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Research Article

Two new approximations to Caputo–Fabrizio fractional equation on non-uniform meshes and its applications

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Abstract

We present two numerical approximations with non-uniform meshes to the Caputo–Fabrizio derivative of order α $(0 < \alpha < 1)$. First, the L1 formula is obtained by using the linear interpolation approximation for constructing the second-order approximation. Next, the quadratic interpolation approximation is used for improving the accuracy in the temporal direction. Besides, we discretize the spatial derivative using the compact finite difference scheme. The accuracy of the suggested schemes is not dependent on the fractional α. The coefficients and the truncation errors are carefully investigated for two schemes, separately. Three examples are carried out to support the convergence orders and show the efficiency of the suggested scheme.

AMS subject classifications (2020): 26A33, 35K57.

Keywords: Numerical approximations, Caputo–Fabrizio fractional derivative, Diffusion equation, Advection equation, Non-uniform meshes.

1 Introduction

In recent years, some approximations have been proposed for the fractional derivative of order α, such as Grünwald–Letnikov, Lubich, and Caputo approximations [15, 16, 24, 28, 30, 27]. The schemes that are proposed until now to discretize the Caputo fractional derivative have been limited to the

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accuracy of order $2-\alpha$ $(0<\alpha<1)$ on uniform meshes. One disadvantage of previously proposed methods is that when $\alpha \approx 1$, its accuracy may lead to poor accuracy. Thus, in this point of view, the numerical solutions of high dimensional partial fractional differential equations require a large number of computations. Besides, from the truncation error estimate of the methods on uniform meshes verify that the accuracy is dependent on the fractional order α. We are able to overcome these difficulties using Caputo–Fabrizio fractional derivative with a non-singular kernel. Afterward, we will obtain the second and third-orders accuracy in time that is independent of the fractional order α.

Let us consider the following time fractional diffusion and advection equations, respectively:

$$
\begin{aligned}
{}_{0}^{\mathrm{CF}}\mathcal{D}_t^{\alpha} u(X,t) &= \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad x \in \Omega,\ 0 \le t \le T, \\
{}_{0}^{\mathrm{CF}}\mathcal{D}_t^{\alpha} u(X,t) &= \frac{\partial u(x,t)}{\partial x} + f(x,t), \quad x \in \Omega,\ 0 \le t \le T,
\end{aligned}
\tag{1}
$$

in which ${}_{0}^{\mathrm{CF}}\mathcal{D}_t^{\alpha}$ is the αth Caputo–Fabrizio fractional derivative defined by

$$
{}_{0}^{\mathrm{CF}}\mathcal{D}_t^{\alpha} u(X,t) = \frac{M(\alpha)}{1-\alpha}\int_0^t u'(X,s)\exp\big(-\alpha\frac{t-s}{1-\alpha}\big)\mathrm{ds}, \quad 0<\alpha<1, \tag{2}
$$

where $M(\alpha)$ is a normalization function such that $M(0)=M(1)=1$.

In 2015, Caputo and Fabrizio [7] suggested a new definition of fractional derivative based on the exponential kernel. They considered two different representations for the temporal and the spatial variables. It is important and interesting that this approach describes the behaviour of classical viscoelastic materials, electromagnetic systems, thermal media, and so on. Another interesting property of this definition is that it opens up new avenues in the mechanical phenomena, related to plasticity, fatigue, damage, and electromagnetic hysteresis [7].

Recently, studies of Caputo–Fabrizio fractional derivatives have been carried out by some authors. Authors of [6] investigated the existence of a solution for two high-order fractional integro-differential equations including the Caputo–Fabrizio derivative. Atangana and Alqahtani [4] considered a numerical approximation of the space and time Caputo–Fabrizio fractional derivative in connection with ground water pollution equation. In [18], the authors presented a Crank–Nicolson finite difference scheme to solve fractional Cattaneo equation by a new fractional derivative. Furthermore, they analyzed the stability and convergence order of the scheme. The main aim of [9] is to prove the existence and uniqueness of the flow of water within a confined aquifer with Caputo–Fabrizio fractional diffusion for the spatial and the temporal variables. In [12], the authors applied the Ritz method with known basis functions for a type of Fokker–Planck equation with Caputo–Fabrizio fractional derivative. In 2017, Mirza and Vieru [21] proposed the fundamental solutions to time-fractional advection-diffusion equation without singular kernel. They applied the Laplace transform and Fourier transforms with respect to the temporal variable and the space coordinates, respectively. In [19], the authors constructed the shifted Legendre polynomials operational matrix in order to solve problems with left-sided Caputo–Fabrizio operator. A second-order scheme for the space fractional diffusion equation with Caputo–Fabrizio is provided in [26]. The main aim of [10] is to solve two problems in nonlocal quantum mechanics wherein the nonlocal shrödinger equation has been transformed to an ordinary linear differential equation. Other interesting papers in the field of the Caputo–Fabrizio derivative are found in [1, 2, 3, 5, 8, 11, 13, 14, 20, 22, 23, 25, 17].

The main goal of this paper is to derive two new formulas to approximation the Caputo–Fabrizio derivative of order α $(0 < \alpha < 1)$. For this purpose, we use the linear and the quadratic interpolation approximations on non-uniform meshes for obtaining the second and the third orders accuracy. Besides, we discretize the spatial derivative using the compact finite difference scheme. The advantages of the present paper are in the following two aspects, that is, the obtained accuracy is independent of the fractional α and gives a new high-order accuracy to the time fractional derivative in Caputo–Fabrizio's sense on non-uniform meshes. In this paper, much attention is paid to the numerical aspects. To our knowledge, our interest in Caputo–Fabrizio derivative is due to the necessity of using a model describing the behaviour of classical viscoelastic materials, thermal media, electromagnetic systems, and so on. In fact, the original definition of fractional derivative appears to be particularly convenient for those mechanical phenomena, related to plasticity, fatigue, damage and with electromagnetic hysteresis. In fact, the Caputo–Fabrizio derivative fits to describe material heterogeneities and structures with different scales. Hence, we have focused on non-uniform meshes.

The rest of the paper is organized as follows. In Section 2, the derivation of the new method on any non-uniform meshes for the Caputo–Fabrizio fractional derivative of order α $(0 < \alpha < 1)$ in both cases of the second and the third order is developed. Three examples are given in Section 3 to support the theoretical analysis. Finally concluding remarks are given in Section 4.

2 Derivation of new method on non-uniform meshes

2.1 The second-order approximation

In this section, we focus our attention on deriving the new fractional numerical differentiation formula in details. By Caputo–Fabrizio fractional derivative, we have

$$
\begin{aligned}
{}_{0}^{\texttt{CF}}\mathrm{D}_t^{\alpha}u(t)|_{t=t_k} &= \frac{1}{1-\alpha}\int_0^t u'(s)\exp\big(-\alpha\frac{t-s}{1-\alpha}\big)\mathrm{ds}\\
&= \frac{1}{1-\alpha}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} u'(s)\exp\big(-\alpha\frac{t_n-s}{1-\alpha}\big)\mathrm{ds}, \qquad (3)
\end{aligned}
$$

where $0 = t_0 < t_1 < \ldots < t_N = T$. We denote the time step by $\tau_n = t_n - t_{n-1}, \quad 1 \leq n \leq N$, and let $\tau_{Max} = \max_{1\leq l\leq N} \tau_l$.

To explain the process of deriving formula, we apply the linear interpolation polynomial using points $\big(t_{k-1}, u(t_{k-1})\big)$ and $\big(t_k, u(t_k)\big)$ for approximating $u'(s)$ as follows:

$$
\begin{aligned}
\Pi_{1,k}u(t) &= u(t_k) + (t-t_k)\frac{u(t_k)-u(t_{k-1})}{t_k-t_{k-1}}\\
&= u(t_{k-1})\frac{t_k-t}{\tau_k} + u(t_k)\frac{t-t_{k-1}}{\tau_k} \qquad (4)
\end{aligned}
$$

and

$$
(\Pi_{1,k}u(t))' = \frac{u(t_k)-u(t_{k-1})}{\tau_k}. \qquad (5)
$$

Then, the interpolation error formula is given by

$$
u(t) - \Pi_{1,k}u(t) = \frac{u''(\xi_k)}{2!}(t-t_{k-1})(t-t_k), \quad t\in[t_{k-1},t_k],\ \xi_k\in(t_{k-1},t_k),\quad 1\leq k\leq n. \qquad (6)
$$

Now, substituting (5) into (3), we obtain the L1 formula as follows:

$$
\begin{aligned}
{}_{0}^{\texttt{CF}}\mathrm{D}_t^{\alpha}u(t)|_{t=t_k} &= \frac{1}{1-\alpha}\sum_{k=1}^{n}\frac{u(t_k)-u(t_{k-1})}{\tau_k}\int_{t_{k-1}}^{t_k}\exp\big(-\alpha\frac{t_n-s}{1-\alpha}\big)\mathrm{ds}\\
&= \frac{1}{\alpha}\sum_{k=1}^{n}(u_k-u_{k-1})M_k^n, \qquad (7)
\end{aligned}
$$

where $M_k^n = \frac{1}{\tau_k}\Bigg(\exp\big(-\alpha\frac{t_n-t_k}{1-\alpha}\big) - \exp\big(-\alpha\frac{t_n-t_{k-1}}{1-\alpha}\big)\Bigg)$.

Lemma 1. For any $1 \leq n \leq N$, we have $M_k^n > 0$ and $M_{k+1}^n > M_k^n$.

Proof. Note that $\frac{-\alpha}{1-\alpha}(t_n-t_k) > \frac{-\alpha}{1-\alpha}(t_n-t_{k-1})$ for $0<\alpha<1$ and $\exp(x)$ is a monotone increasing function. Thus one can verify that

$$
M_k^n = \frac{1}{\tau_k}\Bigg(\exp\big(-\alpha\frac{t_n-t_k}{1-\alpha}\big) - \exp\big(-\alpha\frac{t_n-t_{k-1}}{1-\alpha}\big)\Bigg) > 0.
$$

From the mean value theorem for integrals, we have

$$M_k^n = \frac{1}{\tau_k}\int_{t_{k-1}}^{t_k} \exp\big(-\alpha\frac{t_n-s}{1-\alpha}\big)\mathrm{ds} = \exp\big(-\alpha\frac{t_n-\xi_k}{1-\alpha}\big), \qquad \xi_k \in (t_{k-1},t_k).$$

Clearly, $\exp\big(-\alpha\frac{t_n-s}{1-\alpha}\big)$ is a monotone increasing function, then the second statement of the lemma follows immediately. □

Theorem 1. Suppose $u(t) \in C^2[0,T]$. For any $0<\alpha<1$, it holds that

$$|R(u(t_k))| \le \frac{1}{1-\alpha}\max_{t_0\le t\le t_n}\frac{|u''(t)|}{8}\tau_{\max}^2. \tag{8}$$

Proof. From (2) and (6), we get

$$\begin{aligned}
R(u(t_k)) &= \frac{1}{1-\alpha}\Big[\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}(u(s)-\Pi_{1,k}u(s))'\exp\big(-\alpha\frac{t_n-s}{1-\alpha}\big)\mathrm{ds}\Big]\\
&= \frac{1}{1-\alpha}\sum_{k=1}^{n}\Big[\underbrace{(u(s)-\Pi_{1,k}u(s))\exp\big(-\alpha\frac{t_n-s}{1-\alpha}\big)\Big|_{t_{k-1}}^{t_k}}_{=0}\\
&\quad -\int_{t_{k-1}}^{t_k}(u(s)-\Pi_{1,k}u(s))\exp\big(-\alpha\frac{t_n-s}{1-\alpha}\big)\frac{\alpha}{1-\alpha}\mathrm{ds}\Big]\\
&= \frac{-\alpha}{(1-\alpha)^2}\Big[\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\frac{u''(\nu_k)}{2}(s-t_{k-1})(s-t_k)\exp\big(-\alpha\frac{t_n-s}{1-\alpha}\big)\mathrm{ds}\Big]\\
&\le \frac{\alpha}{(1-\alpha)^2}\frac{|u''(\nu)|}{2}\frac{\tau_k^2}{4}\underbrace{\int_{t_0}^{t_n}\exp\big(-\alpha\frac{t_n-s}{1-\alpha}\big)\mathrm{ds}}_{\le\frac{1-\alpha}{\alpha}}\\
&\le \frac{\alpha}{(1-\alpha)^2}\frac{|u''(\nu)|}{8}\tau_{\max}^2\frac{1-\alpha}{\alpha}, \quad \nu\in(t_0,t_n) \quad \nu_k\in(t_{k-1},t_k).
\end{aligned}$$

This proves the desired formula. □

We apply the time step over the non-uniform mesh defined as [29]

$$\tau_n = (N+1-n)\mu,\ 1\le n\le N, \tag{9}$$

where $\mu = \frac{2T}{N(N+1)}$.

2.2 The third-order approximation

Adding an additional point $\big(t_{k-2}, u(t_{k-2})\big)$ for $k\ge 2$, we obtain a quadratic interpolation function $\Pi_{2,k}u(t)$ of $u(t)$ as follows:

$$\begin{aligned}\Pi_{2,k}u(t) &= \Pi_{1,k}u(t) + \frac{1}{t_k - t_{k-2}}\left[\frac{u(t_k) - u(t_{k-1})}{t_k - t_{k-1}} - \frac{u(t_{k-1}) - u(t_{k-2})}{t_{k-1} - t_{k-2}}\right] \\ &\quad \times (t - t_k)(t - t_{k-1}) \\ &= \Pi_{1,k}u(t) + \frac{1}{\tau_k + \tau_{k-1}}\left[\frac{u(t_k) - u(t_{k-1})}{\tau_k} - \frac{u(t_{k-1}) - u(t_{k-2})}{\tau_{k-1}}\right] \\ &\quad \times (t - t_k)(t - t_{k-1}), \quad t \in [t_{k-1}, t_k], \\ \big(\Pi_{2,k}u(t)\big)' &= \frac{u(t_k) - u(t_{k-1})}{\tau_k} + \big(2t - (t_k + t_{k-1})\big)\mathcal{A}_t u(t_k).\end{aligned} \tag{10}$$

For simplicity in what follows, we define:

$$\mathcal{A}_t u_k = \frac{1}{\tau_k + \tau_{k-1}}\left[\frac{u_k - u_{k-1}}{\tau_k} - \frac{u_{k-1} - u_{k-2}}{\tau_{k-1}}\right],$$

and

$$\begin{aligned}u(t) - \Pi_{2,k}u(t) = \frac{u'''(\eta_k)}{6}(t - t_{k-2})(t - t_{k-1})(t - t_k), \\ t \in [t_{k-1}, t_k],\ \eta_k \in (t_{k-2}, t_k),\ 2 \le k \le n.\end{aligned} \tag{11}$$

We substitute (10) into (2) to obtain a new approximation of the Caputo–Fabrizio derivative as follows:

$$\begin{aligned}{}_{0}^{\mathtt{CF}}\mathcal{D}_t^{\alpha}u(t)|_{t=t_n} &= \frac{1}{1-\alpha}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} u'(s)\exp\big(-\alpha\frac{t_n - s}{1-\alpha}\big)\mathrm{ds} \\ &\approx \frac{1}{1-\alpha}\Big[\int_{t_0}^{t_1}\big(\Pi_{1,1}u(s)\big)'\exp\big(-\alpha\frac{t_n - s}{1-\alpha}\big)\mathrm{ds} \\ &\quad + \sum_{k=2}^{n}\int_{t_{k-1}}^{t_k}\big(\Pi_{2,k}u(s)\big)'\exp\big(-\alpha\frac{t_n - s}{1-\alpha}\big)\mathrm{ds}\Big] \\ &= \frac{1}{1-\alpha}\Big[\frac{u_1 - u_0}{\tau_1}\int_{t_0}^{t_1}\exp\big(-\alpha\frac{t_n - s}{1-\alpha}\big)\mathrm{ds} \\ &\quad + \sum_{k=2}^{n}\int_{t_{k-1}}^{t_k}\big[\frac{u_k - u_{k-1}}{\tau_k} + \mathcal{A}_t u_k(2s - (t_{k-1} + t_k))\big] \\ &\quad \times \exp\big(-\alpha\frac{t_n - s}{1-\alpha}\big)\mathrm{ds}\Big]\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{1-\alpha}\Big[\sum_{k=1}^{n}\frac{u_k-u_{k-1}}{\tau_k}\int_{t_{k-1}}^{t_k}\exp\big(-\alpha\frac{t_n-s}{1-\alpha}\big)\mathrm{ds} \\
&\quad + \sum_{k=2}^{n}\mathcal{A}_t u_k\int_{t_{k-1}}^{t_k}(2s-(t_{k-1}+t_k))\exp\big(-\alpha\frac{t_n-s}{1-\alpha}\big)\mathrm{ds}\Big] \\
&= {}_{0}^{\mathtt{CF}}\mathrm{D}_t^{\alpha}u(t)|_{t=t_n} + \frac{1}{\alpha^2}\sum_{k=2}^{n}B_k^n\mathcal{A}_t u_k, \qquad (12)
\end{aligned}
$$

where

$$
\begin{aligned}
B_k^n &= 2(\alpha-1)\Big[\exp\big(-\alpha\frac{t_n-t_k}{1-\alpha}\big) - \exp\big(-\alpha\frac{t_n-t_{k-1}}{1-\alpha}\big)\Big] \\
&\quad + \alpha\tau_k\Big[\exp\big(-\alpha\frac{t_n-t_k}{1-\alpha}\big) + \exp\big(-\alpha\frac{t_n-t_{k-1}}{1-\alpha}\big)\Big]. \qquad (13)
\end{aligned}
$$

Moreover, ${}_{0}^{\mathtt{CF}}\mathrm{D}_t^{\alpha}$ is the $L1$ method for non-uniform time grid in Section 2, where, we define

$$
{}_{0}^{\mathtt{CF}}\mathbb{D}_t^{\alpha}u(t)|_{t=t_n} = {}_{0}^{\mathtt{CF}}\mathrm{D}_t^{\alpha}u(t)|_{t=t_n} + \frac{1}{\alpha^2}\sum_{k=2}^{n}B_k^n\mathcal{A}_t u_k, \qquad (14)
$$

wherein, we define a new operator $\mathcal{D}_t^{\alpha}u(t)$, which is the new fractional numerical differentiation operator for the Caputo–Fabrizio fractional derivative ${}_{0}^{\mathtt{CF}}\mathbb{D}^{\alpha}$.

The following lemma states the property of coefficients B_k^n.

Lemma 2. For any α $(0<\alpha<1)$, let

$$
\begin{aligned}
B_k^n &= 2(\alpha-1)\Big[\exp\big(-\alpha\frac{t_n-t_k}{1-\alpha}\big) - \exp\big(-\alpha\frac{t_n-t_{k-1}}{1-\alpha}\big)\Big] \\
&\quad + \alpha\tau_k\Big[\exp\big(-\alpha\frac{t_n-t_k}{1-\alpha}\big) + \exp\big(-\alpha\frac{t_n-t_{k-1}}{1-\alpha}\big)\Big], \quad 2\le k\le n.
\end{aligned}
$$

It holds that

$$
B_n^n > B_{n-1}^n > \cdots > B_k^n > B_{k-1}^n > \cdots > B_2^n > 0.
$$

Proof. To prove our statement, we apply the error representation of the trapezoidal formula. For this purpose, we consider the function $-2\alpha\exp\left(-\alpha\frac{t_n-s}{1-\alpha}\right)$ on the interval $[t_{k-1},t_k]$. Then

$$
\begin{aligned}
B_k^n &= -2\alpha\Big[\int_{t_{k-1}}^{t_k} \exp\big(-\alpha\frac{t_n-s}{1-\alpha}\big)\mathrm{ds} - \frac{\tau_k}{2}\big(\exp\big(-\alpha\frac{t_n-t_k}{1-\alpha}\big)\\
&\quad + \exp\big(-\alpha\frac{t_n-t_{k-1}}{1-\alpha}\big)\big)\Big]\\
&= -2\alpha(-\frac{1}{12})\big(\exp\big(-\alpha\frac{t_n-s}{1-\alpha}\big)\big)''\big|_{s=\xi_k}\\
&= \frac{\alpha}{6}\frac{\alpha^2}{(1-\alpha)^2}\exp\big(-\alpha\frac{t_n-\xi_k}{1-\alpha}\big), \quad \xi_k \in (t_{k-1}, t_k).
\end{aligned}
$$

Since $\exp\big(-\alpha\frac{t_n-s}{1-\alpha}\big) > 0$ and $\frac{\alpha}{6}\frac{\alpha^2}{(1-\alpha)^2} > 0$, these imply that $B_k^n > 0$ for $2 \le k \le n$. Besides, $\exp\big(-\alpha\frac{t_n-s}{1-\alpha}\big)$ is a monotone increasing function with respect to s on $[0, T]$ and this completes the proof. □

Theorem 2. Suppose $u(t) \in C^3[0, T]$. For any $0 < \alpha < 1$, it holds that

$$
|R(u(t_k))| \le \Big[\frac{\alpha}{(1-\alpha)^2}\max_{t_0\le t\le t_1}\frac{|u''(t)|}{8} + \max_{t_0\le t\le t_n}\frac{|u'''(t)|}{12}\frac{\alpha}{1-\alpha}\Big]\tau_{\max}^3. \quad (15)
$$

Proof. To prove this, we see from (2), (6), and (11) that

$$
\begin{aligned}
R(u(t_k)) &= \frac{1}{1-\alpha}\Big[\int_{t_0}^{t_1}(u(s)-\Pi_{1,1}u(s))'\exp\big(-\alpha\frac{t_n-s}{1-\alpha}\big)\mathrm{ds}\\
&\quad + \sum_{k=2}^{n}\int_{t_{k-1}}^{t_k}(u(s)-\Pi_{2,k}u(s))'\exp\big(-\alpha\frac{t_n-s}{1-\alpha}\big)\mathrm{ds}\Big]\\
&= \frac{1}{1-\alpha}\Big[\underbrace{(u(s)-\Pi_{1,1}u(s))\exp\big(-\alpha\frac{t_n-s}{1-\alpha}\big)\big|_{t_0}^{t_1}}_{=0}\\
&\quad - \int_{t_0}^{t_1}(u(s)-\Pi_{1,1}u(s))\exp\big(-\alpha\frac{t_n-s}{1-\alpha}\big)\frac{\alpha}{1-\alpha}\mathrm{ds}\Big]\\
&\quad + \frac{1}{1-\alpha}\Big[\underbrace{\sum_{k=2}^{n}\int_{t_{k-1}}^{t_k}(u(s)-\Pi_{2,k}u(s))'\exp\big(-\alpha\frac{t_n-s}{1-\alpha}\big)\big|_{t_{k-1}}^{t_k}}_{=0}\\
&\quad - \sum_{k=2}^{n}\int_{t_{k-1}}^{t_k}(u(s)-\Pi_{2,k}u(s))'\exp\big(-\alpha\frac{t_n-s}{1-\alpha}\big)\frac{\alpha}{1-\alpha}\mathrm{ds}\Big]
\end{aligned}
$$

$$\leq \frac{\alpha}{(1-\alpha)^2}\Bigg[\frac{|u''(\xi_1)|}{2}\frac{\tau^2}{4}\underbrace{\int_{t_0}^{t_1}\exp\big(-\alpha\frac{t_n-s}{1-\alpha}\big)\mathrm{ds}}_{\tau_1}$$
$$+\frac{|u'''(\nu)|}{6}\frac{\tau_k^2}{4}(\tau_{k-1}+\tau_k)\underbrace{\int_{t_1}^{t_n}\exp\big(-\alpha\frac{t_n-s}{1-\alpha}\big)\mathrm{ds}}_{\leq\frac{1-\alpha}{\alpha}}\Bigg],$$
$$\xi_1\in(t_0,t_1),\nu\in(t_0,t_n).$$

The second term of last inequality results by the following remark in [29]:

Remark 1. With regard to $\max_{t_0\leq s\leq t_n}|(s-t_k)(s-t_{k-1})|=\frac{\tau_k^2}{4}$ is obtained at $s=t_{k-1}+\frac{\tau_k}{2}$, consequently we have:

$$\begin{aligned}(s-t_{k-2})(s-t_{k-1})(s-t_k)&=\frac{\tau_k^2}{4}(t_{k-1}+\frac{\tau_k}{2}-t_{k-2})\\&=\frac{\tau_k^2}{4}(\tau_{k-1}+\frac{\tau_k}{2})\leq\frac{\tau_k^2}{4}(\tau_{k-1}+\tau_k).\end{aligned}$$

Besides, we know that $\tau_{Max}=\max_{1\leq l\leq N}\tau_l$. This proves (15).

□

3 Numerical applications of the examples

In the current section, the efficiency of the suggested scheme for the time Caputo–Fabrizio fractional diffusion and advection equations are presented on three numerical examples in one dimension. The accuracy and the stability of the suggested scheme in the paper for different values of M and N are tested. In order to carry out our numerical examples, we have used the Maple 18 software with a PC of 4 GHz CPU and 6 GB memory. The accuracy of the proposed scheme is measured by the following error norm

$$e(N,M)=\max_{1\leq i\leq M-1}|u(x_i,t_N)-u_i^N|.$$

We denote the numerical convergence orders by

$$Rate=\log_2\left(\frac{e(N/2,M)}{e(N,M)}\right).$$

Example 1. Suppose $0<\alpha<1$. Let $u(t)=\sin(4t)$. The exact solution is obtained from the definition of the Caputo–Fabrizio (2) without the variable x. Denote $e(N)=|u(t_N)-u_N|$.

Table 1: Numerical convergence orders in temporal direction for Example 1

α	N	The second-order		The third-order	
		$e(N)$	*Rate*	$e(N)$	*Rate*
0.5	5	5.6188×10^{-2}	–	2.0699×10^{-2}	–
	10	1.5043×10^{-2}	1.9012	2.3793×10^{-3}	3.2100
	20	3.9207×10^{-3}	1.9399	2.6437×10^{-4}	3.1699
	40	1.0028×10^{-3}	1.9671	3.1297×10^{-5}	3.0785
	80	$53722. \times 10^{-4}$	1.9827	3.6530×10^{-6}	3.0989
0.9	5	3.4485×10^{-3}	–	5.9336×10^{-2}	–
	10	9.3840×10^{-4}	1.8777	7.9779×10^{-3}	2.8946
	20	2.4994×10^{-4}	1.9086	1.0192×10^{-3}	2.9886
	40	6.4364×10^{-5}	1.9572	1.2836×10^{-4}	2.9892
	80	1.6314×10^{-5}	1.9801	1.5964×10^{-5}	3.0073

From Table 1, this fact is extracted that the computational orders of our schemes are independent of the fractional order α. This means that with changing values α, the computational orders do not change and the orders of convergence of these two cases the second-order and three-order schemes should be 2 and 3, respectively.

Example 2. Consider the time fractional diffusion equation [29]

$$\begin{cases} {}_0^{\mathrm{C}}\mathcal{D}_t^{\alpha} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), & 0 < x < 1,\ 0 < t \leq 1, \\ u(x,0) = u^0(x), & 0 \leq x \leq 1, \\ u(0,t) = \Phi(t),\ u(1,t) = \varphi(t), & 0 < t \leq 1. \end{cases} \tag{16}$$

The exact solution of (16) is $u(x,t) = \sin(\pi x)t^2$. The functions can be obtained by substituting $u(x,t)$ into (16).

Remark 2. For discretizing the term $\frac{\partial^2 u(x,t)}{\partial x^2}$, we apply the fourth-order CFD scheme as follows:

$$\frac{\partial^2 u(x,t)}{\partial x^2}|_{x=x_i} = \frac{\delta_x^2}{1+\frac{h^2}{12}\delta_x^2} u(x_i,t) + O(h^4).$$

Now, we consider the fourth-order approximation of the second derivative of u at point x_i as follows:

$$\frac{\partial^2 u(x,t)}{\partial x^2}|_{i,n} = \frac{\delta_x^2}{1+\frac{h^2}{12}\delta_x^2} u_i^n,$$

where u_i^n denotes the numerical solution at (x_i, t_n). Then, a difference scheme using third-order formula can be obtained as

$$\mathcal{H}_0^{\mathsf{CF}}\mathbb{D}_t^{\alpha}u_i^n = \delta_x^2 u_i^n + \mathcal{H}f_i^n, \quad 1 \le i \le M-1, \quad 1 \le n \le N, \tag{17}$$

$$u_0^n = \Phi(t_n), \quad u_1^n = \varphi(t_n), \quad , 1 \le n \le N, \tag{18}$$

$$u_i^0 = u^0(x_i), \quad 0 \le i \le M, \tag{19}$$

where

$$\delta_x^2 v_i^n = \frac{1}{h}(\delta_x v_{i+\frac{1}{2}}^n - \delta_x v_{i-\frac{1}{2}}^n).$$

Moreover

$$\mathcal{H}v_i = \begin{cases} \frac{1}{12}(v_{i+1} + 10v_i + v_{i-1}), & 1 \le i \le M-1, \\ v_i, & i = 0 \text{ or } M. \end{cases}$$

It can be seen that

$$\mathcal{H}v_i = \left(I + \frac{h^2}{12}\delta_x^2\right)v_i, \quad 1 \le i \le M-1.$$

Lemma 3. (See [31]). consider the function $g(x) \in \mathcal{C}^6[x_{i-1}, x_{i+1}]$, and let $\xi(s) = 5(1-s)^3 - 3(1-s)^5$. Then

$$\begin{aligned}&\frac{g''(x_{i+1}) + 10g''(x_i) + g''(x_{i-1})}{12} \\ &\quad = \frac{g(x_{i+1}) - 2g(x_i) + g(x_{i-1})}{h^2} + \frac{h^4}{360}\int_0^1 \left[g^6(x_i - sh) + g^6(x_i + sh)\right]\xi(s)\mathrm{ds}.\end{aligned}$$

Having seen Tables 2 and 3, we observe that the third-order scheme produces better results than the second-order scheme. In the Caputo's sense, the accuracy of the presented method is dependent on α. In this case, the computational orders for $\alpha = 0.4, 0.6$ and 0.8 are $1.6, 1.4$ and 1.2, respectively, when the theoretical order is $3 - \alpha$, whereas the accuracy of the presented method is not dependent on the fractional α. Table 4 illustrates the error and CPU time of the third-order and the second-order schemes. Having seen Table 4, we conclude that the third-order scheme produces more accurate results than the second-order scheme. Besides, the third-order scheme needs fewer temporal grid size and less CPU time for bigger N.

Figure 1 exhibits the solution curves at final time $T = 1$ for different values of $\alpha = 0.1, 0.5$ and 0.9 with $M = N = 50$ for Example 2. Figure 2 shows the comparison of the absolute errors wherein the third-order scheme is more accurate than the second order-scheme. The plots of absolute error and the numerical solution for $\alpha = 0.1$ with $M = N = 50$ for Example 2 are shown in Figure 3.

Table 2: Numerical convergence orders in temporal direction with $M = 50$ for Example 2

α	N	The second-order		The third-order	
		$e(N,M)$	*Rate*	$e(N,M)$	*Rate*
0.25	5	3.6444×10^{-4}	–	1.9185×10^{-4}	–
	10	9.9230×10^{-5}	1.8768	3.0437×10^{-5}	2.6561
	20	2.5987×10^{-5}	1.8299	4.4476×10^{-6}	2.7747
	40	6.6615×10^{-6}	1.9639	5.4377×10^{-7}	3.0320
0.5	5	1.1051×10^{-3}	–	5.2234×10^{-4}	–
	10	2.9975×10^{-4}	1.8823	7.9398×10^{-5}	2.7206
	20	7.8515×10^{-5}	1.9327	1.1188×10^{-5}	2.8272
	40	2.0126×10^{-5}	1.9639	1.5315×10^{-6}	2.8689
0.75	5	2.6813×10^{-3}	–	9.2499×10^{-4}	–
	10	7.2606×10^{-4}	1.8848	1.2632×10^{-4}	2.8724
	20	1.8975×10^{-4}	1.9360	1.6398×10^{-5}	2.9455
	40	4.8687×10^{-5}	1.9625	2.2262×10^{-6}	2.8809

Table 3: Numerical convergence orders in temporal direction with $M = 50$ for Example 2

α	N	The second-order		The third-order	
		$e(N,M)$	*Rate*	$e(N,M)$	*Rate*
0.4	5	7.4249×10^{-4}	–	3.7053×10^{-4}	–
	10	2.0170×10^{-4}	1.8802	5.7480×10^{-5}	2.8434
	20	5.2738×10^{-5}	1.9353	8.0087×10^{-6}	2.8434
	40	1.3679×10^{-5}	1.9469	1.2630×10^{-7}	2.6647
0.6	5	1.5958×10^{-3}	–	6.9588×10^{-4}	–
	10	4.3238×10^{-4}	1.8839	1.0289×10^{-4}	2.7577
	20	1.1306×10^{-4}	1.9352	1.3961×10^{-5}	2.8816
	40	2.8973×10^{-5}	1.9643	1.7908×10^{-6}	2.9627
0.8	5	3.1781×10^{-3}	–	9.4828×10^{-4}	–
	10	8.6131×10^{-4}	1.8836	1.2325×10^{-4}	2.9437
	20	2.2530×10^{-4}	1.9347	1.5675×10^{-5}	2.9751
	40	5.7678×10^{-5}	1.9658	1.9745×10^{-6}	2.9889

Table 4: The errors and CPU time (seconds) of the third-order and the second-order schemes for Example 2

α	The third-order ($M=50$)			The second-order ($M=50$)		
	N	$e(N,M)$	$CPU(s)$	N	$e(N,M)$	$CPU(s)$
0.7	8	9.3694×10^{-4}	0.03	5	8.6414×10^{-4}	0.09
	24	1.1214×10^{-4}	0.06	12	7.2084×10^{-5}	0.1
	72	1.2902×10^{-5}	13.10	24	9.6323×10^{-6}	5.75
0.8	8	1.3174×10^{-3}	0.06	5	9.4828×10^{-4}	0.09
	24	1.5769×10^{-4}	0.09	10	1.2325×10^{-4}	0.1
	72	1.8033×10^{-5}	14.30	20	1.5675×10^{-5}	3.86
0.9	8	1.9637×10^{-3}	0.07	4	1.6778×10^{-3}	0.07
	24	2.3611×10^{-4}	0.09	8	1.7168×10^{-4}	0.09
	72	2.7014×10^{-5}	14.53	16	1.9223×10^{-5}	3.05

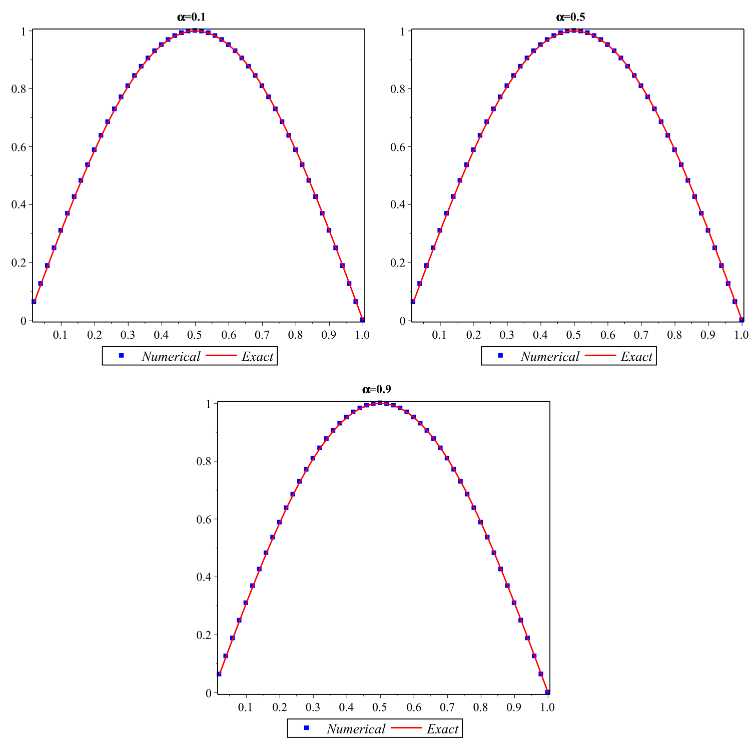

Figure 1: The solution curves at $T=1$ with $M=N=50$ for Example 2

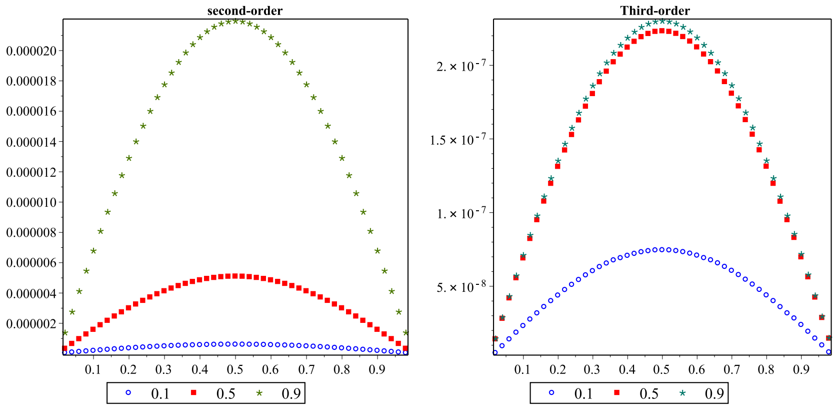

Figure 2: Comparison of the absolute errors for the second-order (left) and the third-order (right) schemes with $M = N = 50$ for Example 2

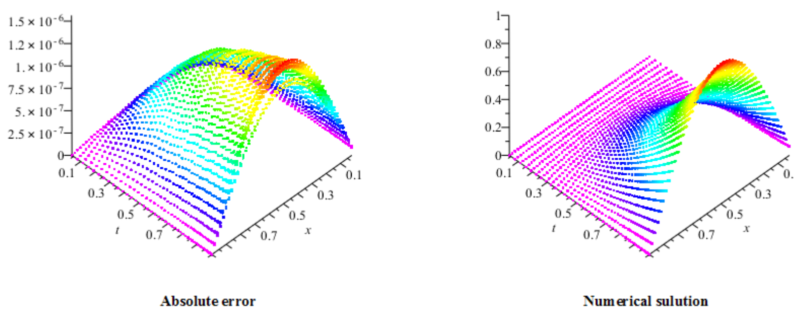

Figure 3: Plots of the absolute error (left) with $M = N = 50$ and the numerical solution (right) for Example 2

Example 3. Consider the time fractional advection equation [29]

$$\begin{cases} {}_0^{\mathrm{C}}\mathcal{D}_t^{\alpha} u(x,t) = \frac{\partial u(x,t)}{\partial x} + f(x,t), & 0 < x < 1,\ 0 < t \leq 1, \\ u(x,0) = u^0(x), & 0 \leq x \leq 1, \\ u(0,t) = \Phi(t),\ u(1,t) = \varphi(t), & 0 < t \leq 1. \end{cases} \tag{20}$$

Remark 3. For discretizing the term $\frac{\partial u(x,t)}{\partial x}$, we apply the fourth-order CFD scheme [29] as follows:

Table 5: Numerical convergence orders in temporal direction with $\alpha = 0.5$ and $M = 50$ for Example 3

N	The second-order		The third-order	
	$e(N,M)$	*Rate*	$e(N,M)$	*Rate*
5	1.3158×10^{-3}	–	5.7401×10^{-3}	–
10	3.4968×10^{-3}	1.9118	8.8998×10^{-4}	2.6892
20	9.0919×10^{-4}	1.9434	1.2614×10^{-4}	2.8187
40	2.3234×10^{-4}	1.9683	1.7195×10^{-5}	2.8750

$$\frac{\partial u(x,t)}{\partial x}|_{i,n} = \frac{\delta_{\widehat{x}}}{1+\frac{h^2}{6}\delta_x^2}u_i^n.$$

Difference scheme using third-order formula can be obtained as

$$\mathcal{A}\,{}_0^{\mathsf{CF}}\mathbb{D}_t^{\alpha} u_i^n = \delta_{\widehat{x}} u_i^n + \mathcal{A} f_i^n, 1 \le i \le M-1, \quad 1 \le n \le N, \tag{21}$$

$$u_0^n = \Phi(t_n), \quad u_1^n = \varphi(t_n), \quad ,1 \le n \le N, \tag{22}$$

$$u_i^0 = u^0(x_i), \quad 0 \le i \le M, \tag{23}$$

where

$$\delta_{\widehat{x}} v_i = \frac{1}{2h}(v_{i+1} - v_{i-1}),$$

and

$$\mathcal{A} v_i = \begin{cases} \frac{1}{6}(v_{i+1} + 4v_i + v_{i-1}), & 1 \le i \le M-1, \\ v_i, & i = 0 \text{ or } M. \end{cases}$$

It can be seen that

$$\mathcal{A} v_i = \left(I + \frac{h^2}{6}\delta_x^2\right)v_i, \quad 1 \le i \le M-1.$$

Like Remark 2 but with slight change, we consider the fourth-order approximation of the first derivative of u at point x_i as follows:

$$\frac{\partial u(x,t)}{\partial x}|_{x=x_i} = \frac{\delta_{\widehat{x}}}{1+\frac{h^2}{6}\delta_x^2}u(x_i,t) + O(h^4). \tag{24}$$

The exact solution of (20) is $u(x,t) = \sin(\pi x)t^5$. Functions can be obtained by substituting $u(x,t)$ into (20).

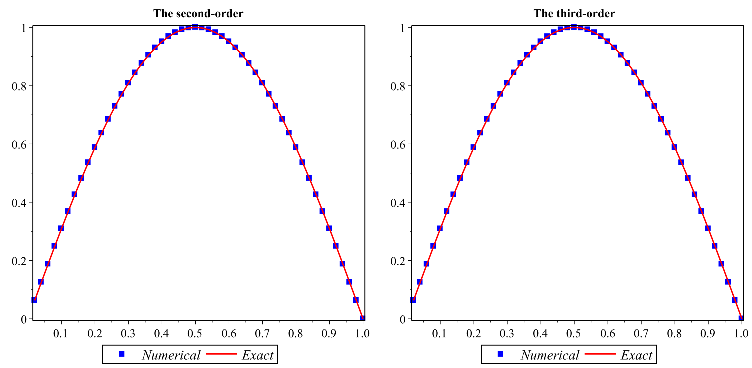

Figure 4: The solution curves at $T = 1$ with $M = 50, N = 40$ for Example 3

Table 5 confirms that the numerical convergence orders in both the second-order and the third-order schemes are close to theoretical results. Having seen Table 5, we conclude that the third-order produces better results for $e(N, M)$ than that the second-order both in error and accuracy. In [29], the accuracy of the presented method is dependent on α. In this case, the computational orders for $\alpha = 0.5$ is 1.5 with the theoretical order of $3-\alpha$, whereas the accuracy of our schemes is not dependent on the fractional α. Figure 4 illustrates the plot of numerical solution and exact solution with $\alpha = 0.5, M = 50$ and $N = 40$ at final time $T = 1$ for Example 3.

4 Conclusions

In the current paper, we have obtained two new fractional numerical differentiation formulas to approximate the time Caputo–Fabrizio fractional derivative of order α $(0 < \alpha < 1)$ on non-uniform meshes. First, the linear and quadratic interpolation approximations are considered for the integrand $u(t)$ because of obtaining the new formulas. Then, a fourth-order CFD scheme is employed for spatial discretization. This difference scheme is led to the third-order (second-order) and the fourth-order accuracy in the temporal and the spatial variables, respectively. Numerical results are carried out to support the convergence orders and show the efficiency of the suggested scheme. What distinguishes this paper from our previous studies is its accuracy aspect because the accuracy of the suggested schemes is not dependent on the fractional α.

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