin{cases} \beta_0, & i=j, \\ \beta_1, & |i-j|=1, \\ \vdots & \\ \beta_{k}, & |i-j|=k, \\ 0, & otherwise. \end{cases}$$

Since the coefficients of the mentioned samples of consistency relations can be observed in the rows of the consecutive multiplication of matrix C by itself, it is expected that the coefficients of (3) is obtained from multiplying matrix A by the matrix C. Therefore, for each $k=1,2,\ldots,$ the general matrix form can be defined for (3) as follows:

$$(AC)_{k+1,*}V + h^2(B)_{k+1,*}V'' = 0, \quad i=k,k+1,\ldots,n-k. \tag{5}$$

where $_{k+1,*}$ denotes $(k+1)$th row of the above matrices. Proportional to k, we also define the column vectors V and V'' as follows:

$$V_j = \begin{cases} u_{i-k+(j-1)}, & 1\le j\le 2k+1, \\ 0, & 2k+1<j\le n-1, \end{cases}$$

and

$$V''_j = \begin{cases} u''_{i-k+(j-1)}, & 1\le j\le 2k+1, \\ 0, & 2k+1<j\le n-1, \end{cases}$$

where V_j and V''_j are the jth element of vectors V and V'', respectively.

Now, according to (5), we can obtain a set of parametric spline methods by changing k, without need to know the nature of the elements of matrices A and B. The numerical values of these elements are determined by expanding

(5) in Taylor's series around x_i. For this purpose, we first rewrite (5) in the following form:

$$
\begin{aligned}
&(AC)_{k+1,1}(u_{i-k}+u_{i+k})+(AC)_{k+1,2}(u_{i-k+1}+u_{i+k-1})+\dots\\
&+(AC)_{k+1,k+1}u_i+h^2((B)_{k+1,1}(u''_{i-k}+u''_{i+k})+(B)_{k+1,2}(u''_{i-k+1}+u''_{i+k-1})\\
&+\cdots+(B)_{k+1,k+1}u''_i)\\
&\quad=0, \qquad i=k,k+1,\dots,n-k.
\end{aligned}
$$

Note that in the above relation, we used the equalities $(AC)_{k+1,j}=(AC)_{k+1,2k+(2-j)}$ and $(B)_{k+1,j}=(B)_{k+1,2k+(2-j)}$, $1\le j\le k$. These are directly obtained from the definition of band matrices A, B and C.

Now the local truncation error T_i, corresponding to Taylor's series of (5) can be obtained as

$$
\begin{aligned}
T_i=&(AC)_{k+1,1}(u_i-khu'_i+\frac{(-kh)^2}{2!}u''_i+\cdots+u_i+khu'_i+\frac{(kh)^2}{2!}u''_i+\dots)\\
&+(AC)_{k+1,2}(u_i+(-k+1)hu'_i+\frac{((-k+1)h)^2}{2!}u''_i+\dots\\
&+u_i+(k-1)hu'_i+\frac{((k-1)h)^2}{2!}u''_i+\dots)+\cdots+(AC)_{k+1,k+1}u_i\\
&+h^2((B)_{k+1,1}(u''_i-khu_i^{(3)}+\frac{(-kh)^2}{2!}u_i^{(4)}+\cdots+u''_i+khu_i^{(3)}\\
&+\frac{(kh)^2}{2!}u_i^{(4)}+\dots)+(B)_{k+1,2}(u''_i+(-k+1)hu_i^{(3)}+\frac{((-k+1)h)^2}{2!}u_i^{(4)}\\
&+\cdots+u''_i+(k-1)hu_i^{(3)}+\frac{((k-1)h)^2}{2!}u_i^{(4)}+\dots)+\dots\\
&+(B)_{k+1,k+1}u''_i).
\end{aligned}
$$

On simplifying, we get

$$
\begin{aligned}
T_i=&\,(2(AC)_{k+1,1}+2(AC)_{k+1,2}+\cdots+2(AC)_{k+1,k}+(AC)_{k+1,k+1})u_i\\
&+(k^2(AC)_{k+1,1}+(k-1)^2(AC)_{k+1,2}+\cdots+(k-(k-1))^2(AC)_{k+1,k}\\
&+2(B)_{k+1,1}+2(B)_{k+1,2}+\cdots+2(B)_{k+1,k}+(B)_{k+1,k+1})h^2u''_i\\
&+\cdots.
\end{aligned}
$$

Equations (2) and (4) give us $2\sum_{j=1}^{k}(AC)_{k+1,j}+(AC)_{k+1,k+1}=0$. Therefore, the first term of the above truncation error, that is, the term with coefficient u_i, is removed. We can obtain classes of the method, namely several orders of convergence, by utilizing the above truncation error and eliminating the coefficients of the various powers of h for different choices of α_j's and β_j's. However, since our goal is to obtain the highest order of convergence, so we choose α_j's and β_j's that have the following conditions:

(i) They satisfy the following relation

$$\sum_{j=1}^{k}(k-(j-1))^2(AC)_{k+1,j}+2\sum_{j=1}^{k}(B)_{k+1,j}+(B)_{k+1,k+1}=0,$$

which eliminates the second term of the truncation error, that is, the term with coefficient h^2u_i''. The above relation can be written as follows:

$$\alpha_0+2\sum_{j=1}^{k-1}\alpha_j=\beta_0+2\sum_{j=1}^{k}\beta_j,$$

because from the definition of matrices A and B, we have:

$$\sum_{j=1}^{k}(k-(j-1))^2(AC)_{k+1,j}=-\alpha_0-2\sum_{j=1}^{k-1}\alpha_j,$$
$$2\sum_{j=1}^{k}(B)_{k+1,j}+(B)_{k+1,k+1}=\beta_0+2\sum_{j=1}^{k}\beta_j.$$

In accordance with the papers related to splines, the following relation is provided for α_0 and β_0:

$$\alpha_0+2\sum_{j=1}^{k-1}\alpha_j=1=\beta_0+2\sum_{j=1}^{k}\beta_j.$$

In other words,

$$\alpha_0=1-2\sum_{j=1}^{k-1}\alpha_j, \tag{6}$$

and

$$\beta_0=1-2\sum_{j=1}^{k}\beta_j. \tag{7}$$

For more details, the interested readers are advised to see [11, 16, 20, 19, 21, 22, 25] and other related papers.

(ii) The remaining unknown elements, namely the following $(2k-1)$ coefficients are chosen in such a way that the terms with coefficient $h^4u_i^{(4)}$ to $h^{4k}u_i^{(4k)}$, in the truncation error, are eliminated:

$$\alpha_1,\alpha_2,\ldots,\alpha_{k-1},\beta_1,\beta_2,\ldots,\beta_k.$$

Therefore the local truncation error associated with (5) is $O(h^{4k+2})$, $k=1,2,\ldots$. Consequently, the proposed method is convergent of order

$O(h^{4k})$, $k = 1, 2, \ldots$.

2.3 Spline solution

Now we discretize (1) as $u_i'' = f_i + g_i u_i,\ i = 1, 2, \ldots, n-1$, at the grid points, where $f_i = f(x_i),\ g_i = g(x_i)$. As a result, the vector V'' can be rewritten as

$$V_j'' = \begin{cases} f_{i-k+(j-1)} + g_{i-k+(j-1)} u_{i-k+(j-1)}, & 1 \le j \le 2k+1, \\ 0, & 2k+1 < j \le n-1. \end{cases}$$

If we substitute V'' in (5) for $i = k, k+1, \ldots, n-k$, then a system with $(n-2k+1)$ linear algebraic equations and $(n-1)$ unknowns as $u_1, u_2, \ldots, u_{n-1}$ is obtained. Note that $u_0 = \lambda$ and $u_n = \gamma$. It can be represented in the matrix form as follows:

$$\widetilde{D}U = \widetilde{R}, \tag{8}$$

where $U = [u_1, u_2, \ldots, u_{n-1}]^T$ is a column vector with $(n-1)$ elements. Moreover, $\widetilde{D}$ is a matrix of order $(n-2k+1) \times (n-1)$, indicated as follows:

$$\widetilde{D} = \overline{(AC)} + h^2 \overline{B} G,$$

with $G = diag(g_1, g_2, \ldots, g_{n-1})$. Matrices $\overline{AC}$ and $\overline{B}$ are also defined by omitting the $(k-1)$ first and last rows of matrices AC and B, respectively. In other words, for $1 \le i \le n-2k+1$ and $1 \le j \le n-1$, we have

$$(\overline{AC})_{i,j} = (AC)_{k+(i-1),j}, \quad (\overline{B})_{i,j} = (B)_{k+(i-1),j}.$$

Finally, the vector $\widetilde{R}$ is given by

$$\widetilde{R}_i = -h^2 \sum_{j=1}^{2k+1} (B)_{k+1,j} f_{j+i-2} + \begin{cases} -u_0 \left((AC)_{k+1,1} + h^2 g_0 (B)_{k+1,1} \right), & i = 1, \\ 0, & 2 \le i \le n-2k, \\ -u_n \left((AC)_{k+1,1} + h^2 g_n (B)_{k+1,1} \right), & i = n-2k+1. \end{cases}$$

2.4 Development of the boundary formulas

To obtain a unique solution for the system (8), we need $(2(k-1))$ more equations; thus, we define them in the following form:

$$\sum_{j=0}^{k+i+1} \widehat{\alpha}_j^i u_j + h^2 \sum_{j=0}^{4k-1} \widehat{\beta}_j^i u_j'' = 0, \qquad i = 1, 2, \ldots, k-1,$$
$$\sum_{j=0}^{k+i+1} \widehat{\alpha}_{n-j}^i u_{n-j} + h^2 \sum_{j=0}^{4k-1} \widehat{\beta}_{n-j}^i u_{n-j}'' = 0, \quad i = n-(k-1), \ldots, n-2, n-1.$$

In order to use the band matrices in the new system, that is, system (8) along with the above equations, we use the following replacements for $j = 1, 2, \ldots, i+k$:

$$\widehat{\alpha}_j^i = (AC)_{i,j}, \qquad i = 1, 2, \ldots, k-1,$$
$$\widehat{\alpha}_{n-j}^i = (AC)_{i,n-j}, \qquad i = n-(k-1), \ldots, n-2, n-1.$$

These replacements simplify the convergence analysis of the method. The other unknown coefficients, $\widehat{\beta}_j^i$'s and $\widehat{\beta}_{n-j}^i$'s, are determined by considering the local truncation error of order $O(h^{4k+2})$ for the added equations and using Taylor's expansion of these equations for $i = 1, 2, \ldots, k-1$ around x_0 (or for $i = n-(k-1), \ldots, n-2, n-1$ around x_n).

On the other hand, from (2) and (4), we have $(AC)_{i,j} = (AC)_{n-i,n-j}$. Consequently, $\widehat{\alpha}_j^i = \widehat{\alpha}_{n-j}^{n-i}$ and $\widehat{\beta}_j^i = \widehat{\beta}_{n-j}^{n-i}$. Considering these justifications, system (8) is converted to the following system:

$$DU = R, \tag{9}$$

with

$$D = AC + h^2 \widehat{B} G, \tag{10}$$

where the matrix $\widehat{B}$ is $(n-1) \times (n-1)$-dimensional as the following form:

$$(\widehat{B})_{i,j} = \begin{cases} (\widehat{B})_{n-i,n-j} = \widehat{\beta}_j^i, & 1 \le i \le k-1, \quad 1 \le j \le 4k-1 \\ (B)_{i,j}, & \textit{for other } i, j. \end{cases} \tag{11}$$

For the column vector R with $(n-1)$ elements, we have

$$R_i = \begin{cases} -u_0(\widehat{\alpha}_0^i + h^2 g_0 \widehat{\beta}_0^i) - h^2(\widehat{\beta}_0^i f_0 + \sum_{j=1}^{4k-1} (\widehat{B})_{i,j} f_j), & 1 \le i \le k-1, \\ \widetilde{R}_{i-(k-1)}, & k \le i \le n-k, \\ -u_n(\widehat{\alpha}_n^i + h^2 g_n \widehat{\beta}_n^i) - h^2(\widehat{\beta}_n^i f_n + \sum_{j=1}^{4k-1} (\widehat{B})_{i,j} f_{n-j}), & n-k+1 \le i \le n-1. \end{cases}$$

Finally, by solving the system (9), we obtain the solution vector U, the elements of which are approximately equal to the solution of (1) at nodes $x_1, x_2, \ldots, x_{n-1}$.

3 Numerical results

In order to test the viability of the proposed method and to demonstrate its convergence computationally, some BVPs including the cases of linear, nonlinear, perturbed, and system are considered. We measure the accuracy in the discrete maximum norm

$$\|E\| = \|U - U_{exact}\| = \max_{1\leq i\leq n-1} |U_i - (U_{exact})_i|,$$

and the convergence rate for linear and perturbed cases

$$CR = \log_2\left(\frac{\|E^n\|}{\|E^{2n}\|}\right),$$

where $\|E^n\|$ and $\|E^{2n}\|$ are the maximum absolute errors on n and $2n$ grid points, respectively. The results are listed in tables for different choices of n and k. From the tables, we see that the quantity CR is close to $4k$ for each k. In other words, by reducing the step size from h to $\frac{h}{l}$, the observed errors are approximately reduced by a factor $(\frac{1}{l})^{4k}$ verifying the convergence order of the presented method, that is, $O(h^{4k})$, $k = 1, 2, \ldots$. For example, in the rows related to $n = 16$ and $n = 64$ from Table (1), it is observed that the maximum absolute error is decreased by a factor $(\frac{1}{4})^{4k}$ when $n = 16$ is varied to $n = 64$. Namely, we have

$$for\ k = 1: \qquad (6.72 * (10^{-9})) * (\frac{1}{4})^{4*1} \simeq 2.63 * (10^{-11}),$$
$$for\ k = 2: \qquad (3.12 * (10^{-15})) * (\frac{1}{4})^{4*2} \simeq 8.09 * (10^{-21}),$$

The outcomes indicate that our presented method produces more accurate results in comparison with those obtained by other methods. It should be mentioned that the computations associated with the examples in this paper were performed using Mathematica 8.0. Run applications were done in just a few minutes. The numerical results in tables were written just for some values of k, but we could solve the examples for other values and the results are quite satisfactory as was already expected.

In addition, we have used some plots to illustrate the behavior of the numerical solutions. Furthermore, since $\widehat{B}$ is $(n-1)\times(n-1)$-dimensional, in (11) we should have $4k - 1 \leq n - 1$ and $k - 1 \leq n - 1$, which results in $4k \leq n$. This can be seen in the results tables.

Example 1. We consider the following linear two-point BVP:

$$u''(x) - u(x) = x^2 - 2,$$

$$u(0) = 0, \quad u(1) = 1.$$

The exact solution is

$$u(x) = 2\left(\frac{\sinh(x)}{\sinh(1)}\right) - x^2.$$

The corresponding maximum absolute errors and convergence rates in our computed solutions are listed in Tables (1) and (2), respectively. Rashidinia, Jalilian, and Mohammadi [21] solved this problem by using the nonpolynomial quintic spline. Although their method is similar to ours for $k = 2$, the only difference is that they used a lower order of convergence of the method, see Table (3).

The graph of the exact and approximate solutions of Example (1) for $n = 20$ and $k = 1, 2, 3, 4, 5$ is depicted in Figure (1).

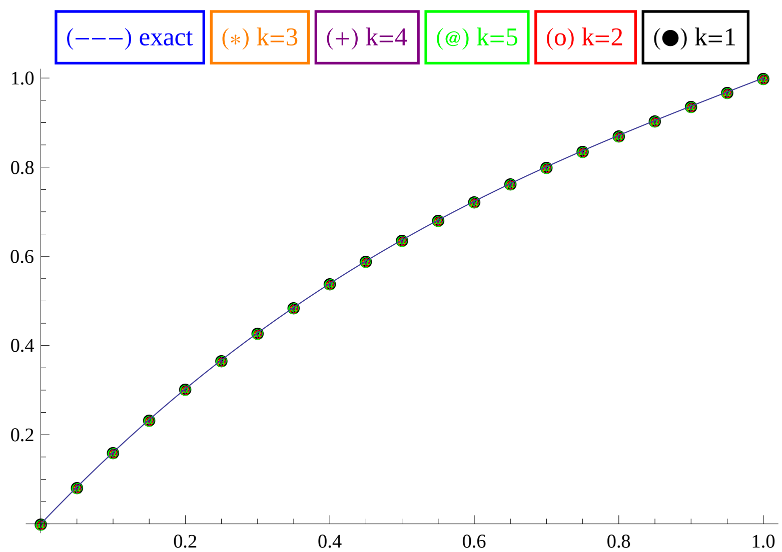

Figure 1: Plot of the exact and numerical solutions of Example (1) for $k = 1, 2, 3, 4, 5$ and $n = 20$

.

Example 2. We consider the following nonlinear BVP, the classical Bratu's problem:

$$u''(x) + \eta e^{u(x)} = 0,$$
$$u(0) = u(1) = 0,$$

where $\eta > 0$. The exact solution is

$$u(x) = -2\ln\left(\frac{\cosh((x-\frac{1}{2})\frac{\theta}{2})}{\cosh(\frac{\theta}{4})}\right),$$

where $\theta = \sqrt{2\eta}\cosh(\frac{\theta}{4})$. The Bratu's problem has zero, one, or two solutions when $\eta > \eta_c$, $\eta = \eta_c$, and $\eta < \eta_c$, respectively, where the critical value η_c satisfies the equation $1 = \frac{1}{4}\sqrt{2\eta_c}\sinh(\frac{\theta}{4})$ and it was evaluated in [5, 12] that the critical value η_c is given by $\eta_c = 3.513830719$.

We have solved this example for $\eta = 1, 2$, and 3.51 using our method with different values of k and tabulated the results in Tables (4), (5), and (6). Note that, in this example, we have used the Newton–Raphson algorithm, just with two iterations. Thus, there are errors related to the initial conjecture and the number of iterations, in addition to the error of our method. Tables (7) and (8) contain the comparison of our results and the results in [6, 13, 31]. The method in [31] is the same as our method for $k = 2$ with a lower order of convergence. Note that, the mentioned references have presented the outputs of their methods only for $n = 10$, so for the sake of comparison, in Table (7), we have to show the results of our method only for this value of n. We can use both $k = 1$ and $k = 2$ (according to condition $4k \leq n$), but $k = 2$ provides better results. Thus, we display its maximum absolute error.

Figure (2) plots the graphs of analytic and approximate solutions of Example (2) for $n = 20$, $\eta = 1$, and $k = 1, 2, 3, 4, 5$.

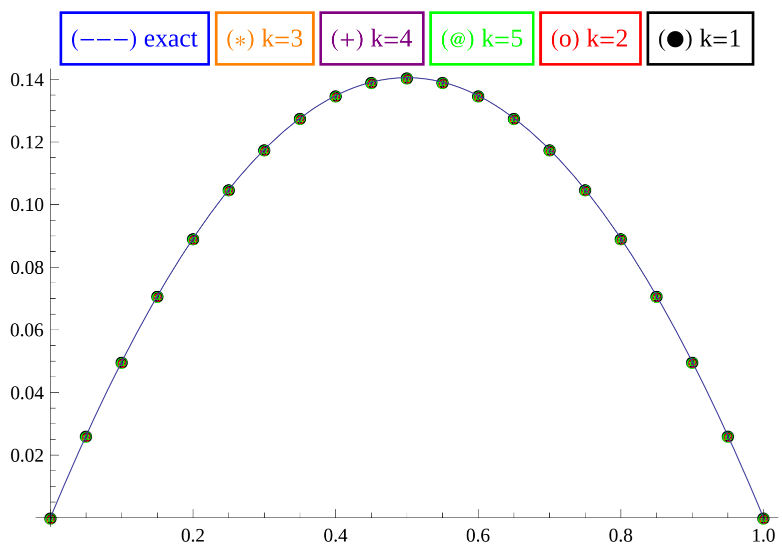

Figure 2: Plot of the exact and numerical solutions of Example (2) for $k = 1, 2, 3, 4, 5$ and $n = 20$, $\eta = 1$

.

Example 3. We consider the following singularly Perturbed BVP:

$$\epsilon u''(x) = u(x) + \cos^2(\pi x) + 2\epsilon\pi^2 \cos(2\pi x),$$
$$u(0) = u(1) = 0.$$

The exact solution is given by

$$u(x) = \frac{\exp(\frac{-(1-x)}{\sqrt{\epsilon}}) + \exp(\frac{-x}{\sqrt{\epsilon}})}{1 + \exp(\frac{-1}{\sqrt{\epsilon}})} - \cos^2(\pi x).$$

The maximum absolute errors and convergence rates for $\epsilon = \frac{1}{16}$ are tabulated in Tables (9) and (10), respectively. The results for this example from [4, 8, 17, 22, 27] are listed in Table (11). Note that the method used in [4, 22] is the same as our method for $k = 2$, but with a lower order of convergence. The results of [8] are also the same as the results of our method for $k = 4$.

We observe from Figure (3) that the graphic of the approximate solution of Example (3) for $n = 20$ and $k = 1, 2, 3, 4, 5$ coincides with the graphic of the exact solution.

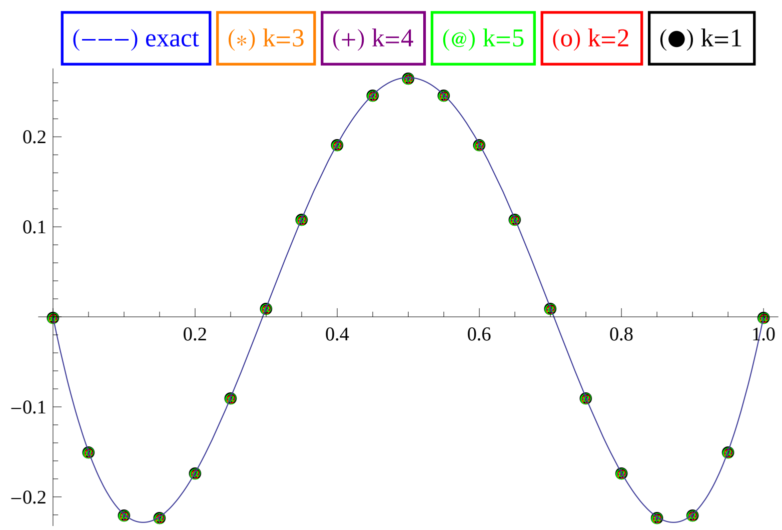

Figure 3: Plot of the exact and numerical solutions of Example (3) for $k = 1, 2, 3, 4, 5$ and $n = 20$

.

Example 4. We consider a BVP in calculus of variations, that is, the problem of finding the extremal of the functional [23]:

$$J[u_I(x), u_{II}(x)] = \int_0^{\frac{\pi}{2}} \left(u_I'^2(x) + u_{II}'^2(x) + 2u_I(x)u_{II}(x)\right) dx,$$

with boundary conditions

$$\begin{cases} u_I(0) = 0, & u_I(\frac{\pi}{2}) = 1, \\ u_{II}(0) = 0, & u_{II}(\frac{\pi}{2}) = -1. \end{cases}$$

The exact solution is given by $u_I(x) = -u_{II}(x) = \sin(x)$. For this problem, the corresponding Euler–Lagrange equations are

$$\begin{cases} u_I''(x) = u_{II}(x), \\ u_{II}''(x) = u_I(x), \end{cases}$$

that is, a system of equations such as (1). It should be mentioned that in this example, we compute $\|E_{u_I}\|$ and $\|E_{u_{II}}\|$, but one of them is displayed in Table (12), because, the value of both norms is the same. This example has already been solved by using cubic [29] and quintic [30] parametric spline methods, namely, the same as our method for $k = 1$ and $k = 2$ (with a lower convergence order), respectively. The sinc-Galerkin method [28] is also the other method that has been used for solving the above problem. The mentioned references have provided the numerical results only for $n = 5, 10, 20, \ldots, 50$. To make a proper comparison with these methods, we have shown our results only for these values, in Table (13). We have selected k's that apply to condition $4k \leq n$ and give the best outputs. For instance, for $n = 50$, we could display the numerical results of our method for $k = 1, 2, 3, \ldots, 12$, but since $k = 12$ gives the best result, we display its maximum absolute error.

The numerical results of Example (4) for $n = 20$ and $k = 1, 2, 3, 4, 5$ are plotted in Figure (4). Note that we have displayed this graph just for $u_I(x)$. Similarly, it can be shown for $u_{II}(x)$.

4 Conclusion

A long process is needed to obtain the differential relations of spline-based methods. Therefore, it is important to use a method that has considerably less computational effort with high accuracy and improves the spline methods. In this paper, for the first time, a generalized form of methods based on parametric splines of degree $(2k+1)$, $k = 1, 2, \ldots,$ was introduced that has all the mentioned properties. A very good accuracy of this method was demonstrated for solving some linear, nonlinear, perturbed, and system of BVPs. We mention some advantages of our method in the following remarks.

Remark 1. It is necessary to obtain the criterion of spline function in the spline methods. For instance, in the parametric spline method, this criterion

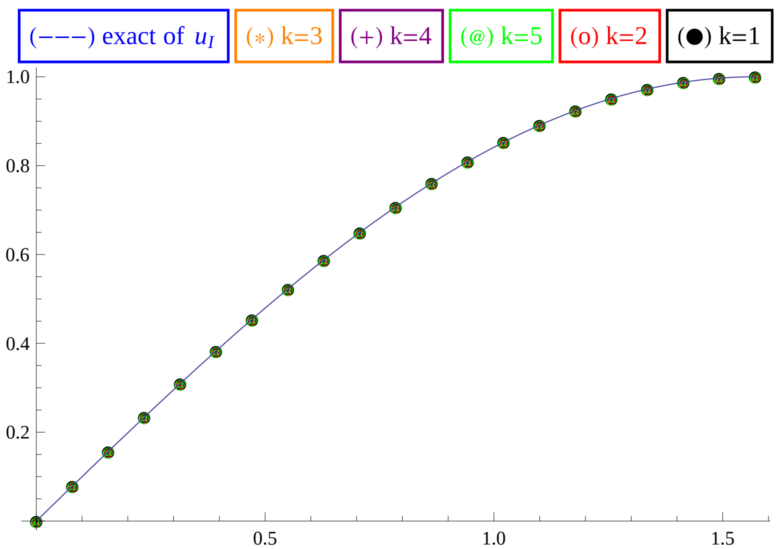

Figure 4: Plot of the exact and numerical solutions of Example (4) for $k = 1, 2, 3, 4, 5$ and $n = 20$

.

Table 1: Maximum absolute errors for Example (1).

n	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7 \dots$
4	1.66E−6	–	–	–	–	–	–
8	1.07E−7	2.24E−12	–	–	–	–	–
12	2.13E−8	4.92E−14	1.13E−18	–	–	–	–
16	6.72E−9	3.12E−15	2.43E−20	2.10E−25	–	–	–
20	2.76E−9	3.62E−16	1.19E−21	4.44E−27	1.74E−32	–	–
24	1.33E−9	6.17E−17	9.97E−23	1.84E−28	3.64E−34	7.42E−40	–
28	7.18E−10	1.37E−17	1.21E−23	1.23E−29	1.34E−35	1.52E−41	1.77E−47
⋮							
64	2.63E−11	8.09E−21	1.34E−28	5.35E−36	2.28E−43	1.02E−50	4.75E−58
128	1.64E−12	2.61E−23	8.75E−33	2.22E−41	6.07E−50	1.74E−58	5.16E−67
256	1.02E−13	9.71E−26	5.52E−37	8.84E−47	1.52E−56	2.76E−66	5.18E−76
512	6.43E−15	3.74E−28	4.03E−41	3.44E−52	3.73E−63	4.25E−74	5.01E−85
1024	4.02E−16	1.45E−30	6.27E−45	1.32E−57	9.02E−70	6.43E−82	4.75E−94
⋮							

Table 2: Convergence Rates, Example (1).

n	$k=1$	$k=2$	$k=3$	$k=4$	$k=5\ldots$
64	4.00	8.27	13.90	17.87	21.84
128	4.00	8.07	13.95	17.93	21.92
256	3.98	8.02	13.74	17.97	22.10
512	3.99	8.01	12.65	17.99	21.83

Table 3: Maximum absolute errors in [21] for Example (1).

n	Second-order [21]	Fourth-order[21]	Sixth-order[21]
8	1.09E−4	5.22E−8	8.75E−11
16	3.06E−5	2.31E−9	5.74E−13
32	8.11E−6	1.34E−10	2.30E−14
64	2.09E−6	8.42E−12	3.68E−14

Table 4: Maximum absolute errors for Example (2), $\eta=1$.

n	$k=1$	$k=2$	$k=3$	$k=4$	$k=5\ldots$
4	1.17E−5	–	–	–	–
8	7.23E−7	1.78E−9	–	–	–
12	1.42E−7	1.72E−11	8.37E−13	–	–
16	4.50E−8	5.50E−13	6.32E−15	5.89E−16	–
20	1.84E−8	4.20E−14	2.94E−16	3.08E−16	3.98E−16
24	8.89E−9	6.63E−15	2.22E−16	2.77E−16	2.91E−16
28	4.79E−9	1.38E−15	4.16E−16	4.99E−16	2.49E−16
32	2.81E−9	1.41E−15	5.82E−16	4.30E−16	4.44E−16
36	1.75E−9	3.33E−16	1.05E−15	6.10E−16	1.38E−16
⋮					

Table 5: Maximum absolute errors for Example (2), $\eta=2$.

n	$k=1$	$k=2$	$k=3$	$k=4$	$k=5\ldots$
4	1.58E−4	–	–	–	–
8	9.55E−6	1.22E−7	–	–	–
12	1.87E−6	4.09E−10	3.32E−10	–	–
16	5.92E−7	8.14E−11	2.38E−12	1.07E−12	–
20	2.42E−7	1.09E−11	3.06E−13	1.78E−14	3.88E−15
⋮					

Table 6: Maximum absolute errors for Example (2), $\eta = 3.51$.

n	$k=1$	$k=2$	$k=3$	$k=4$	$k=5\ldots$
4	5.55E−1	−	−	−	−
8	2.88E−3	2.29E−4	−	−	−
12	5.51E−4	2.46E−5	9.43E−6	−	−
16	1.73E−4	6.92E−7	9.35E−7	4.08E−7	−
20	7.08E−5	1.18E−8	4.85E−9	4.00E−8	1.74E−8
⋮					

Table 7: Comparison of $\|E\|$ for Example (2), $\eta = 1$ $n = 10$.

n	Our method for $k=2$	Method[31]	Method[6]
10	1.44E−10	5.87E−10	8.89E−6

Table 8: Comparison of $\|E\|$ for Example (2) with $\eta = 1, 2$ and 3.51.

	Our method			Method[13]		
n	$\eta=1$	$\eta=2$	$\eta=3.51$	$\eta=1$	$\eta=2$	$\eta=3.51$
8	1.78E−9(k= 2)	1.22E−7(k= 2)	2.29E−4(k= 2)	5.64E−9	4.53E−8	3.51E−5
16	5.89E−16(k= 4)	1.07E−12(k= 4)	4.08E−7(k= 4)	4.66E−11	1.76E−9	1.45E−7
32	4.30E−16(k= 4)	6.71E−15(k= 3)	9.32E−10(k= 2)	8.33E−13	2.13E−11	1.02E−9
64	4.71E−16(k= 6)	3.60E−15(k= 3)	1.37E−9(k= 2)	9.21E−15	2.87E−13	1.48E−11
128	2.22E−15(k= 6)	4.99E−16(k= 6)	1.37E−9(k= 2)	−	2.47E−14	1.58E−13

Table 9: Maximum absolute errors for Example (3), $\epsilon = \frac{1}{16}$.

n	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7\ldots$
4	1.15E−2	−	−	−	−	−	−
8	6.65E−4	4.45E−5	−	−	−	−	−
12	1.29E−4	7.49E−8	5.02E−8	−	−	−	−
16	4.07E−5	2.73E−8	4.75E−10	1.72E−11	−	−	−
20	1.66E−5	5.10E−9	1.50E−12	2.17E−13	2.45E−15	−	−
24	8.01E−6	1.05E−9	1.03E−12	3.93E−15	3.53E−17	1.73E−19	−
28	4.32E−6	2.58E−10	2.17E−13	1.51E−17	7.65E−19	2.67E−21	6.76E−24
⋮							
64	1.58E−7	9.10E−14	4.72E−18	2.66E−22	1.51E−26	8.18E−31	3.78E−35
128	9.87E−9	2.26E−16	3.28E−22	1.28E−27	5.36E−33	2.31E−38	1.02E−43
256	6.17E−10	9.47E−19	2.08E−26	5.21E−33	1.40E−39	3.95E−46	1.14E−52
512	3.86E−11	3.76E−21	1.29E−30	2.02E−38	3.43E−46	6.10E−54	1.12E−61
1024	2.41E−12	1.47E−23	7.92E−35	7.78E−44	8.25E−53	9.18E−62	1.05E−70
⋮							

Table 10: Convergence Rates, Example (3).

n	$k=1$	$k=2$	$k=3$	$k=4$	$k=5\ldots$
64	4.00	8.65	13.81	17.66	21.42
128	3.99	7.89	13.94	17.90	21.86
256	3.99	7.97	13.97	17.97	21.96
512	4.00	7.99	13.99	17.98	21.98

Table 11: Comparison of $\|E\|$ for Example (3), $\epsilon=\frac{1}{16}$.

n	Method[8]	Method[17]	Method[4]	Method[22]	Method[27]
16	1.72E−11	1.22E−6	1.57E−5	4.07E−5	1.20E−4
32	1.52E−17	6.45E−9	8.79E−7	2.53E−6	7.47E−6
64	2.66E−22	3.40E−11	5.32E−8	1.58E−7	4.67E−7
128	1.28E−27	1.03E−12	3.30E−9	9.87E−9	2.90E−8

Table 12: Maximum absolute errors for Example (4).

n	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7\ldots$
4	2.76E−5	–	–	–	–	–	–
8	1.72E−6	1.80E−10	–	–	–	–	–
12	3.41E−7	3.37E−12	5.75E−16	–	–	–	–
16	1.08E−7	1.97E−13	1.03E−17	6.79E−22	–	–	–
20	4.44E−8	2.32E−14	4.72E−19	1.22E−23	3.53E−28	–	–
24	2.14E−8	4.16E−15	3.74E−20	4.60E−25	6.41E−30	9.33E−35	–
28	1.15E−8	1.00E−15	4.37E−21	2.92E−26	2.16E−31	1.69E−36	1.37E−41
⋮							
512	1.03E−13	3.66E−26	1.97E−38	5.96E−49	3.95E−59	2.75E−69	1.98E−79
⋮							

Table 13: Comparison of $\|E\|$ for Example (4).

n	Our method	Method[30]	Method[29]	Method[28]
5	1.12E−5(k= 1)	–	1.12E−5	–
10	2.02E−11(k= 2)	6.70E−10	7.05E−7	2.72E−4
20	3.53E−28(k= 5)	7.07E−12	4.44E−8	8.69E−6
30	1.74E−42(k= 7)	8.10E−13	8.77E−9	5.65E−7
40	2.61E−63(k= 10)	1.55E−13	2.78E−9	5.47E−8
50	9.59E−67(k= 12)	4.21E−14	–	6.93E−9

is obtained by solving a special ordinary differential equation. Or nonpolynomial spline is a function with unknown coefficients that should be determined accordingly. In all these cases, some time-consuming calculations are needed, while the criterion and coefficients of no function are required in our method.

Remark 2. The continuity property of spline and its derivatives in grid points plays a major role in all of the spline methods. One can use this property to obtain the required spline relations. In this paper, instead of using the properties of spline directly, to save time and reduce calculations, we derived the consistency relations from a special algorithm and then obtained its matrix form by defining some band matrices.

Remark 3. The approximate solution converges to the exact solution of order $O(h^{4k})$. It follows that $\|E\| \to 0$ as $h \to 0$. The convergence occurs more quickly when k is a larger number. Indeed, the order of error is not fixed and decreases by increasing the value of k. It is regarded as one of our method's advantages. In addition, since we have $h = \frac{b-a}{n}$, it concludes that $\|E\| = O((\frac{b-a}{n})^{4k})$. This indicates that an increasing k is more effective than that n in reducing error. It can be seen in the tables containing numerical results.

Remark 4. We claim that the proposed method can be applied to solve other similar differential equations in particular as $u^{(2m)}(x) = f(x) + g(x)u(x)$, where m indicates a positive integer. This will be considered in our future research. Moreover, because of the adequate flexibility and expandability of this method, there is a possibility of achieving the generalized form of the methods based on splines with the degree $(2k)$, $k = 1, 2, \dots$.

Acknowledgements

The authors are grateful to the anonymous referees for their time, effort, and extensive comments which improve the quality of the presentation of the paper.

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How to cite this article

Sarvari. Z, A generalized form of the parametric spline methods of degree $(2k + 1)$ for solving a variety of two-point boundary value problems. *Iran. J. Numer. Anal. Optim.*, 2023; 13(4): 578-603. https://doi.org/10.22067/ijnao.2023.79288.1192