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Research Article

Solving fuzzy multiobjective linear bilevel programming problems based on the extension principle

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Abstract

Fuzzy multiobjective linear bilevel programming (FMOLBP) problems are studied in this paper. The existing methods replace one or some deterministic model(s) instead of the problem and solve the model(s). Doing this work, we lose much information about the compromise decision, and it does not make sense for the uncertain conditions. To overcome the difficulties, Zadeh's extension principle is applied to solve the FMOLBP problems. Two crisp multiobjective linear three-level programming problems are proposed to find the lower and upper bound of its objective values in different levels. The problems are reduced to some linear optimization problems using one of the scalarization approaches, called the weighting method, the dual theory, and the vertex enumeration method. The lower and upper bounds are estimated by the resolution of the corresponding linear optimization problems. Hence, the membership functions of compromise objective values are produced, which is the main contribution of this paper. This technique is applied for the problem for the first time. This method applies all information of a fuzzy number and does not estimate it by a crisp number. Hence, the compromise decision resulted from the proposed method is consistent with reality. This point can minimize the gap between theory and practice. The results are compared with the results of existing approaches. It shows the efficiency of the proposed approach.

AMS subject classifications (2020): 90C70, 90C29, 90C46.

Keywords: Fuzzy mathematical programming; Fuzzy number; Fuzzy multiobjective bilevel programming; Extension principle; Vertex points; Weighting method.

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1 Introduction

A bilevel multiobjective programming problem (BL-MOPP) is applied to model decentralized decision problems that contain the objectives at the upper level, called leader, and the objectives at the lower level, called the follower. The decisions are made in a hierarchical order from the upper level to the lower level. The decision-makers in each level try to optimize their objective functions. However, the decision in a level may be affected by the objective values in another level. The objectives in both levels may have confliction objectives, which should be optimized simultaneously. The coefficients in the objective functions and constraints are determined by experts; see [3, 46]. In the most real-world situations, the possible values of the coefficients are usually estimated imprecisely or vaguely by experts. These values cannot be presented in terms of crisp values because we miss much information. The most appropriate form to express the values is fuzzy sets.

Many researchers have been studied bilevel programming problems since it was introduced by Von Stackelberg [36]. A class of algorithms and approaches was well designed to solve the problem like the Kuhn–Tucker method [7, 8, 9, 16], the vertex enumeration method [9], the K-best approach [9, 2], and the branch and bound procedure [14]. The BL-MOPPs were studied to formulate the problems of the real-world, for example, the location planning problem for stone industrial parks; see [5]. Their coefficients are expressed by crisp numbers. On the other hand, the coefficients cannot usually be expressed in terms of a real number in the uncertain conditions. Also, multilevel programming problems (MLPPs) in [15, 35, 34] were discussed for hierarchical decentralized planning problems. An overview of bilevel programming problem was provided by Colson, Marcotte, and Savard [13]. The fuzzy approach for MLPP in [34] was extended to solve bilevel and three-level nonlinear multiobjective programming problems by Abo-Sinna and Baky [2]. Interactive fuzzy programming was developed to solve fuzzy multilevel linear programming problem in [29]. The balance space method was proposed for the nonlinear multiobjective bilevel programming problem in [2]. The fuzzy goal programming (FGP) algorithm was used to solve BL-MOPP in [6]. An algorithm was designed based on FGP to solve MLPP in [9]. An FGP model was proposed to solve BL-MOPP by Pramanik and Dey [4]. This model does not follow the hierarchical structure of bilevel programming and does not consider the upper level decisions, and it is as a single-level multiobjective programming problem. In bilevel programming problems, the decisions of leader should be dominant over that of followers. Mohamed [5] well developed an efficient method to solve the multiobjective programming problems using FGP approach. Moitra and Pal [15] then extended the concept of FGP approach to finding a satisfactory solution for bilevel programming problems. Moreover, FGP approach was extended to solve a deterministic decentralized BL-MOPP in [6] and MLPP in [4]. Due to the existence of uncertainty in real-world situations, some or all parameters in objective functions and/or in constraints of the leader and/or the follower can be described by fuzzy numbers. Budnitzkia [5] designed a solution procedure to solve the linear fuzzy bilevel programming using α-cuts of the fuzzy polytopes. Zhang and his colleagues [17, 35, 37] utilized the extended solution concept and theorems of bilevel programming and presented procedures to solve the fuzzy linear bilevel programming (FLBP) in special cases of membership functions such as the triangular and the general form [42, 43, 44, 45]. When the leader, the follower, or both of them have multiple objectives, the fuzzy set technique was applied to investigate FLBP problems by Zhang, Lu, and Dillon [46]. They also extended their previous research to the fuzzy multiobjective linear bilevel programming (FMOLBP) problem and developed an approximation branch-and-bound algorithm to solve the FMOLBP problem; see [46]. A new definition proposed in [32, 35] implies that the upper level constraints containing the second variable move to the lower level. This point changes the nature of the problem; see [1, 8]. On the other hand, the method given in [46] solves several crisp multiobjective nonlinear programming problems along with many crisp objectives and constraints by the approximation branch and bound method until the termination criterion holds. Hence, it will have high computational complexity and we have to check the complementary slackness conditions in each of the iterations, which is time-consuming.

Toksari and Bilim [13] focused on decentralized bilevel multiobjective fractional programming problems (DBL-MOFPPs) with a single decision-maker at the first level and multiple decision-makers at the second level, where the coefficients are crisp numbers. They presented an FGP based on Jacobian matrix for DBL-MOFPP. In the approach, membership functions of fuzzy goals are constructed for all objectives at two levels, and they are linearized by using the Jacobian matrix. Then the FGP approach is applied to obtain the highest degree of each of the membership goals by finding the most satisfactory solution for all decision-makers. This algorithm was applied for the crisp problems based on FGP. Peric, Babic, and Omerovi [15] studied a similar problem with crisp coefficients and presented a method based on the FGP. The approaches fail to solve the proposed model of this paper due to its fuzzy coefficients. Liu and Yang [18] proposed an interactive programming method to obtain the compromise optimal solution of the multilevel multiobjective linear programming problem with crisp parameters. The resolution process is done in two stages: analysis and decision-making. If the objective values were not satisfactory, then the decision-maker improves the values by giving a concession to the unsatisfied values. When the satisfactory degree in upper levels is mat, the problem of lower levels will be solved. The proposed method in [18] was designed for the case that the coefficients are crisp numbers. The proposed method cannot be applied to the proposed model with fuzzy coefficients. Kamal et al. [19] applied crisp bilevel multiobjective programming for production planning problems. Then an FGP algorithm was designed to solve the problem. This approach cannot also be applied to solve FMOLBP due to fuzziness of its coefficients. Abdelaziz and Mejri

[23] formulated a shared inventory model as a bilevel programming problem. Both emergency and backorders are considered in the model. The interaction between decision-makers is done in an uncertain environment of probability type. They designed a decentralized bilevel programming problem, where the leader has multiple objectives. The numerical study was focused on the simulation. We cannot use the approach to solve the proposed model with fuzzy coefficients because the approach in [23] discusses with the uncertainty of the probability type. Baky Eid, and El Sayed [3] extended the FGP to solve BL-MOPP with fuzzy demands that are given as triangular fuzzy numbers. The FGP algorithm is applied to attain the highest degree of each of the membership goals by minimizing their deviational variables and does not present the membership functions of the objectives of leader and follower, completely. In this paper, we will answer the following questions about the proposed method in [3]:

1- How can we find the membership functions of objectives of leader and follower instead of obtaining the highest degree of each of the membership goals?

2- Finding the required goals is not always easy, especially, when the dimension of the problem increases. How can we overcome these difficulties?

The answers to the above questions are the motivations of this paper. In this paper, we will consider the FMOLBP problem in a general case, where all coefficients are fuzzy numbers. Zadeh's extension principle [6, 40, 41] is applied to solve the problem. In this approach, a pair of crisp multiobjective linear three-level mathematical programming problem is formulated to compute the lower and upper bounds of the α-level of the objective values. Then, one of the scalarization methods, called weighting method, dual theory, and vertex enumeration approach are applied to solve the crisp multiobjective linear three-level programming problems. Finally, the membership function of the fuzzy compromise objective value is derived numerically by enumerating different values of α. This is the first contribution of the paper. In this approach, the KKT conditions are not used because of its nonlinearity nature. When the constraint functions at the upper level have an arbitrary linear form, the K-best algorithm cannot always find the optimal solution; see [31]. Hence, the most suitable method will be the vertex enumeration approach along with the weighting method and dual theory. These techniques preserve the linearity property of the problem. In addition, the results of the proposed approach are compared with the results of existing approaches. It shows the efficiency of the proposed approach with respect to the existing methods. This point is the second contribution of the paper.

The organization of this paper is as follows: Section 2 is formed of three subsections. The notations and properties of fuzzy numbers are reminded. Also, the weighting method is reviewed to solve multiobjective linear programming problem in the second subsection. In the third subsection, the multiobjective linear bilevel programming problem is formulated and an approach is presented to solve it based on the weighting method, dual theory, and vertex enumeration approach. Section 3 is divided into two subsections. The first subsection formulates the FMOLBP problem. The second subsection designs a solution procedure to solve the problem based on Zadeh's extension principle. Section 4 designs an efficient algorithm to solve the problem of FMOLBP based on the results of Section 3. A numerical example is presented to illustrate the algorithm in Section 5. Section 6 presents a comparison among the proposed algorithm and the existing methods to show the efficiency of the proposed algorithm. Conclusions are presented in Section 7.

2 Preliminaries

This section provides the required preliminaries in terms of three subsections. The preliminaries are about fuzzy numbers, weighting method in multiobjective linear programming problem, and multiobjective linear bilevel programming.

2.1 Fuzzy numbers

Let $\mathbb{R}$ be the set of all real numbers, let $\mathbb{R}^n$ be n-dimensional Euclidean space, and let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ be two vectors, where $x_i, y_i \in \mathbb{R}$, $i = 1, \ldots, n$. For any two vectors $x, y \in \mathbb{R}^n$, we write $x \geq y$ if and only if $x_i \geq y_i$, for all $i = 1, \ldots, n$. The notation "o" displays the zero vector. Its dimension is determined in the context.

Assume $X \subseteq \mathbb{R}$. In this paper, it is supposed that the set X is equal to $\mathbb{R}$ or $[0, 1]$. If $\tilde{a}$ is a fuzzy subset of X with membership function $\mu_{\tilde{a}} : X \to [0, 1]$, then the α-level set of $\tilde{a}$ is as $\tilde{a}_\alpha = \{x \in X \mid \mu_{\tilde{a}}(x) \geq \alpha\}$ for all $\alpha \in (0, 1]$ and $\tilde{a}_0 = cl(\{x \in X \mid \mu_{\tilde{a}}(x) > 0\})$.

Definition 1. (a) Assume that $FN(X)$ is the set of all fuzzy subsets $\tilde{a}$ of X with $\mu_{\tilde{a}}$ satisfying the following conditions:
(i) There exists $x \in X$ such that $\mu_{\tilde{a}}(x) = 1$;
(ii) $\mu_{\tilde{a}}(\lambda.x + (1 - \lambda).y) \geq \min\{\mu_{\tilde{a}}(x), \mu_{\tilde{a}}(y)\}$ for all $x, y \in X$ and $\lambda \in [0, 1]$;
(iii) $\{x \in X \mid \mu_{\tilde{a}}(x) \geq \alpha\} = \tilde{a}_\alpha$ is a closed subset for each $\alpha \in (0, 1]$;
(iv) The 0-level set $\tilde{a}_0$ is a compact subset of X.
(b) A triangular fuzzy number is a fuzzy number that is presented by $\tilde{a} = (a_1, a_2, a_3)$, where $a_1 \leq a_2 \leq a_3$ and its membership function is as follows:

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & \text{if } a_1 \leq x \leq a_2, \\ \frac{a_3-x}{a_3-a_2} & \text{if } a_2 \leq x \leq a_3, \\ 0 & otherwise. \end{cases}$$

With regard to Definition 1, the α-level set $\tilde{a}$ can be written as $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$; see [40]. In the triangular case, the α-level set $\tilde{a}$ is as $\tilde{a}_\alpha = [a_1 + (a_2 - a_1).\alpha, a_3 - (a_3 - a_2).\alpha]$. If $\tilde{a}$ is a crisp number with the value m, then its membership function is as follows:

$$\mu_{\tilde{a}}(x) = \begin{cases} 1 & \text{if } x = m, \\ 0 & otherwise. \end{cases} \tag{1}$$

The set of all finite fuzzy numbers on $\mathbb{R}$ is denoted by $FN^*(\mathbb{R})$.

Definition 2. Let $\tilde{a}_i \in FN(\mathbb{R})$ for $i = 1, \ldots, n$. We define $\tilde{a} = (\tilde{a}_1, \ldots, \tilde{a}_n)$ and $\mu_{\tilde{a}} : \mathbb{R}^n \to [0, 1]$ by $\mu_{\tilde{a}}(x) = \min\limits_{i=1,\ldots,n} \mu_{\tilde{a}_i}(x_i)$, where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and $\tilde{a}$ is called an n-dimensional fuzzy number on $\mathbb{R}^n$. If $\tilde{a}_i \in FN^*(\mathbb{R})$, $i = 1, \ldots, n$, then $\tilde{a}$ is called an n-dimensional finite fuzzy number on $\mathbb{R}^n$. Let $FN(\mathbb{R}^n)$ and $FN^*(\mathbb{R}^n)$ be the set of all n-dimensional fuzzy numbers and the set of all n-dimensional finite fuzzy numbers on $\mathbb{R}^n$, respectively.

2.2 Weighting method in multiobjective linear programming (MOLP) problem

Consider a multiobjective linear programming (MOLP) problem

$$\min \quad z(x) = (z_1(x), \ldots, z_k(x)), \tag{2}$$
$$\text{s.t.} \quad x \in X \subseteq \mathbb{R}^n, \tag{3}$$

where $x \in \mathbb{R}^n$ and X denote the decision-making variables and feasible set, respectively. Moreover, $Z : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ and $z_i(x)$, $i = 1, \ldots, k$, are real-valued functions over X.

Definition 3 ([17])**.** The vector x^* is said to be a Pareto optimal solution (or compromise solution) if and only if there does not exist another $x \in X$ such that $z_i(x) \leq z_i(x^*)$ for all i and $z_j(x) \neq z_j(x^*)$ for at least one j.

The weighting method for obtaining a Pareto optimal solution is to solve the weighting problem formulated by taking the weighted sum of all of the objective functions of the original MOLP (1)–(3). Thus, the weighting problem is defined by

$$\min \quad W.z(x) = \sum_{i=1}^{k} w_i.z_i(x), \tag{4}$$

$$s.t. \qquad x \in X, \quad \sum_{i=1}^{k} w_i = 1, \quad w_i \geq 0, \quad for \quad i = 1, \ldots, k, \tag{5}$$

where $W = (w_1, \ldots, w_k)$ is the vector of weighting coefficients assigned to the objective functions, and it is assumed that $W = (w_1, \ldots, w_k) \geq o$. The following theorem presents a relationship between the Pareto optimal solution of problem (1)–(3) and the optimal solution of problem (2)–(5).

Theorem 1 ([17]). If $x^* \in X$ is an optimal solution of the weighting problem (2)–(5) for some $W > o$, then x^* is a Pareto optimal solution of the MOLP (1)–(3). Conversely, if $x^* \in X$ is a Pareto optimal solution of the MOLP (1)–(3), then x^* is an optimal solution of the weighting problem (2)–(5) for some $W = (w_1, \ldots, w_k) \geq o$.

2.3 Multiobjective linear bilevel programming(MOLBP)

First of all, throughout this paper, we assume that $X = \{x \in \mathbb{R}^n | x \geq o\}$ and $Y = \{y \in \mathbb{R}^m | y \geq o\}$. The MOLBP problem is formulated as follows:
For $x \in X \subseteq \mathbb{R}^n$, $y \in Y \subseteq \mathbb{R}^m$, $F : X \times Y \longrightarrow \mathbb{R}^s$, and $f : X \times Y \longrightarrow \mathbb{R}^t$, we define

$$Z(c_1, \ldots, c_s, d_1, \ldots, d_s, c'_1, \ldots, c'_t, d'_1, \ldots, d'_t, A_1, B_1, A_2, B_2, k_1, k_2)$$
$$= \min_{x \in X} F(x, y) = (c_1.x + d_1.y, \ldots, c_s.x + d_s.y)^T, \tag{6}$$
$$s.t. \quad A_1.x + B_1.y \leq k_1, \tag{7}$$
$$\min_{y \in Y} f(x, y) = (c'_1.x + d'_1.y, \ldots, c'_t.x + d'_t.y)^T, \tag{8}$$
$$s.t. \quad A_2.x + B_2.y \leq k_2, \tag{9}$$

where

$$c_i = (c_{1i}, \ldots, c_{ni}),\ c'_j = (c'_{1j}, \ldots, c'_{nj}) \in \mathbb{R}^n, \quad i = 1, \ldots, s, j = 1, \ldots, t,$$

$$d_i = (d_{1i}, \ldots, d_{mi}),\ d'_j = (d'_{1j}, \ldots, d'_{mj}) \in \mathbb{R}^m, \quad i = 1, \ldots, s,\ j = 1, \ldots, t,$$

$k_1 = (k_{11}, \ldots, k_{p1}) \in \mathbb{R}^p$, $k_2 = (k_{12}, \ldots, k_{q2}) \in \mathbb{R}^q$, $A_1 = [a_{ij}] \in \mathbb{R}^{p \times n}$, $B_1 = [b_{ij}] \in \mathbb{R}^{p \times m}$, $A_2 = [e_{ij}] \in \mathbb{R}^{q \times n}$, and $B_2 = [s_{ij}] \in \mathbb{R}^{q \times m}$.
Let $S = \{(x, y) \in X \times Y | A_1.x + B_1.y \leq k_1, A_2.x + B_2.y \leq k_2\}$ and $S(x) = \{y \in Y \mid B_2.y \leq k_2 - A_2.x\}$. Denote by $R(x)$ the set of the Pareto optimal solutions of the lower level problem for any fixed x. Then we have

the following definitions.

Definition 4 ([8])**.** For a fixed $(x, y) \in S$, if y is a Pareto optimal solution to the lower level problem, then (x, y) is a feasible solution to problem (6)–(6).

Definition 5 ([8])**.** If (x, y) is a feasible solution to problem (6)–(6) and x is a Pareto optimal solution to the upper level problem for fixed y, then (x, y) is a Pareto optimal solution to problem (6)–(6).

We now apply the weighting method and rewrite problem (6)–(6) as follows:

$$\overline{Z}(c_1, \ldots, c_s, d_1, \ldots, d_s, c'_1, \ldots, c'_t, d'_1, \ldots, d'_t, A_1, B_1, A_2, B_2, k_1, k_2)$$

$$= \min_{x \in X} \quad W.F(x, y) = \sum_{i=1}^{s} w_i.(c_i.x + d_i.y), \tag{10}$$

$$s.t. \quad A_1.x + B_1.y \leq k_1, \sum_{i=1}^{s} w_i = 1, w_i \geq 0, \quad for \quad i = 1, \ldots, s, \tag{11}$$

$$\min_{y \in Y} \quad W'.f(x, y) = \sum_{i=1}^{t} w'_i.(c'_i.x + d'_i.y), \tag{12}$$

$$s.t. \quad A_2.x + B_2.y \leq k_2, \sum_{i=1}^{t} w'_i = 1, w'_i \geq 0, \quad for \quad i = 1, \ldots, t, . \tag{13}$$

For some fixed $x \geq o$, the compromise solution of the lower programming can be found by solving the following linear programming with a given weight $W' \geq o$.

$$\min_{y} \quad \sum_{i=1}^{t} w'_i.(c'_i.x + d'_i.y), \tag{14}$$

$$s.t. \quad B_2.y \leq k_2 - A_2.x, \tag{15}$$

$$\sum_{i=1}^{t} w'_i = 1, w'_i \geq 0, \quad for \quad i = 1, \ldots, t, y \geq o. \tag{16}$$

Since $c'_i.x$, for $i = 1, \ldots, t$, are constant, we can ignore the terms. Hence, a dual multiplier u_i is considered for each constraint (15), the primal variables are in terms of the vector of y, and other expressions are considered as constant. Thus, we can get the dual of problem (14)–(16) according to the dual definition for linear programming problems in [40] as follows:

$$\max_{u} \quad (k_2 - A_2.x)^T.u, \tag{17}$$

$$s.t. \quad B_2^T.u \geq \sum_{i=1}^{t} w'_i.d'_i, \tag{18}$$

$$\sum_{i=1}^{t} w'_i = 1,\ w'_i \geq 0, \quad for \quad i = 1, \ldots, t,\ u \geq o, \tag{19}$$

where $u \in \mathbb{R}^q$ is the dual multipliers. The following theorem converts MOLBP problem (6)–(6) to a one-level programming problem with one objective function.

Theorem 2. The pair (x^*, y^*) is the compromise solution of problem (6)–(6) if and only if there exists u^* such that (x^*, y^*, u^*) is the solution of the following program:

$$\min \quad F(x,y) = \sum_{i=1}^{s} w_i.(c_i.x + d_i.y), \tag{20}$$

$$s.t. \quad A_1.x + B_1.y \leq k_1, \tag{21}$$

$$B_2^T.u \geq \sum_{i=1}^{t} w'_i.d'_i, \tag{22}$$

$$\sum_{i=1}^{t} w'_i.(d'_i.y) - (k_2 - A_2.x)^T.u = 0, \tag{23}$$

$$\sum_{i=1}^{t} w'_i = 1, w'_i \geq 0, \quad for \quad i = 1, \ldots, t, \quad x, y, u \geq o. \tag{24}$$

Proof. The proof is similar to the proof of Theorem 1 in [33]. □

Let $U = \{u | B_2^T.u \geq \sum_{i=1}^{t} w'_i.d'_i, u \geq o\}$ denote the feasible region to linear programming (17)–(19). If $U \neq \emptyset$, then the set U has at least one vertex and at most finite vertices. Moreover, if there is an optimal solution to the linear programming (17)–(19), then it must be one of the vertex points of set U (see [28]). With attention to the above results, we can convert problem (20)–(24) to a series of the following linear programming problems by obtaining all vertices of U, denoted by $U^v = \{u^1, u^2, \ldots, u^r\}$, according to the method in linear programming problem from [28].

$$LP(u^p): \min_{x,y} \quad F(x,y)=\sum_{i=1}^{s} w_i.(c_i.x+d_i.y), \tag{25}$$

$$s.t. \quad A_1.x+B_1.y \le k_1, \tag{26}$$

$$\sum_{i=1}^{t} w'_i.(d'_i.y)-(k_2-A_2.x)^T.u^p=0, \tag{27}$$

$$\sum_{i=1}^{t} w'_i=1, w'_i \ge 0, \quad for \quad i=1,\dots,t, \quad x,y \ge o. \tag{28}$$

We can solve the above $LP(u^p)$ for $p=1,\dots,r$. Let $I \subseteq \{1,\dots,r\}$ denote the Indices of problems that $LP(u^i)$, for $i \in I$, has an optimal solution. For $i \in I$, let (x^i, y^i) be the optimal solution for problem $LP(u^i)$ and $F(x^k,y^k)=\min\{F(x^i,y^i)|i \in I\}$. Therefore, we have the following theorem.

Theorem 3. The pair (x^k, y^k) is an optimal solution for problem (20)–(24) and a Pareto optimal solution for problem (6)–(6).

We are now ready to formulate and solve fuzzy multiobjective linear bilevel programming.

3 Fuzzy multiobjective linear bilevel programming

In this section, the FMOLBP problem is formulated and a procedure is designed to solve the problem based on Zadeh's extension principle [6, 40, 41].

3.1 The formulation of FMOLBP

For $x \in X \subseteq \mathbb{R}^n, y \in Y \subseteq \mathbb{R}^m$, $\tilde{F}: X \times Y \longrightarrow FN^*(\mathbb{R}^s)$, and $\tilde{f}: X \times Y \longrightarrow FN^*(\mathbb{R}^t)$, the problem of FMOLBP is formulated by

$$\tilde{Z}=\min_{x \in X} \quad \tilde{F}(x,y)=(\tilde{c}_1.x+\tilde{d}_1.y,\dots,\tilde{c}_s.x+\tilde{d}_s.y)^T, \tag{29}$$

$$s.t. \quad \tilde{A}_1.x+\tilde{B}_1.y \le \tilde{k}_1, \tag{30}$$

$$\min_{y \in Y} \quad \tilde{f}(x,y)=(\tilde{c}'_1.x+\tilde{d}'_1.y,\dots,\tilde{c}'_t.x+\tilde{d}'_t.y)^T, \tag{31}$$

$$s.t. \quad \tilde{A}_2.x+\tilde{B}_2.y \le \tilde{k}_2, \tag{32}$$

$$x \ge o, \quad y \ge o, \tag{33}$$

where

$$\tilde{c}_i=(\tilde{c}_{1i},\dots,\tilde{c}_{ni}), \tilde{c}'_j=(\tilde{c}'_{1j},\dots,\tilde{c}'_{nj}) \in FN^*(\mathbb{R}^n), \ i=1,\dots,s, \ j=1,\dots,t,$$

$\tilde{d}_i = (\tilde{d}_{1i}, \ldots, \tilde{d}_{mi}), \tilde{d}'_j = (\tilde{d}'_{1j}, \ldots, \tilde{d}'_{mj}) \in FN^*(\mathbb{R}^m),\ i = 1, \ldots, s,\ j = 1, \ldots, t,$

$\tilde{k}_1 = (\tilde{k}_{11}, \ldots, \tilde{k}_{p1}) \in FN^*(\mathbb{R}^p),\ \tilde{k}_2 = (\tilde{k}_{12}, \ldots, \tilde{k}_{q2}) \in FN^*(\mathbb{R}^q),$
$\tilde{A}_1 = [\tilde{a}_{ij}]_{p\times n},\ \tilde{a}_{ij} \in FN^*(\mathbb{R}),\ \tilde{B}_1 = [\tilde{b}_{ij}]_{p\times m},\ \tilde{b}_{ij} \in FN^*(\mathbb{R}),$
$\tilde{A}_2 = [\tilde{e}_{ij}]_{q\times n},\ \tilde{e}_{ij} \in FN^*(\mathbb{R})$, and $\tilde{B}_2 = [\tilde{s}_{ij}]_{q\times m},\ \tilde{s}_{ij} \in FN^*(\mathbb{R})$.
The feasible solution set of FMOLBP is defined as follows:

$$S = \{(x, y) \in X \times Y | \tilde{A}_1.x + \tilde{B}_1.y \leq \tilde{k}_1, \tilde{A}_2.x + \tilde{B}_2.y \leq \tilde{k}_2\}. \qquad (34)$$

In an FMOLBP problem, for each $(x, y) \in S$, the value of the objective functions $\tilde{F}(x, y) = (\tilde{F}_1(x, y), \ldots, \tilde{F}_s(x, y))$ and $\tilde{f}(x, y) = (\tilde{f}_1(x, y), \ldots, \tilde{f}_t(x, y))$ of leader and follower is s-dimension and t-dimension fuzzy vectors, respectively.

3.2 The solution procedure for the problem (29)–(33)

We will now consider problem (29)–(33). Let $\mu_{\tilde{c}_{1i}}, \ldots, \mu_{\tilde{c}_{ni}}, \mu_{\tilde{c}'_{1j}}, \ldots, \mu_{\tilde{c}'_{nj}}, \mu_{\tilde{d}_{1i}}, \ldots, \mu_{\tilde{d}_{mi}}, \mu_{\tilde{d}'_{1j}}, \ldots, \mu_{\tilde{d}'_{mj}},\ i = 1, \ldots, s,\ j = 1, \ldots, t,\ \mu_{\tilde{k}_{i1}}, \mu_{\tilde{k}_{j2}},\ i = 1, \ldots, p,\ j = 1, \ldots, q,\ \mu_{\tilde{a}_{ij}},\ i = 1, \ldots, p, j = 1, \ldots, n,$ $\mu_{\tilde{b}_{ij}},\ i = 1, \ldots, p, j = 1, \ldots, m,\ \mu_{\tilde{e}_{ij}},\ i = 1, \ldots, q, j = 1, \ldots, n$, and $\mu_{\tilde{s}_{ij}}$, $i = 1, \ldots, q, j = 1, \ldots, m$, denote the membership functions $\tilde{c}_{1i}, \ldots, \tilde{c}_{ni}$, $\tilde{c}'_{1j}, \ldots, \tilde{c}'_{nj},\ \tilde{d}_{1i}, \ldots, \tilde{d}_{mi},\ \tilde{d}'_{1j}, \ldots, \tilde{d}'_{mj},\ i = 1, \ldots, s, j = 1, \ldots, t,\ \tilde{k}_{i1},\ \tilde{k}_{j2}$, $i = 1, \ldots, p, j = 1, \ldots, q,\ \tilde{a}_{ij},\ i = 1, \ldots, p, j = 1, \ldots, n,\ \tilde{b}_{ij},\ i = 1, \ldots, p, j = 1, \ldots, m,\ \tilde{e}_{ij},\ i = 1, \ldots, q, j = 1, \ldots, n$, and $\tilde{s}_{ij},\ i = 1, \ldots, q, j = 1, \ldots, m$, respectively.

We have $\tilde{v} = \{(v, \mu_{\tilde{v}}(v)) \mid v \in S(\tilde{v})\}$, where $\tilde{v}$ is one of the parameters $\tilde{c}_{1i}, \ldots, \tilde{c}_{ni},\ \tilde{c}'_{1j}, \ldots, \tilde{c}'_{nj},\ \tilde{d}_{1i}, \ldots, \tilde{d}_{mi},\ \tilde{d}'_{1j}, \ldots, \tilde{d}'_{mj},\ i = 1, \ldots, s, j = 1, \ldots, t,$ $\tilde{k}_{i1},\ \tilde{k}_{j2},\ i = 1, \ldots, p, j = 1, \ldots, q,\ \tilde{a}_{ij},\ i = 1, \ldots, p, j = 1, \ldots, n,\ \tilde{b}_{ij},\ i = 1, \ldots, p, j = 1, \ldots, m,\ \tilde{e}_{ij},\ i = 1, \ldots, q, j = 1, \ldots, n,\ \tilde{s}_{ij},\ i = 1, \ldots, q, j = 1, \ldots, m$, and $S(\tilde{v})$ is the support of $\tilde{v}$. Denote the α-cuts of $\tilde{v}$ by

$$(\tilde{v})_\alpha = [(\tilde{v})^L_\alpha, (\tilde{v})^U_\alpha] = [\min_v \{v \in S(\tilde{v}) \mid \mu_{\tilde{v}}(v) \geq \alpha\},$$
$$\max_v \{v \in S(\tilde{v}) \mid \mu_{\tilde{v}}(v) \geq \alpha\}], \qquad (35)$$

where $\tilde{v}$ is one of the above fuzzy parameters. These intervals show that the coefficients of objective function and constraints in problem (29)–(33) lie at the α-level set. Here, we intend to obtain the membership function $\tilde{Z}$. To do this, Zadeh's extension principle [6, 40, 41] is used. Based on the extension principle, $\mu_{\tilde{Z}}$ can be written by

$$\mu_{\tilde{Z}}(z) = \sup_{\substack{c_i,\, c'_j,\, d_i,\, d'_j,\\ k_1,\, k_2,\, A_1,\, B_1,\\ A_2,\, B_2.}} \min\{\mu_{\tilde{c}_{1i}}(c_{1i}), \ldots, \mu_{\tilde{c}_{ni}}(c_{ni}), \mu_{\tilde{c}'_{1j}}(c'_{1j}), \ldots, \mu_{\tilde{c}'_{nj}}(c'_{nj}),$$
$$\mu_{\tilde{d}_{1i}}(d_{1i}), \ldots, \mu_{\tilde{d}_{mi}}(d_{mi}), \mu_{\tilde{d}'_{1j}}(d'_{1j}), \ldots, \mu_{\tilde{d}'_{mj}}(d'_{mj}), \mu_{\tilde{k}_{i1}}(k_{i1}),$$
$$\mu_{\tilde{k}_{j2}}(k_{j2}), \mu_{\tilde{a}_{ij}}(a_{ij}), \mu_{\tilde{b}_{ij}}(b_{ij}), \mu_{\tilde{e}_{ij}}(e_{ij}), \mu_{\tilde{s}_{ij}}(s_{ij}), \text{ for all } i,j \mid$$
$$z = Z(c_1, \ldots, c_s, d_1, \ldots, d_s, c'_1, \ldots, c'_t, d'_1, \ldots, d'_t, A_1, B_1, A_2, B_2, k_1, k_2)\}, \tag{36}$$

where matrices $c_i = [c_{li}]_{1\times n}$, $c'_j = [c'_{lj}]_{1\times n}$, $d_i = [d_{li}]_{1\times m}$, $d'_j = [d'_{lj}]_{1\times m}$, $i = 1, \ldots, s, j = 1, \ldots, t$, $k_1 = [k_{i1}]_{p\times 1}$, $k_2 = [k_{j2}]_{q\times 1}$, $A_1 = [a_{ij}]_{p\times n}$, $B_1 = [b_{ij}]_{p\times m}$, $A_2 = [e_{ij}]_{q\times n}$, and $B_2 = [s_{ij}]_{q\times m}$, are the supports of $\tilde{c}_i, \tilde{c}'_j, \tilde{d}_i, \tilde{d}'_j, i = 1, \ldots, s, j = 1, \ldots, t, \tilde{k}_1, \tilde{k}_2, \tilde{A}_1, \tilde{B}_1, \tilde{A}_2$, and $\tilde{B}_2$, respectively. Also $Z(c_1, \ldots, c_s, d_1, \ldots, d_s, c'_1, \ldots, c'_t, d'_1, \ldots, d'_t, A_1, B_1, A_2, B_2, k_1, k_2)$ is defined as follows:

$$\begin{aligned}
&Z(c_1, \ldots, c_s, d_1, \ldots, d_s, c'_1, \ldots, c'_t, d'_1, \ldots, d'_t, A_1, B_1, A_2, B_2, k_1, k_2)\\
&= \min_{x\in X} F(x,y) = (c_1.x + d_1.y, \ldots, c_s.x + d_s.y)^T,\\
&s.t. \quad A_1.x + B_1.y \le k_1,\\
&\min_{y\in Y} f(x,y) = (c'_1.x + d'_1.y, \ldots, c'_t.x + d'_t.y)^T,\\
&s.t. \quad A_2.x + B_2.y \le k_2.
\end{aligned} \tag{37}$$

If the α-cuts of $\tilde{Z}$ at all α values are converted to the same point, then the objective functions are crisp numbers. Otherwise, they are fuzzy numbers. According to (36), $\mu_{\tilde{Z}}$ is the minimum of $\mu_{\tilde{v}}(v)$ for $v = c_i, c'_j, d_i, d'_j, k_{i1}, k_{j2}, a_{ij}, b_{ij}, e_{ij}, s_{ij},$ for all i,j. We need $\mu_{\tilde{v}}(v) \ge \alpha$, for $v = c_i, c'_j, d_i, d'_j, k_{i1}, k_{j2}, a_{ij}, b_{ij}, e_{ij}, s_{ij},$ for all i,j, and at least one $\mu_{\tilde{v}}(v) \ge \alpha$, for $v = c_i, c'_j, d_i, d'_j, k_{i1}, k_{j2}, a_{ij}, b_{ij}, e_{ij}, s_{ij},$ for all i,j, equal to α such that $z = Z(c_1, \ldots, c_s, d_1, \ldots, d_s, c'_1, \ldots, c'_t, d'_1, \ldots, d'_t, A_1, B_1, A_2, B_2, k_1, k_2)$ to satisfy $\mu_{\tilde{Z}}(z) = \alpha$. To compute the membership function $\mu_{\tilde{Z}}$, we intend to compute the left and right shape function of $\mu_{\tilde{Z}}$. It is equivalent to finding $\tilde{Z}^L_\alpha$ and $\tilde{Z}^U_\alpha$ of the α-cuts of $\tilde{Z}$. Since $\tilde{Z}^L_\alpha$ is the minimum of $Z(c_1, \ldots, c_s, d_1, \ldots, d_s, c'_1, \ldots, c'_t, d'_1, \ldots, d'_t, A_1, B_1, A_2, B_2, k_1, k_2)$ and $\tilde{Z}^U_\alpha$ is the maximum of $Z(c_1, \ldots, c_s, d_1, \ldots, d_s, c'_1, \ldots, c'_t, d'_1, \ldots, d'_t, A_1, B_1, A_2, B_2, k_1, k_2)$, we have

$$\begin{aligned}
\tilde{Z}^L_\alpha = \min\{&Z(c_1, \ldots, c_s, d_1, \ldots, d_s, c'_1, \ldots, c'_t, d'_1, \ldots, d'_t, A_1, B_1, A_2, B_2, k_1, k_2)\\
&\mid (\tilde{v})^L_\alpha \le v \le (\tilde{v})^U_\alpha, where\ v = c_{1i}, \ldots, c_{ni}, d_{1i}, \ldots, d_{mi}, c'_{1j}, \ldots, c'_{nj},\\
&d'_{1j}, \ldots, d'_{mj}, a_{ij}, b_{ij}, e_{ij}, s_{ij}, k_{i1}, k_{j2}, \text{ for all } i,j\}
\end{aligned} \tag{38}$$

and

$$\tilde{Z}^U_\alpha = \max\{Z(c_1,\dots,c_s,d_1,\dots,d_s,c'_1,\dots,c'_t,d'_1,\dots,d'_t,A_1,B_1,A_2,B_2,k_1,k_2) \mid \tilde{v}^L_\alpha \le v \le \tilde{v}^U_\alpha, where \;\; v = c_{1i},\dots,c_{ni},d_{1i},\dots,d_{mi},c'_{1j},\dots,c'_{nj}, d'_{1j},\dots,d'_{mj},a_{ij},b_{ij},e_{ij},s_{ij},k_{i1},k_{j2}, \text{ for all } i,j\}. \tag{39}$$

They can equivalently be rewritten as two problems below:

$$\tilde{Z}^L_\alpha = \min_{\substack{(\tilde{v})^L_\alpha \le v \le (\tilde{v})^U_\alpha, where \\ v = c_{1i},\dots,c_{ni},d_{1i},\dots,d_{mi}, \\ c'_{1j},\dots,c'_{nj},d'_{1j},\dots,d'_{mj}, \\ a_{ij},b_{ij},e_{ij},s_{ij},k_{i1},k_{j2}, \text{ for all } i,j.}} \begin{cases} \min\limits_{x\in X}(c_1x+d_1y,\dots,c_sx+d_sy)^T, \\ \quad s.t. \;\; A_1x+B_1y \le k_1, \\ \qquad \min\limits_{y\in Y}(c'_1x+d'_1y,\dots,c'_tx+d'_ty)^T, \\ \qquad\quad s.t. \;\; A_2x+B_2y \le k_2, \end{cases} \tag{40}$$

and

$$\tilde{Z}^U_\alpha = \max_{\substack{(\tilde{v})^L_\alpha \le v \le (\tilde{v})^U_\alpha, where \\ v = c_{1i},\dots,c_{ni},d_{1i},\dots,d_{mi}, \\ c'_{1j},\dots,c'_{nj},d'_{1j},\dots,d'_{mj}, \\ a_{ij},b_{ij},e_{ij},s_{ij},k_{i1},k_{j2}, \text{ for all } i,j.}} \begin{cases} \min\limits_{x\in X}(c_1x+d_1y,\dots,c_sx+d_sy)^T, \\ \quad s.t. \;\; A_1x+B_1y \le k_1, \\ \qquad \min\limits_{y\in Y}(c'_1x+d'_1y,\dots,c'_tx+d'_ty)^T, \\ \qquad\quad s.t. \;\; A_2x+B_2y \le k_2. \end{cases} \tag{41}$$

We can use the weighting method for the problem to find the compromise solutions of problem (40). Hence, we can rewrite problem (40) as follows:

$$\bar{\tilde{Z}}^L_\alpha = \max_{\substack{(\tilde{v})^L_\alpha \le v \le (\tilde{v})^U_\alpha, where \\ v = c_{1i},\dots,c_{ni},d_{1i},\dots,d_{mi}, \\ c'_{1j},\dots,c'_{nj},d'_{1j},\dots,d'_{mj}, \\ a_{ij},b_{ij},e_{ij},s_{ij},k_{i1},k_{j2}, \text{ for all } i,j.}} \begin{cases} \min\limits_{x\in X} \sum\limits_{i=1}^{s} w_i(c_i.x) + \sum\limits_{i=1}^{s} w_i(d_i.y), \\ \quad s.t. \;\; A_1x+B_1y \le k_1, \\ \qquad \sum\limits_{i=1}^{s} w_i = 1, w_i \ge 0, for \;\; i=1,\dots,s, \\ \qquad \min\limits_{y\in Y} \sum\limits_{i=1}^{t} w'_i(c'_i.x) + \sum\limits_{i=1}^{t} w'_i(d'_i.y), \\ \qquad s.t. \;\; A_2x+B_2y \le k_2, \\ \qquad \sum\limits_{i=1}^{t} w'_i = 1, w'_i \ge 0, for \;\; i=1,\dots,t. \end{cases} \tag{42}$$

Since model (42) finds the minimum of all minimum objective values, we should consider the objective function of leader in the best case, that is, we set c_i and d_i to their lower bound, in other words, $(\tilde{c}_i)^L_\alpha$ and $(\tilde{d}_i)^L_\alpha$, for each i, respectively. Therefore, we can rewrite problem (42) as follows:

$$\overline{\tilde{Z}}_\alpha^L = \max_{\substack{(\tilde{v})_\alpha^L \le v \le (\tilde{v})_\alpha^U,\, where \\ v = c'_{1j}, \dots, c'_{nj}, d'_{1j}, \dots, d'_{mj}, \\ a_{ij}, b_{ij}, e_{ij}, s_{ij}, k_{i1}, k_{j2},\ \text{for all } i, j.}} \begin{cases} \min\limits_{x \in X} \sum\limits_{i=1}^{s} w_i((\tilde{c}_i)_\alpha^L . x) + \sum\limits_{i=1}^{s} w_i((\tilde{d}_i)_\alpha^L . y), \\ \quad s.t. \;\; A_1 x + B_1 y \le k_1, \\ \qquad \sum\limits_{i=1}^{s} w_i = 1, w_i \ge 0, for \;\; i = 1, \dots, s, \\ \qquad \min\limits_{y \in Y} \sum\limits_{i=1}^{t} w'_i(c'_i . x) + \sum\limits_{i=1}^{t} w'_i(d'_i . y), \\ \qquad s.t. \;\; A_2 x + B_2 y \le k_2, \\ \qquad \sum\limits_{i=1}^{t} w'_i = 1, w'_i \ge 0, for \;\; i = 1, \dots, t. \end{cases} \tag{43}$$

Notation 1. Assume that $\tilde{A} = [\tilde{a}_{ij}]_{m \times n}$ is a fuzzy matrix with fuzzy components. Then $(\tilde{A})_\alpha^L = [(\tilde{a}_{ij})_\alpha^L]_{m \times n}$ and $(\tilde{A})_\alpha^U = [(\tilde{a}_{ij})_\alpha^U]_{m \times n}$.

To obtain the minimum objective value, we should consider the values of $A_1, B_1, A_2, B_2, k_1,$ and k_2 in $(\tilde{A}_1)_\alpha^U, (\tilde{B}_1)_\alpha^U, (\tilde{A}_2)_\alpha^U, (\tilde{B}_2)_\alpha^U, (\tilde{k}_1)_\alpha^L,$ and $(\tilde{k}_2)_\alpha^L$, respectively, because we can obtain the largest feasible solution set, in this case. Hence, the least value is found for the objective function. To this end, problem (43) is converted to the following problem.

$$\overline{\tilde{Z}}_\alpha^L = \min_{\substack{(\tilde{v})_\alpha^L \le v \le (\tilde{v})_\alpha^U,\, where \\ v = c'_{1j}, \dots, c'_{nj}, d'_{1j}, \dots, d'_{mj}, \\ \text{for all } j.}} \begin{cases} \min\limits_{x \in X} \sum\limits_{i=1}^{s} w_i((\tilde{c}_i)_\alpha^L . x) + \sum\limits_{i=1}^{s} w_i((\tilde{d}_i)_\alpha^L . y), \\ \quad s.t. \;\; (\tilde{A}_1)_\alpha^U x + (\tilde{B}_1)_\alpha^U y \le (\tilde{k}_1)_\alpha^L, \\ \qquad \sum\limits_{i=1}^{s} w_i = 1, w_i \ge 0, for \;\; i = 1, \dots, s, \\ \qquad \min\limits_{y \in Y} \sum\limits_{i=1}^{t} w'_i(c'_i . x) + \sum\limits_{i=1}^{t} w'_i(d'_i . y), \\ \qquad s.t. \;\; (\tilde{A}_2)_\alpha^U x + (\tilde{B}_2)_\alpha^U y \le (\tilde{k}_2)_\alpha^L, \\ \qquad \sum\limits_{i=1}^{t} w'_i = 1, w'_i \ge 0, for \;\; i = 1, \dots, t. \end{cases} \tag{44}$$

With regard to Theorem 2, we can write the inner problem as follows:

$$\overline{\tilde{Z}}_{\alpha}^{L} = \min_{\substack{(\tilde{v})_{\alpha}^{L} \leq v \leq (\tilde{v})_{\alpha}^{U},\\ where \;\; v = c'_{1j},\ldots,c'_{nj},\\ d'_{1j},\ldots,d'_{mj}, \;\; \text{for all } j.}} \begin{cases} \min \sum_{i=1}^{s} w_i((\tilde{c}_i)_{\alpha}^{L}.x) + \sum_{i=1}^{s} w_i((\tilde{d}_i)_{\alpha}^{L}.y), \\ s.t. \;\; (\tilde{A}_1)_{\alpha}^{U} x + (\tilde{B}_1)_{\alpha}^{U} y \leq (\tilde{k}_1)_{\alpha}^{L}, \\ \quad \sum_{i=1}^{s} w_i = 1, w_i \geq 0, for \; i = 1,\ldots,s, \\ \quad \sum_{i=1}^{t} w'_i(d'_i.y) - ((\tilde{k}_2)_{\alpha}^{L} - (\tilde{A}_2)_{\alpha}^{U} x)^T.u = 0, \\ \quad ((\tilde{B}_2)_{\alpha}^{U})^T.u \geq \sum_{i=1}^{t} w'_i.d'_i, \\ \quad \sum_{i=1}^{t} w'_i = 1, w'_i \geq 0, for \; i = 1,\ldots,t. \end{cases} \tag{45}$$

Or equivalently, we can rewrite problem (45) as follows:

$$\overline{\tilde{Z}}_{\alpha}^{L} = \min \sum_{i=1}^{s} w_i((\tilde{c}_i)_{\alpha}^{L}.x) + \sum_{i=1}^{s} w_i((\tilde{d}_i)_{\alpha}^{L}.y), \tag{46}$$

$$s.t. \;\; (\tilde{A}_1)_{\alpha}^{U} x + (\tilde{B}_1)_{\alpha}^{U} y \leq (\tilde{k}_1)_{\alpha}^{L}, \tag{47}$$

$$\sum_{i=1}^{t} w'_i(d'_i.y) - ((\tilde{k}_2)_{\alpha}^{L} - (\tilde{A}_2)_{\alpha}^{U} x)^T.u = 0, \tag{48}$$

$$((\tilde{B}_2)_{\alpha}^{U})^T.u \geq \sum_{i=1}^{t} w'_i.d'_i, \tag{49}$$

$$(\tilde{d}'_i)_{\alpha}^{L} \leq d'_i \leq (\tilde{d}'_i)_{\alpha}^{U}, \;\; \text{for all } i = 1,\ldots,t, \tag{50}$$

$$\sum_{i=1}^{s} w_i = 1, \sum_{i=1}^{t} w'_i = 1, w_i, w'_j \geq 0, for \;\; i = 1,\ldots,s, \; j = 1,\ldots,t,$$

$$x, y, u \geq o. \tag{51}$$

We can now remove the constraint (50) and equivalently rewrite problem (46)–(51) as follows:

$$\overline{\tilde{Z}}_{\alpha}^{L} = \min \sum_{i=1}^{s} w_i((\tilde{c}_i)_{\alpha}^{L}.x) + \sum_{i=1}^{s} w_i((\tilde{d}_i)_{\alpha}^{L}.y), \tag{52}$$

$$s.t. \ \ (\tilde{A}_1)_{\alpha}^{U} x + (\tilde{B}_1)_{\alpha}^{U} y \leq (\tilde{k}_1)_{\alpha}^{L}, \tag{53}$$

$$\sum_{i=1}^{t} w'_i((\tilde{d'}_i)_{\alpha}^{L}.y) \leq ((\tilde{k}_2)_{\alpha}^{L} - (\tilde{A}_2)_{\alpha}^{U} x)^T.u \leq \sum_{i=1}^{t} w'_i((\tilde{d'}_i)_{\alpha}^{U}.y), \tag{54}$$

$$((\tilde{B}_2)_{\alpha}^{U})^T.u \geq \sum_{i=1}^{t} w'_i.(\tilde{d}'_i)_{\alpha}^{U}, \tag{55}$$

$$\sum_{i=1}^{s} w_i = 1, \sum_{i=1}^{t} w'_i = 1, w_i, w'_j \geq 0, for \ \ i = 1, \ldots, s, \ j = 1, \ldots, t,$$

$$x, y, u \geq o. \tag{56}$$

Let $U = \{u \geq o \mid ((\tilde{B}_2)_{\alpha}^{U})^T.u \geq \sum_{i=1}^{t} w'_i.(\tilde{d}'_i)_{\alpha}^{U}\}$. Assume that the vertex points of U are denoted by the set as $U^V = \{u^1, \ldots, u^r\}$. Hence, we should solve at most r linear programming problems as follows:

$$LP(u^k) : \overline{\tilde{Z}}_{\alpha,k}^{L} = \min \sum_{i=1}^{s} w_i((\tilde{c}_i)_{\alpha}^{L}.x) + \sum_{i=1}^{s} w_i((\tilde{d}_i)_{\alpha}^{L}.y), \tag{57}$$

$$s.t. \ \ (\tilde{A}_1)_{\alpha}^{U} x + (\tilde{B}_1)_{\alpha}^{U} y \leq (\tilde{k}_1)_{\alpha}^{L}, \tag{58}$$

$$\sum_{i=1}^{t} w'_i((\tilde{d'}_i)_{\alpha}^{L}.y) \leq ((\tilde{k}_2)_{\alpha}^{L} - (\tilde{A}_2)_{\alpha}^{U} x)^T.u^k \leq \sum_{i=1}^{t} w'_i((\tilde{d'}_i)_{\alpha}^{U}.y), \tag{59}$$

$$\sum_{i=1}^{s} w_i = 1, \sum_{i=1}^{t} w'_i = 1, w_i, w'_j \geq 0,$$

$$for \ \ i = 1, \ldots, s, \ = 1, \ldots, t, \ x, y \geq o, \tag{60}$$

where $k = 1, \ldots, r$. Solving r linear programming problems $LP(u^k)$, we can find the optimal solution of the problem for each $\alpha \in [0, 1]$ as it was mentioned in Subsection 2.3. Finding the optimal values of variables x and y for each $\alpha \in [0, 1]$, we can estimate values $(\tilde{c}_i)_{\alpha}^{L}.x + (\tilde{d}_i)_{\alpha}^{L}.y$ for each $\alpha \in [0, 1]$ and $i = 1, \ldots, s$. Hence, we can obtain $\tilde{Z}_{\alpha}^{L}$.
We now focus on obtaining the $\tilde{Z}_{\alpha}^{U}$. To do this, the weighting method is applied to the problem. Thus, problem (41) can be rewritten as follows:

$$
\overline{\tilde{Z}}^U_\alpha = \max_{\substack{(\tilde{v})^L_\alpha \leq v \leq (\tilde{v})^U_\alpha,\, where \\ v = c_{1i}, \ldots, c_{ni}, d_{1i}, \ldots, d_{mi}, \\ c'_{1j}, \ldots, c'_{nj}, d'_{1j}, \ldots, d'_{mj}, \\ a_{ij}, b_{ij}, e_{ij}, s_{ij}, k_{i1}, k_{j2}, \text{ for all } i, j.}} \begin{cases} \min\limits_{x \in X} \sum\limits_{i=1}^{s} w_i(c_i.x) + \sum\limits_{i=1}^{s} w_i(d_i.y), \\ \quad s.t. \;\; A_1 x + B_1 y \leq k_1, \\ \qquad \sum\limits_{i=1}^{s} w_i = 1, w_i \geq 0, for \;\; i = 1, \ldots, s, \\ \qquad \min\limits_{y \in Y} \sum\limits_{i=1}^{t} w'_i(c'_i.x) + \sum\limits_{i=1}^{t} w'_i(d'_i.y), \\ \qquad s.t. \;\; A_2 x + B_2 y \leq k_2, \\ \qquad \sum\limits_{i=1}^{t} w'_i = 1, w'_i \geq 0, for \;\; i = 1, \ldots, t. \end{cases} \tag{61}
$$

Since model (61) finds the maximum of all minimum objective values, we should consider the objective function of leader in the worst case, that is, we set c_i and d_i to their upper bound, in other words, $(\tilde{c}_i)^U_\alpha$ and $(\tilde{d}_i)^U_\alpha$, respectively.

$$
\overline{\tilde{Z}}^U_\alpha = \max_{\substack{(\tilde{v})^L_\alpha \leq v \leq (\tilde{v})^U_\alpha,\, where \\ v = c'_{1j}, \ldots, c'_{nj}, d'_{1j}, \ldots, d'_{mj}, \\ a_{ij}, b_{ij}, e_{ij}, s_{ij}, k_{i1}, k_{j2}, \text{ for all } i, j.}} \begin{cases} \min\limits_{x \in X} \sum\limits_{i=1}^{s} w_i((\tilde{c}_i)^U_\alpha.x) + \sum\limits_{i=1}^{s} w_i((\tilde{d}_i)^U_\alpha.y), \\ \quad s.t. \;\; A_1 x + B_1 y \leq k_1, \\ \qquad \sum\limits_{i=1}^{s} w_i = 1, w_i \geq 0, for \;\; i = 1, \ldots, s, \\ \qquad \min\limits_{y \in Y} \sum\limits_{i=1}^{t} w'_i(c'_i.x) + \sum\limits_{i=1}^{t} w'_i(d'_i.y), \\ \qquad s.t. \;\; A_2 x + B_2 y \leq k_2, \\ \qquad \sum\limits_{i=1}^{t} w'_i = 1, w'_i \geq 0, for \;\; i = 1, \ldots, t. \end{cases} \tag{62}
$$

To obtain an upper bound (or a maximum) for the minimum objective value, we should consider the values of components of the matrices A_1, B_1, A_2, B_2, k_1, and k_2 in $(\tilde{A}_1)^L_\alpha, (\tilde{B}_1)^L_\alpha, (\tilde{A}_2)^L_\alpha, (\tilde{B}_2)^L_\alpha, (\tilde{k}_1)^U_\alpha$, and $(\tilde{k}_2)^U_\alpha$, respectively, because we can find the smallest feasible region, in this case. Hence, the upper bound is obtained for the objective function. Doing this work, problem (19) is converted to the following problem.

$$\overline{\tilde{Z}}_{\alpha}^{U} = \max_{\substack{(\tilde{v})_{\alpha}^{L} \leq v \leq (\tilde{v})_{\alpha}^{U},\, where \\ v = c'_{1j}, \ldots, c'_{nj}, d'_{1j}, \ldots, d'_{mj}, \\ \text{for all } j.}} \begin{cases} \min_{x \in X} \sum_{i=1}^{s} w_i((\tilde{c}_i)_{\alpha}^{U}.x) + \sum_{i=1}^{s} w_i((\tilde{d}_i)_{\alpha}^{U}.y), \\ \quad s.t. \ \ (\tilde{A}_1)_{\alpha}^{L}.x + (\tilde{B}_1)_{\alpha}^{L}.y \leq (\tilde{k}_1)_{\alpha}^{U}, \\ \quad \sum_{i=1}^{s} w_i = 1, w_i \geq 0, for \ i = 1, \ldots, s, \\ \quad \min_{y \in Y} \sum_{i=1}^{t} w'_i(c'_i.x) + \sum_{i=1}^{t} w'_i(d'_i.y), \\ \quad s.t. \ \ (\tilde{A}_2)_{\alpha}^{L}.x + (\tilde{B}_2)_{\alpha}^{L}.y \leq (\tilde{k}_2)_{\alpha}^{U}, \\ \quad \sum_{i=1}^{t} w'_i = 1, w'_i \geq 0, for \ i = 1, \ldots, t. \end{cases} \tag{63}$$

We apply Theorem 2 for problem (3) and rewrite the problem as follows:

$$\overline{\tilde{Z}}_{\alpha}^{U} = \max_{\substack{(\tilde{v})_{\alpha}^{L} \leq v \leq (\tilde{v})_{\alpha}^{U}, \\ where \ v = c'_{1j}, \ldots, c'_{nj}, \\ d'_{1j}, \ldots, d'_{mj}, \quad \text{for all } j.}} \begin{cases} \min \sum_{i=1}^{s} w_i((\tilde{c}_i)_{\alpha}^{U}.x) + \sum_{i=1}^{s} w_i((\tilde{d}_i)_{\alpha}^{U}.y), \\ \quad s.t. \ \ (\tilde{A}_1)_{\alpha}^{L}.x + (\tilde{B}_1)_{\alpha}^{L}.y \leq (\tilde{k}_1)_{\alpha}^{U}, \\ \quad \sum_{i=1}^{t} w'_i(d'_i.y) - ((\tilde{k}_2)_{\alpha}^{U} - (\tilde{A}_2)_{\alpha}^{L}.x)^T.u = 0, \\ \quad ((\tilde{B}_2)_{\alpha}^{L})^T.u \geq \sum_{i=1}^{t} w'_i.d'_i, \\ \quad \sum_{i=1}^{s} w_i = 1, \sum_{i=1}^{t} w'_i = 1, \\ \quad w_i, w'_j \geq 0, for \ i = 1, \ldots, s, \ \ j = 1, \ldots, t, \\ \quad \quad x, y, u \geq o. \end{cases} \tag{64}$$

Let $U = \{u \geq o \mid ((\tilde{B}_2)_{\alpha}^{L})^T.u \geq \sum_{i=1}^{t} w'_i.(\tilde{d}'_i)_{\alpha}^{U}\}$. Suppose that the vertex points of set U are denoted by a set $U^V = \{u^1, \ldots, u^{r'}\}$. Hence, we should solve at most r' linear programming problems as follows:

$$LP(u^k): \overline{\tilde{Z}}^U_{\alpha,k} = \min \sum_{i=1}^{s} w_i((\tilde{c}_i)^U_\alpha.x) + \sum_{i=1}^{s} w_i((\tilde{d}_i)^U_\alpha.y), \tag{65}$$

$$s.t.\ \ (\tilde{A}_1)^L_\alpha.x + (\tilde{B}_1)^L_\alpha.y \leq (\tilde{k}_1)^U_\alpha, \tag{66}$$

$$\sum_{i=1}^{t} w'_i(\tilde{d}'_i)^L_\alpha.y \leq ((\tilde{k}_2)^U_\alpha - (\tilde{A}_2)^L_\alpha.x)^T.u^k \leq \sum_{i=1}^{t} w'_i(\tilde{d}'_i)^U_\alpha.y, \tag{67}$$

$$\sum_{i=1}^{s} w_i = 1, \sum_{i=1}^{t} w'_i = 1, w_i, w'_j \geq 0, for\ i = 1,\dots,s,\ = 1,\dots,t,$$

$$x, y \geq o, \tag{68}$$

where $k = 1,\dots,r'$. Finding the optimal values of variables x and y for each $\alpha \in [0,1]$, we can estimate values $(\tilde{c}_i)^U_\alpha.x + (\tilde{d}_i)^U_\alpha.y$ for each $\alpha \in [0,1]$ and $i = 1,\dots,s$. Hence, we can obtain $\tilde{Z}^U_\alpha$. Having $\tilde{Z}^L_\alpha$ and $\tilde{Z}^U_\alpha$, we can easily compute the compromise vector $\tilde{Z}$. We will illustrate the process of obtaining $\tilde{Z}^L_\alpha$, $\tilde{Z}^U_\alpha$, and $\tilde{Z}$ by a numerical example, in Section 5.

Remark 1. As it was considered the formulation of FMOLBP in Subsection 3.1 and the solution procedure for problem (29)–(33) in Subsection 3.2, the fuzzy numbers can be arbitrary according to Definition 1(a) and there is no restriction on the fuzzy number type. The solution procedure does not depend on the form or type of fuzzy numbers.

In the next section, we design an efficient algorithm to solve the problem of FMOLBP using the points in this section.

4 An algorithm for resolution of the problem of FMOLBP as (29)–(33)

An efficient algorithm is designed to solve problem (29)–(33). This algorithm produces the fuzzy compromise objective values of the problem as fuzzy compromise objective values in the leader and follower and fuzzy optimal weighted objective value. The step-by-step algorithm is as follows:

Step 1. Determine the weight vectors $W = (w_1,\dots,w_s)$ and $W' = (w'_1,\dots,w'_t)$ such that $\sum_{i=1}^{s} w_i = 1$, $\sum_{i=1}^{t} w'_i = 1$, and $w_i \geq 0, w'_j \geq 0$, for $i = 1,\dots,s,\ j = 1,\dots,t$.

Step 2. Determine the fuzzy parameters $\tilde{v} = \{(v, \mu_{\tilde{v}}(v)) \mid v \in S(\tilde{v})\}$ of problem (29)–(33), where $\tilde{v}$ is one of the parameters $\tilde{c}_{1i},\dots,\tilde{c}_{ni}$, $\tilde{c}'_{1j},\dots,\tilde{c}'_{nj}$, $\tilde{d}_{1i},\dots,\tilde{d}_{mi}$, $\tilde{d}'_{1j},\dots,\tilde{d}'_{mj}$, $i = 1,\dots,s, j = 1,\dots,t$, $\tilde{a}_{ij}$, $i = 1,\dots,p, j = 1,\dots,n$, $\tilde{b}_{ij}$, $i = 1,\dots,p, j = 1,\dots,m$, $\tilde{e}_{ij}$, $i = 1,\dots,q, j = 1,\dots,n$, $\tilde{s}_{ij}$,

$i = 1, \dots, q, j = 1, \dots, m$, $\tilde{k}_{i1}$, $\tilde{k}_{j2}$, $i = 1, \dots, p, j = 1, \dots, q$, and $S(\tilde{v})$ is the support of $\tilde{v}$.

Step 3. Create α-cuts of the fuzzy parameters $\tilde{v}$ as follows:

$$(\tilde{v})_\alpha = [(\tilde{v})^L_\alpha, (\tilde{v})^U_\alpha] = [\min_v \{v \in S(\tilde{v}) \mid \mu_{\tilde{v}}(v) \geq \alpha\}, \\ \max_v \{v \in S(\tilde{v}) \mid \mu_{\tilde{v}}(v) \geq \alpha\}].$$

Step 4. Create the following system:

$$\begin{cases} ((\tilde{B}_2)^U_\alpha)^T.u \geq \sum_{i=1}^{t} w'_i.(\tilde{d}'_i)^U_\alpha, \\ u \geq o. \end{cases}$$

Step 5. Select some discrete points for α from $[0, 1]$ as $\alpha_t = 0.1 \times t$ for $t = 0, 1, \dots, 10$ or $\alpha_t = 0.25 \times t$ for $t = 0, 1, 2, 3, 4$. The selection depends on the decision-maker or the required accuracy (it can be selected one of them or another partition from $[0, 1]$ for choosing the values of α).
Step 6. Compute the vertex points of the system of Step 4 for values α_t of Step 5. Denote the vertex points by $U^v_{\alpha_t} = \{u^{1,\alpha_t}, \dots, u^{r,\alpha_t}\}$. If the system of Step 4 is empty for each $\alpha \in [0, 1]$, then problem (29)–(33) has no compromise solutions. The feasible solution set is empty. End.
Step 7. Create the linear programming problem (57)–(60), for $U^v_{\alpha_t} = \{u^{1,\alpha_t}, \dots, u^{r,\alpha_t}\}$ and value α_t. Solve r linear programming problems and select the minimum objective value among them.
Step 8. Find the lower bound of α-cut of the fuzzy compromise objective values of leader and follower or the fuzzy optimal weighted objective value using Step 7, for values α_t's of Step 5.
Step 9. Create the following system:

$$\begin{cases} ((\tilde{B}_2)^L_\alpha)^T.u \geq \sum_{i=1}^{t} w'_i.(\tilde{d}'_i)^U_\alpha, \\ u \geq o. \end{cases}$$

Step 10. Compute the vertex points of the system of Step 9 for values α_t of Step 5. Denote the vertex points by $(U')^v_{\alpha_t} = \{(u')^{1,\alpha_t}, \dots, (u')^{r',\alpha_t}\}$. If the system of Step 9 is empty for each $\alpha \in [0, 1]$, then problem (29)–(33) has no compromise solutions. The feasible solution set is empty. End.
Step 11. Create the linear programming (65)–(68), for $(U')^v_{\alpha_t} = \{(u')^{1,\alpha_t}, \dots, (u')^{r',\alpha_t}\}$ and value α_t. Solve r' linear programming problems and select the maximum objective value among them.
Step 12. Find the upper bound of α-cut of the fuzzy compromise objective values of leader and follower or the fuzzy optimal weighted objective value using Step 11, for values α_t's of Step 5.
Step 13. Create the fuzzy compromise objective values of leader and follower

or the optimal weighted objective value by two Steps 8 and 12.
Step 14. End.

Some numerical examples are presented to illustrate the algorithm and compare it with other existing approaches, in Sections 5 and 6. The comparison shows the efficiency of the proposed algorithm.

5 Numerical example

In this section, we illustrate the proposed algorithm in Section 5 to solve FMOLBP by a numerical example.

Example 1. Consider the following problem:

$$\begin{aligned}
\tilde{Z} = \min_{x \in X} \tilde{F}(x, y) &= (\tilde{c}_1.x + \tilde{d}_1.y, \tilde{c}_2.x + \tilde{d}_2.y)^T, \\
s.t. \quad & \tilde{A}_1.x + \tilde{B}_1.y \leq \tilde{k}_1, \\
& \min_{y \in Y} \tilde{f}(x, y) = (\tilde{c}'_1.x + \tilde{d}'_1.y, \tilde{c}'_2.x + \tilde{d}'_2.y)^T, \\
& \quad s.t. \quad \tilde{A}_2.x + \tilde{B}_2.y \leq \tilde{k}_2,
\end{aligned}$$

where $\tilde{c}_1 = [\tilde{c}_{11}] = [(1, 2, 3)]$, $\tilde{d}_1 = [\tilde{d}_{11}] = [(-1, 1, 2)]$, $\tilde{c}_2 = [\tilde{c}_{12}] = [(2, 3, 5)]$, $\tilde{d}_2 = [\tilde{d}_{12}] = [(4, 5, 6)]$, $\tilde{c}'_1 = [\tilde{c}'_{11}] = [(2, 3, 4)]$, $\tilde{d}'_1 = [\tilde{d}'_{11}] = [(3, 4, 5)]$, $\tilde{c}'_2 = [\tilde{c}'_{12}] = [(-1, 0, 1)]$, $\tilde{d}'_2 = [\tilde{d}'_{12}] = [(1.5, 2.3, 3.5)]$, $\tilde{A}_1 = \begin{pmatrix} \tilde{a}_{11} \\ \tilde{a}_{21} \end{pmatrix} = \begin{pmatrix} (-3, -2, -1) \\ (-2, -1, 0) \end{pmatrix}$, $\tilde{B}_1 = \begin{pmatrix} \tilde{b}_{11} \\ \tilde{b}_{21} \end{pmatrix} = \begin{pmatrix} (-1, 0, 1) \\ (1.5, 2.5, 3.5) \end{pmatrix}$, $\tilde{k}_1 = \begin{pmatrix} \tilde{k}_{11} \\ \tilde{k}_{21} \end{pmatrix} = \begin{pmatrix} (5, 6, 7) \\ (6, 7, 9) \end{pmatrix}$, $\tilde{A}_2 = [\tilde{e}_{11}] = [(1, 1, 1)]$, $\tilde{B}_2 = [\tilde{s}_{11}] = [(2.2, 3.4, 4.6)]$, and $\tilde{k}_2 = [\tilde{k}_{12}] = [(5, 6, 8)]$.
Step 1. The weight vectors are $w_1 = w_2 = w'_1 = w'_2 = 0.5$.
Step 2. The fuzzy parameters $\tilde{v} = \{(v, \mu_{\tilde{v}}(v)) \mid v \in S(\tilde{v})\}$ of problem (29)–(33) for this example are given above.
Step 3. The α-cuts of the fuzzy parameters of $\tilde{v} = (v_1, v_2, v_3)$ can be computed the following formulas: $(\tilde{v})_\alpha = [(\tilde{v})^L_\alpha, (\tilde{v})^U_\alpha]$, where $(\tilde{v})^L_\alpha = v_1 + (v_2 - v_1)\alpha$ and $(\tilde{v})^U_\alpha = v_3 - (v_3 - v_2)\alpha$.
Step 4. The system corresponding to this step for this example is
$U = \{u \geq o \mid (4.6 - 1.2\alpha).u \geq w'_1(5 - \alpha) + w'_2(3.5 - 1.2\alpha)\}$.
Step 5. The selected values for α are as 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, and 1.
Step 6. The vertex points of set U^v_α are easily obtained for each $\alpha = 0, 0.1, \ldots, 1$ as $u_\alpha = \frac{0.5(5-\alpha)+0.5(3.5-1.2\alpha)}{(4.6-1.2\alpha)}$. Moreover U^v_α is singleton for each α.
Step 7. The linear programming problem (57)–(60), for U^v_α, and the value α is created and the problem related to $\overline{\overline{Z}}^L_{\alpha,k}$ is as follows:

$$
\begin{aligned}
LP(u^k):\bar{\tilde{Z}}^L_{\alpha,k} &= \min(w_1(1+\alpha)+w_2(2+\alpha))x+(w_1(-1+2\alpha)+w_2(4+\alpha))y,\\
s.t.\quad &\begin{pmatrix}(-1-\alpha)\\(-\alpha)\end{pmatrix}.x+\begin{pmatrix}(-\alpha+1)\\(-\alpha+3.5)\end{pmatrix}.y\le\begin{pmatrix}(5+\alpha)\\(6+\alpha)\end{pmatrix},\\
&(w'_1(3+\alpha)+w'_2(1.5+0.8\alpha))y\le((5+\alpha)-x)u_\alpha\\
&\qquad\le(w'_1(5-\alpha)+w'_2(3.5-1.2\alpha))y,\\
&\qquad x,y\ge o.
\end{aligned}
$$

Step 8. The lower bound of α-cut of the fuzzy compromise objective values of leader using Step 7, for values α_t's of Step 5, is solved. Its results display in Table 1.

Step 9. The system corresponding to this step for this example is $U'=\{u\ge o\mid(2.2+1.2\alpha).u\ge w'_1(5-\alpha)+w'_2(3.5-1.2\alpha)\}$.

Step 10. The vertex points of set $(U')^v_\alpha$ are easily obtained for each $\alpha=0,0.1,\ldots,1$ as $u'_\alpha=\frac{0.5(5-\alpha)+0.5(3.5-1.2\alpha)}{(2.2+1.2\alpha)}$. Moreover $(U')^v_\alpha$ is singleton for each α.

Step 11. The linear programming problem (65)–(68), for $(U')^v_\alpha$, and the value α is created, and the problem related to $\bar{\tilde{Z}}^U_{\alpha,k}$ is as follows:

$$
\begin{aligned}
&LP(u^k):\\
&\bar{\tilde{Z}}^U_{\alpha,k}=\min(w_1.(3-\alpha)+w_2.(5-2\alpha)).x+(w_1.(2-\alpha)+w_2.(6-\alpha)).y,\\
&s.t.\quad\begin{pmatrix}(-3+\alpha)\\(-2+\alpha)\end{pmatrix}.x+\begin{pmatrix}(-1+\alpha)\\(1.5+\alpha)\end{pmatrix}.y\le\begin{pmatrix}(7-\alpha)\\(9-2\alpha)\end{pmatrix},\\
&(w'_1(3+\alpha)+w'_2(1.5+0.8\alpha))y\le((8-2\alpha)-x).u'_\alpha\\
&\qquad\le(w'_1(5-\alpha)+w'_2(3.5-1.2\alpha))y,\\
&\qquad x,y\ge o.
\end{aligned}
$$

Step 12. The upper bound of α-cut of the fuzzy compromise objective values of leader using Step 11, for values α_t's of Step 5, is solved. Its results display in Table 1.

Step 13. The fuzzy compromise objective values of leader by two Steps 8 and 12 are created in Table 1. End.

Figure 1 displays the fuzzy objective values of leader using the data of Table 1. Using the data of columns 1, 2, and 3, the fuzzy objective values of follower can be easily computed.

In the next section, the fuzzy compromise objectives of leader and follower, for an example from [3] as a benchmark, are computed and compared with other existing methods. It shows the efficiency of the proposed algorithm in this paper.

Table 1: Estimation of compromise vector $\tilde{Z} = (\tilde{Z}_1, \tilde{Z}_2)$ using $(\tilde{Z}_i)^L_\alpha$ and $(\tilde{Z}_i)^U_\alpha$, for $i = 1, 2$, with different α-level values.

α	(x^*, y^*) **optimal of problem related to** $\overline{\tilde{Z}}^L_\alpha$	(x^*, y^*) **optimal of problem related to** $\overline{\tilde{Z}}^U_\alpha$	$\tilde{Z}_1$		$\tilde{Z}_2$	
			$(\tilde{Z}_1)^L_\alpha = (\tilde{c}_1)^L_\alpha x + (\tilde{d}_1)^L_\alpha y$	$(\tilde{Z}_1)^U_\alpha = (\tilde{c}_1)^U_\alpha x + (\tilde{d}_1)^U_\alpha y$	$(\tilde{Z}_2)^L_\alpha = (\tilde{c}_2)^L_\alpha x + (\tilde{d}_2)^L_\alpha y$	$(\tilde{Z}_2)^U_\alpha = (\tilde{c}_2)^U_\alpha x + (\tilde{d}_2)^U_\alpha y$
0	(0,1.087)	(0,3.636)	-1.087	7.272	4.348	21.816
0.1	(0,1.138)	(0,3.362)	-0.9104	6.3878	4.6658	19.8358
0.2	(0,1.193)	(0,3.115)	-0.7158	5.607	5.0106	18.067
0.3	(0,1.25)	(0,2.891)	-0.5	4.9147	5.375	16.4787
0.4	(0,1.311)	(0,2.687)	-0.2622	4.2992	5.7684	15.0472
0.5	(0,1.375)	(0,2.5)	0	3.75	6.1875	13.75
0.6	(0,1.443)	(0,2.329)	0.2886	3.2606	6.6378	12.5766
0.7	(0,1.516)	(0,2.171)	0.6064	2.8223	7.1252	11.5063
0.8	(0,1.593)	(0,2.025)	0.9558	2.43	7.6464	10.53
0.9	(0,1.676)	(0,1.89)	1.3408	2.079	8.2124	9.639
1.0	(0,1.765)	(0,1.765)	1.765	1.765	8.825	8.825

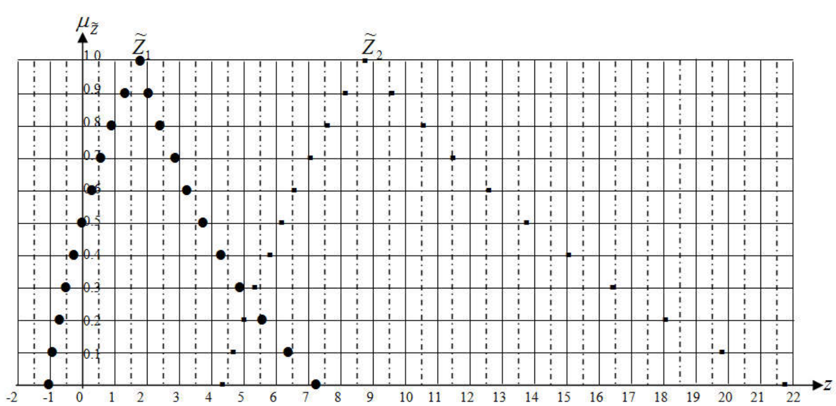

Figure 1: Membership function of compromise vector $\tilde{Z} = (\tilde{Z}_1, \tilde{Z}_2)$, where $\tilde{Z}_1$ and $\tilde{Z}_2$ are computed by $\left((\tilde{Z}_1)^L_\alpha, (\tilde{Z}_1)^U_\alpha\right)$ and $\left((\tilde{Z}_2)^L_\alpha, (\tilde{Z}_2)^U_\alpha\right)$, respectively.

6 Comparison with other existing methods

In this section, we present an example from [3, 4] as a benchmark to show the efficiency of the proposed method in this paper with the existing methods.

Example 2. Consider the following problem:

$$\begin{aligned}
&\min_{x_1,x_2} \begin{pmatrix} \tilde{f}_{11}(x) = (x_1 + \tilde{3}x_2 + \tilde{2}x_3 + \tilde{3}x_4) \\ \tilde{f}_{12}(x) = (\tilde{2}x_1 + \tilde{9}x_2 + \tilde{3}x_3 + \tilde{5}x_4) \\ \tilde{f}_{13}(x) = (\tilde{3}x_1 + \tilde{9}x_2 + \tilde{9}x_3 + x_4) \end{pmatrix} \\
&s.t. \quad \min_{x_3,x_4} \begin{pmatrix} \tilde{f}_{21}(x) = (\tilde{6}x_1 + \tilde{3}x_2 + \tilde{2}x_3 + \tilde{2}x_4) \\ \tilde{f}_{22}(x) = (\tilde{5}x_1 + \tilde{9}x_2 - \tilde{9}x_3 + \tilde{6}x_4) \end{pmatrix} \\
&\qquad s.t. \quad \tilde{3}x_1 - x_2 + x_3 + \tilde{3}x_4 \le \tilde{48}, \\
&\qquad\qquad \tilde{2}x_1 + \tilde{4}x_2 + \tilde{2}x_3 - \tilde{2}x_4 \le \tilde{35}, \\
&\qquad\qquad x_1 + \tilde{2}x_2 - x_3 + x_4 \le \tilde{30}, \\
&\qquad\qquad x_1, x_2, x_3, x_4 \ge 0.
\end{aligned}$$

The fuzzy numbers in [3, 4] are assumed to be triangular fuzzy numbers and given as follows:
$\tilde{2} = (0,2,3), \tilde{3} = (2,3,4), \tilde{4} = (3,4,5), \tilde{5} = (4,5,6), \tilde{6} = (5,6,7)$, $\tilde{8} = (6,8,10), \tilde{9} = (8,9,10), \tilde{30} = (28,30,32), \tilde{35} = (33,35,37)$, and $\tilde{48} = (45,48,49)$.
The problems related to $\overline{\tilde{Z}}_{\alpha}^{L}$ and $\overline{\tilde{Z}}_{\alpha}^{U}$ are as follows:

$$\begin{aligned}
LP(u^k) : \overline{\tilde{Z}}_{\alpha,k}^{L} = &\min \frac{1}{3}(1 + (2\alpha) + (2+\alpha))x_1 + \frac{1}{3}((2+\alpha) + (8+\alpha) + (8+\alpha))x_2 \\
&+ \frac{1}{3}((2\alpha) + (2+\alpha) + (8+\alpha))x_3 + \frac{1}{3}((2+\alpha) + (4+\alpha) + 1)x_4, \\
&s.t.\ 0.5((2\alpha) + (-8-\alpha))x_3 + 0.5((2\alpha) + (5+\alpha))x_4 \\
&\qquad \le ((45+3\alpha) - (4-\alpha)x_1 + x_2)u_1^k \\
&\qquad\quad + ((33+2\alpha) - (3-\alpha)x_1 - (5-\alpha)x_2)u_2^k \\
&\qquad\quad + ((-32+2\alpha) + x_1 - (-2\alpha)x_2)u_3^k \\
&\qquad \le 0.5((3-\alpha) + (-8-\alpha))x_3 + 0.5((3-\alpha) + (7-\alpha))x_4, \\
&\qquad\qquad x_1, x_2, x_3, x_4 \ge 0,
\end{aligned}$$

where u^k's are the vertex points of the set
$U = \{u = (u_1, u_2, u_3) \ge o \mid u_1 + (3-\alpha)u_2 + u_3 \ge -2.5 - \alpha, (4-\alpha)u_1 + (-2\alpha)u_2 - u_3 \ge 5 - \alpha\}$ and

$$
\begin{aligned}
LP(u^k): \overline{\tilde{Z}}^U_{\alpha,k} = &\min \frac{1}{3}(1+(3-\alpha)+(4-\alpha))x_1 \\
&+\frac{1}{3}((4-\alpha)+(10-\alpha)+(10-\alpha))x_2 \\
&+\frac{1}{3}((3-\alpha)+(4-\alpha)+(10-\alpha))x_3 \\
&+\frac{1}{3}((4-\alpha)+(6-\alpha)+1)x_4, \\
s.t.\ & 0.5((-10+3\alpha)x_3+(5+3\alpha)x_4) \\
&\le ((49-\alpha)-(2+\alpha)x_1+x_2)u_1^k \\
&\quad +((37-2\alpha)-2\alpha.x_1-(3+\alpha)x_2)u_2^k \\
&\quad +((-28-2\alpha)+x_1+(3-\alpha)x_2) \\
&\le 0.5((-5-2\alpha)x_3+(10-2\alpha)x_4), \\
&x_1,x_2,x_3,x_4 \ge 0,
\end{aligned}
$$

where u^k's are the vertex points of the following set:
$U = \{u=(u_1,u_2,u_3) \ge o \mid u_1+(2\alpha)u_2+u_3 \ge -2.5-\alpha, (2+\alpha)u_1+(-3+\alpha)u_2-u_3 \ge 5-\alpha\}$.
In the above models, $\alpha \in [0,1]$, $w_1 = w_2 = w_3 = \frac{1}{3}$, and $w'_1 = w'_2 = 0.5$. The compromise values $\tilde{f}_{11}, \tilde{f}_{12}, \tilde{f}_{13}, \tilde{f}_{21}$, and $\tilde{f}_{22}$ are computed using the values of $(\tilde{f}_{11})^L_\alpha, (\tilde{f}_{11})^U_\alpha, (\tilde{f}_{12})^L_\alpha, (\tilde{f}_{12})^U_\alpha, (\tilde{f}_{13})^L_\alpha, (\tilde{f}_{13})^U_\alpha$, $(\tilde{f}_{21})^L_\alpha, (\tilde{f}_{21})^U_\alpha, (\tilde{f}_{22})^L_\alpha, (\tilde{f}_{22})^U_\alpha$, and x_i^* obtained from solving two models $LP(u^k)$. The optimal solutions of problems related to $\bar{\tilde{Z}}^L_\alpha$ and $\bar{\tilde{Z}}^U_\alpha$ for different values α are presented in Table 2. The values of leader and follower objectives for different values α are showed in Tables 3 and 4, respectively.

Table 2: Optimal solutions of problems related to $\bar{\tilde{Z}}^L_\alpha$ and $\bar{\tilde{Z}}^U_\alpha$ for different values α.

α	x^* **optimal of problem related to** $\bar{\tilde{Z}}^L_\alpha$	x^* **optimal of problem related to** $\bar{\tilde{Z}}^U_\alpha$
0	(11.25,0,0,0)	(24.5,0,0,0)
0.25	(12.2,0,0,0)	(21.66667,0,0,0)
0.5	(13.28571,0,0,0)	(19.4,0,0,0)
0.75	(14.53846,0,0,0)	(17.54545,0,0,0)
1	(16,0,0,0)	(16,0,0,0)

A comparison among the proposed method and the methods in [3, 4] is presented in Table 5.

As it is shown in Table 5, the results obtained from the proposed method in terms of fuzzy numbers are more accurate than methods of finding a solution with the highest membership degree. In the proposed method, the decision-maker can consider all of the solutions in all the membership de-

Table 3: Values of leader objectives for different values α.

α	$\tilde{f}_{11}$		$\tilde{f}_{12}$		$\tilde{f}_{13}$	
	$\left(\tilde{f}_{11}\right)_\alpha^L$	$\left(\tilde{f}_{11}\right)_\alpha^U$	$\left(\tilde{f}_{12}\right)_\alpha^L$	$\left(\tilde{f}_{12}\right)_\alpha^U$	$\left(\tilde{f}_{13}\right)_\alpha^L$	$\left(\tilde{f}_{13}\right)_\alpha^U$
0	11.25	24.5	0	73.5	22.5	98
0.25	12.2	21.66667	6.1	59.58334	27.45	81.25001
0.5	13.28571	19.4	13.28571	48.5	33.21428	67.9
0.75	14.53846	17.54545	21.80769	39.47726	39.98077	57.02271
1	16	16	32	32	48	48

Table 4: Values of follower objectives for different values α.

α	$\tilde{f}_{21}$		$\tilde{f}_{22}$	
	$\left(\tilde{f}_{21}\right)_\alpha^L$	$\left(\tilde{f}_{21}\right)_\alpha^U$	$\left(\tilde{f}_{22}\right)_\alpha^L$	$\left(\tilde{f}_{22}\right)_\alpha^U$
0	56.25	171.5	45	147
0.25	64.05	146.25	51.85	124.5834
0.5	73.07141	126.1	59.7857	106.7
0.75	83.59615	109.6591	69.05769	92.11361
1	96	96	80	80

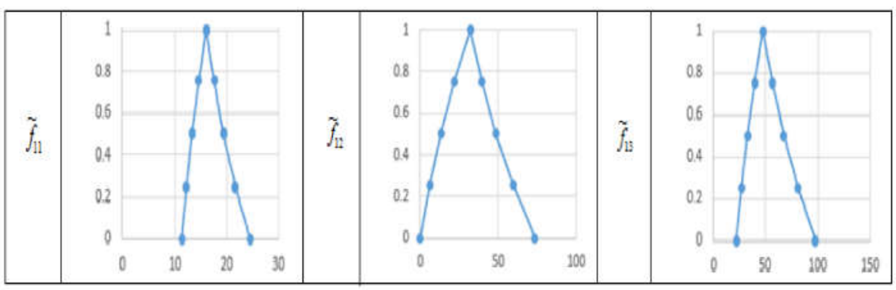

Figure 2: Fuzzy compromise objective values of leaders by the proposed approach

Table 5: Comparison among the proposed method and two other methods.

The method of Baky Eid, and El Sayed, in [3]		**The method of Pramanik and Dey in [4]**		**The proposed method**
$f_{11} = 29$	$\mu_{11} = 1$	$f_{11} = 37$	$\mu_{11} = 0.902$	$\tilde{f}_{11}$ in Figure 2
$f_{12} = 70.9483$	$\mu_{12} = 0.917$	$f_{12} = 90$	$\mu_{12} = 0.815$	$\tilde{f}_{12}$ in Figure 2
$f_{13} = 88.2743$	$\mu_{13} = 0.821$	$f_{13} = 108.25$	$\mu_{13} = 0.692$	$\tilde{f}_{13}$ in Figure 2
$f_{21} = 80.912$	$\mu_{21} = 0.527$	$f_{21} = 78.25$	$\mu_{21} = 0.496$	$\tilde{f}_{21}$ in Figure 3
$f_{22} = 111.27$	$\mu_{22} = 0.81$	$f_{22} = 105.5$	$\mu_{22} = 0.795$	$\tilde{f}_{22}$ in Figure 3

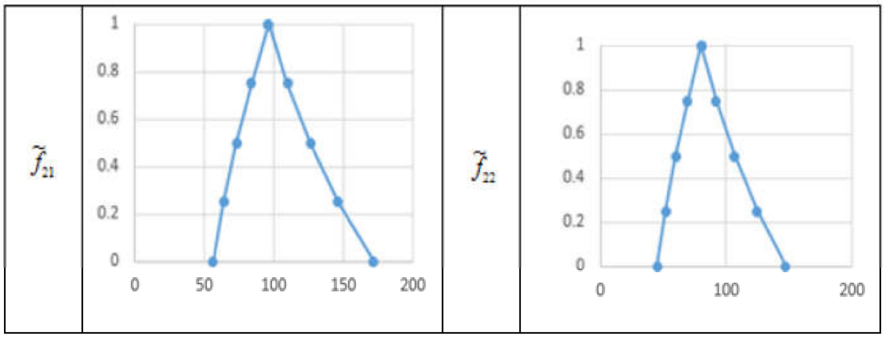

Figure 3: Fuzzy compromise objective values of followers by the proposed approach

grees. Moreover, the method of Pramanik and Dey in [4] did not follow the basic concepts of bilevel programming. Also, the obtained compromise solutions improve the compromise solutions in [3] for degrees 1, 0.917, 0.821, and 0.81, considerably. Moreover, the obtained compromise solution covers the compromise solution in [3] for degree 0.578.

7 Conclusion

This paper showed how Zadeh's extension principle can be efficiently applied to solve the FMOLBP problem. Using the structure of bilevel programming and concept of extension principle, two crisp multiobjective linear three-level programming problems were designed to compute the lower and upper bound of the α-level of the compromise objective value. Due to the strongly NP-hardness and high computational complexity of the problem, there is no efficient algorithm to solve it. The developed methods cannot completely solve the problem. To overcome the deficiencies, a pair of crisp multiobjective linear three-level programming problems was proposed to construct the compromise fuzzy objective functions of leader and follower. The main advantages of this method are access to fuzzy compromise objectives of leader and follower with less computational complexity with respect to approaches based on the rejection of solution and resolve model successively until finding a satisfactory decision or based on FGP. The proposed method is based on the weighting method, dual theory, and vertex enumeration approaches that preserve the linearity of the model. A comparison among the proposed method and the methods of Pramanik and Dey [4] and Baky Eid, and El Sayed, [3] showed the efficiency of our method.

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References

1. Abdelaziz, F.B. and Mejri, S. *Multiobjective bi-level programming for shared inventory with emergency and backorders*, Ann. Oper. Res., 267, (2018) 47–63.

2. Abo-Sinna, M.A. and Baky, I.A. *Interactive balance space approach for solving bilevel multiobjective programming problems*, Adv. Model. Anal. B 49 (3-4) (2006) 43–62.

3. Audet, C., Haddad, J. and Savard, G. *A note on the definition of a linear bilevel programming solution*, Appl. Math. Comput. 181 (2006) 351–355.

4. Baky, I.A. *Fuzzy goal programming algorithm for solving decentralized bilevel multiobjective programming problems*, Fuzzy Set. Syst. 160(18) (2009) 2701–2713.

5. Baky, I.A. *Solving multilevel multiobjective linear programming problems through fuzzy goal programming approach*, Appl. Math. Model. 34(9) (2010) 2377–2387.

6. Baky, I.A., Eid, M.H. and El Sayed, M.A. *Bi-level multiobjective programming problem with fuzzy demands: A fuzzy goal programming algorithm*, OPSEARCH. 51, (2014) 280–296.

7. Bard, J. *Practical bilevel optimization: Algorithms and applications*, Amsterdam, Kluwer Academic Publishers, 1998.

8. Bard, J. and Falk, J. *An explicit solution to the programming problem*, Comput. Oper. Res. 9 (1982) 77–100.

9. Bard, J.F. and Moore, J.T. *A branch and bound algorithm for the bilevel programming problem*, SIAM J. Sci. Stat. Comput. 11 (1990) 281–292.

10. Bazaraa, M.S., Jarvis, J.J. and Sherali, H.D. *Linear programming and network flows, Second edition*, Wiley, New York, 1990.

11. Bellman, R.E. and Zadeh, L.A. *Decision-making in a fuzzy environment*, Manag. Sci. 17 (1970) 141–164.

12. Bialas, W.F. and Karwan, M.H. *Two-level linear programming*, Manag. Sci. 30(8) (1984) 1004–1020.

13. Budnitzkia, A. *The solution approach to linear fuzzy bilevel optimization problems*, Optimization 64(5) (2015) 1195–1209.

14. Candler, W. and Townsley, R. *A linear two-level programming problem*, Comput. Oper. Res. 9 (1982) 59–76.

15. Colson, B. and Marcotte, P. and Savard, G. *An overview of bilevel optimization*, Ann. Oper. Res. 153 (2007) 235–256.

16. Gang, J., Tu, Y., Lev, B., Xu, J., Shen, W. and Yao, L. *A Multi-objective bi-level location planning problem for stone industrial parks*, Comput. Oper. Res. 56 (2015) 8–21.

17. Hansen, P., Jaumard, B. and Savard, G. *New branch-and-bound rules for linear bilevel programming*, SIAM J. Sci. Statist. Comput. 13 (1992) 1194–1217.

18. Kamal, M., Gupta, S., Chatterjee, P., Pamucar, D. and Stevic, Z. *Bi-level multi-objective production planning problem with multi-choice parameters: A fuzzy Goal programming algorithm*, Algorithms 12(7) (2019) 143.

19. Liu, Q.-M. and Yang, Y.-M. *Interactive programming approach for solving multilevel multiobjective linear programming problem*, J. Intell. Fuzzy Syst. 35(1) (2018) 55–61.

20. Lu, J., Shi, C. and Zhang, G. *An extended branch and bound algorithm for bilevel multifollower decision making in a referential-uncooperative situation*, Int. J. Inf. Tech. Decis. Mak. 6(2) (2007) 371–388.

21. Mersha, A.G. and Dempe, S. *Linear bilevel programming with upper level constraints depending on the lower level solution*, Appl. Math. Comput. 180 (2006) 247–254.

22. Moitra, B.N. and Pal, B.B. *A fuzzy goal programming approach for solving bilevel programming problems*, Pal, N.R., Sugeno, M. (eds.) In: Advances in Soft Computing – AFSS. vol. 2275 of Lecture Notes in Computer Science, PP.91–98, Springer, Berlin, Germany, 2002.

23. Mohamed, R.H. *The relationship between goal programming and fuzzy programming*, Fuzzy Set. Syst. 89 (1997) 215–222.

24. Peric, T., Babic, Z. and Omerovi, M. *A fuzzy goal programming approach to solving decentralized bilevel multiobjective linear fractional programming problems*, Croat. Oper. Res. Rev. 10 (2019) 65–74.

25. Pramanik, S. and Dey, P.P. *Bi-level multiobjective programming problem with fuzzy parameters*, Int. J. Comput. Appl. 30(11) (2011) 13–20.

26. Pramanik, S. and Roy, T.K. *Fuzzy goal programming approach to multi-level programming problems*, Eur. J. Oper. Res. 176(2) (2007) 1151–1166.

27. Sakawa, M. *Fuzzy sets and interactive multiobjective optimization*, Plenum press, New York, 1993.

28. Sakawa, M., Nishizaki, I. and Uemura, Y. *Interactive fuzzy programming for two-level linear fractional programming problems with fuzzy parameters*, Fuzzy Set. Syst. 115 (2000) 93–103.

29. Sakawa, M., Nishizaki, I. and Uemura, Y. *Interactive fuzzy programming for multilevel linear programming problems with fuzzy parameters*, Fuzzy Set. Syst. 109 (2000) 3–19.

30. Shi, C., Lu, J. and Zhang, G. *An extended Kth-best approach for linear bilevel programming*, Appl. Math. Comput. 164 (2005) 843–855.

31. Shi, C., Lu, J., Zhang, G. and Zhou, H. *An extended branch and bound algorithm for linear bilevel programming*, Appl. Math. Comput. 180 (2006) 529–537.

32. Shi, C., Zhang, G. and Lu, J. *On the definition of linear bilevel programming solution*, Appl. Math. Comput. 160(1) (2005) 169–176.

33. Shi, C., Zhang, G. and Lu, J. *An extended Kuhn-Tucker approach for linear bilevel programming*, Appl. Math. Comput. 162(1) (2005) 51–63.

34. Shih, H.S., Lai, Y.J. and Lee, E.S. *Fuzzy approach for multilevel programming problems*, Comput. Oper. Res. 23 (1983) 73–91.

35. Shih, H.S. and Lee, E.S. *Compensatory fuzzy multiple level decision making*, Fuzzy Set. Syst. 14 (2000) 71–87.

36. Toksari, M.D. and Bilim, Y. *Interactive fuzzy goal programming based on Jacobian matrix to solve decentralized bi-level multi-objective fractional programming problems*, Int. J. Fuzzy Syst. 17 (2015) 499–508.

37. Von Stackelberg, H. *The theory of the market economy*, Oxford University Press, York, Oxford, 1952.

38. Wan, Zh.P. and Fei, P.Sh. *The theory and algorithm of optimization*, The Wuhan University Press, 2004.

39. Wang, G., Wang, X., Wan, Z. and Lv, Y. *A globally convergent algorithm for a class of bilevel nonlinear programming problem*, Appl. Math. Comput. 188 (2007) 166–172.

40. Zadeh, L.A. *Fuzzy sets*, Inform. Control, 8 (1965) 338–353.

41. Zadeh, L.A. *Fuzzy sets as a basis for a theory of possibility*, Fuzzy Set. Syst. 1 (1978) 3–28.

42. Zhang, G. and Lu, J. *The definition of optimal solution and an extended Kuhn-Tucker approach for fuzzy linear bilevel programming*, The IEEE Computational Intelligence Bulletin 5 (2005) 1–7.

43. Zhang, G. and Lu, J. *Model and approach of fuzzy bilevel decision making for logistics planning problem*, J. Enterp. Inf. Manag. 20 (2007) 178–197.

44. Zhang, G., Lu, J. and Dillon, T. *A branch-and-bound algorithm for fuzzy bilevel decision making*, in: D. Ruan, P. D'hondt, P. Fantoni, M. De Cock, M. Nachtegael, E. Kerre (Eds.), Applied Artificial Intelligence, World Scientific, Singapore, 2006, pp. 291–298.

45. Zhang, G., Lu, J. and Dillon, T. *An approximation branch and bound approach for fuzzy linear bilevel decision making*, The 1st International Symposium Advances in Artifical Intelligence and Applications (AAIA 06) Wisla, Poland, November 6–10, 2006.

46. Zhang, G., Lu, J. and Dillon, T. *Decentralized multiobjective bilevel decision making with fuzzy demands*, Knowl-Based Syst. 20 (2007) 495–507.

47. Zheng, Y., Wan, Z. and Wang, G. *A fuzzy interactive method for a class of bilevel multiobjective programming problem*, Expert Syst. Appl. 38 (2011) 10384–10388.