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Research Article

Monotonicity-preserving splitting schemes for solving balance laws

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Abstract

In this paper, some monotonicity-preserving (MP) and positivity-preserving (PP) splitting methods for solving the balance laws of the reaction and diffusion source terms are investigated. To capture the solution with high-accuracy and resolution, the original equation with reaction source term is separated through the splitting method into two sub-problems including the homogeneous conservation law and a simple ordinary differential equation (ODE). The resulting splitting methods preserve monotonicity and positivity property for a normal CFL condition. A trenchant numerical analysis made it clear that the computing time of the proposed methods decreases when the so-called MP process for the homogeneous conservation law is imposed. Moreover, the proposed methods are successful in recapturing the solution of the problem with high-resolution in the case of both smooth and non-smooth initial profiles. To show the efficiency of proposed methods and to verify the order of convergence and capability of these methods, several numerical experiments are performed through some prototype examples.

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1 Introduction

Balance laws appear in many different areas such as fluid dynamics, gas dynamics, chemical reactions [13, 19, 1, 14, 29, 35, 37]. These equations are hyperbolic and are sometimes called inhomogeneous conservation laws in the literature on physics. In particular, these equations are described as follows:

$$\begin{cases} u_t + g(u)_x = f(u), \\ \\ u_0(x) = u(x,0),\ x \in I, \end{cases} \tag{1}$$

where $g(u)$ and $f(u)$ are smooth functions of u and I is the computational domain. The characterization and design of high-order and high-resolution schemes is a challenging task in the numerical analysis [12, 19, 1, 24, 25, 26, 29, 4]. The main goal of this paper is, therefore, to design schemes with high-accuracy and resolution to obtain the true solution of the balance laws including reaction and diffusion terms [19, 14, 7]. One way to deal with such problems is to split the original problem into two sub-problems and then solve each problem independently by applying an appropriate technique, which is sometimes called the fractional step approach [5, 28, 29, 32]. To this end, the problem (1) splits into the following sub-problems

$$\begin{cases} P_1 :\ u_t + g(u)_x = 0, \\ \\ P_2 :\ u_t = f(u). \end{cases}$$

In other words, the original problem (1) splits into the homogeneous conservation law P_1 and the ODE problem P_2. Mathematically, each sub-problem is computationally solved by a standard numerical scheme. However, solving the initial value sub-problem P_1 by a suitable numerical approach such as high-resolution shock-capturing methods, and then considering its solution as an initial profile for the sub-problem P_2 will eventually lead to the solution of the original problem as we solve sub-problem P_2 with an ODE solver such as Runge-Kutta (RK) or multi-step methods. For a semi-discrete discretization such as linear multi-step methods, we can study the global error propagation [10]. This procedure is regarded as a simple method to deal with this problem and often produces good results. Also, using this process the high-resolution methods such as the total variation diminishing (TVD), weighted-ENO (WENO) and MP methods can be applied directly for solving the homogeneous conservation law P_1 [1, 29].

In the past few decades, some numerical methods for evaluating the solution of the balance laws have been applied. For instance, Strang [4] proposed a splitting method for solving the balance law equations. Hundsdorfer and Verwer in [19] investigated one-step and multi-step methods using a splitting method to solve the advection-reaction equations. Moreover, they analyzed and extended the operator splitting method for a wide range of balance laws

including advection-diffusion-reaction equations in one- and two-dimensional spaces [1]. LeVeque in [29] introduced an analysis of numerical methods for solving balance equations. In [23], the corrected operator splitting method for solving nonlinear parabolic equations of a convection-diffusion type has been presented. Langseth et al. in [28] studied the order of convergence for operator splitting for balance laws. In [35], an A-stability achievement for operator splitting type time integration methods applied to advection-diffusion-reaction equations with possibly indefinite source terms has been achieved. Nuri et al. obtained the numerical behavior of a modified Burger's equation using cubic B-spline collocation finite element method after splitting the equation with Strang splitting technique in [40]. Holly and Polatera in [18] presented a numerical method to model the contaminant dispersion in two-dimensional tidal currents. In fact, they applied the high-order bi-cubic Hermite interpolation with characteristics to obtain the true solution in the presence of the advection part as well as the well-known Crank-Nicolson scheme for the diffusion part. One of the methods of discretization of the homogeneous conservation laws is the use of MP methods. Some numerical schemes with the monotonicity-preserving property are provided in the references [15, 5, 9, 2, 15, 17, 31, 17]. Suresh and Huynh [5] proposed an accurate MP scheme to homogeneous conversation laws. In the MP scheme, starting with an original numerical flux computed by any high-order scheme, a local interval is computed by enlarging the first-order MP interval. Then, the original numerical flux is preserved or replaced based on the relation between the original numerical flux and this local [8, 6, 16, 2, 13, 5]. This process is, however, capable of preserving the monotonicity property and, in contrast to conventional TVD methods, maintains the accuracy of solution in the extremum points. The MP scheme has a simple algorithm to achieve a high-order of accuracy. Moreover, this scheme is widely applied to solve hyperbolic partial differential equations [1, 6, 2, 15, 16, 5]. In this paper, we focus on designing the MP splitting schemes based on the idea of the MP scheme for solving balance laws. This process has some advantages including preserving monotonicity and positivity with high-accuracy, having a high speed of implementation due to the MP process [5], higher resolution at discontinuous points and yielding accurate results in the smooth region.

The outline of this paper is as follows. Details on the construction of the operator splitting schemes are presented in Section 2. In Section 3, the construction of the MP process for homogeneous conservation laws will be described in detail. The main results for the MP and PP splitting process for balance laws will be discussed in Section 4. To examine the constructed methods, some numerical experiments are given in Section 5, and finally, some concluding remarks are given in Section 6.

2 Operator splitting schemes

Here, we discuss the discretization of the problem (1) corresponding to sub-problems P_1 and P_2. For this reason, we consider two types of the operator splitting schemes. The first type is the so-called *basic splitting scheme* and is applied to the sub-problems P_1 and P_2 as follows:

$$\frac{d}{dt}u^*(t) = L\big(t, u^*\big)(t), \qquad \text{for} \quad t_n < t \le t_{n+1}, \;\; u^*(t_n) = u_n, \tag{2}$$

$$\frac{d}{dt}u^{**}(t) = F\big(t, u^{**}\big)(t), \quad \text{for} \;\; t_n < t \le t_{n+1}, \;\; u^{**}(t_n) = u^*(t_{n+1}), \tag{3}$$

where u_n stands for the approximating $u(t_n)$, $L\big(t, u(t)\big)$ denotes the discretized homogeneous operator for conservation laws and $F\big(t, u(t)\big)$ is the operator for reaction source term. Finally, the solution of the problem (1) is obtained as we set $u^{n+1} := u^{**}(t_{n+1})$ in the interval $[t_n, t_{n+1}]$. Also, the basic splitting scheme for the sub-problems P_1 and P_2 provides an error bound for the solution of order $\mathcal{O}(\Delta t)$ [19, 4], where Δt is the splitting time step. The second type is the *Strang splitting method* and is implemented as follows:

$$\frac{d}{dt}u^*(t) = L\big(t, u^*\big)(t), \quad t_n < t \le t_{n+\frac{1}{2}}, \;\; u^*(t_n) = u_n, \tag{4}$$

$$\frac{d}{dt}u^{**}(t) = F\big(t, u^{**}\big)(t), \quad t_n < t \le t_{n+1}, \;\; u^{**}(t_n) = u^*(t_{n+\frac{1}{2}}), \tag{5}$$

$$\frac{d}{dt}u^{***}(t) = L\big(t, u^{***}\big)(t), \quad t_{n+\frac{1}{2}} < t \le t_{n+1}, \;\; u^{***}(t_{n+\frac{1}{2}}) = u^{**}(t_{n+1}), \tag{6}$$

where the operators $L\big(\cdot, u(\cdot)\big)$ and $F\big(\cdot, u(\cdot)\big)$ are the same as those in basic splitting scheme. The final solution of the problem (1) is obtained by setting $u^{n+1} = u^{***}(t_{n+1})$. Strang splitting method introduces an order $\mathcal{O}(\Delta t^2)$ [19, 29, 40]. The homogeneous conservation laws in the sub-problem P_1 are discretized via step-size $\Delta x = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$ in a uniform grid and is defined by the points $x_j = j\Delta x$, $j = 1, 2, ..., N$, with cell boundaries given by $x_{j+\frac{1}{2}} = x_j + \frac{\Delta x}{2}$ in the conservation form

$$\frac{du_j(t)}{dt} = -\frac{1}{\Delta x}\big(f_{j+\frac{1}{2}} - f_{j-\frac{1}{2}}\big),$$

where $f_{j\pm\frac{1}{2}}$ are numerical fluxes. Our aim is to calculate u_j^{n+1} at time level t_{n+1}. It is worth noting that the algorithm for solving equation involving time is based on the well-known three-stage, third-order TVD-RK3 method [15, 36, 5]. Consider $\hat{f}_{j\pm\frac{1}{2}}$ as the original numerical fluxes which are approximations of the point value $u\big(x_{j\pm\frac{1}{2}}, t\big)$ at the time level t_n. We use an original numerical flux with a five-point stencil. Since, we need at least five points

Table 1: The constant coefficients β_l for $k = 3, 4, 5$.

k	β_{-5}	β_{-4}	β_{-3}	β_{-2}	β_{-1}	β_0	β_1	β_2	β_3	β_4	β_5
3			$-\frac{1}{140}$	$\frac{5}{84}$	$-\frac{101}{420}$	$\frac{319}{420}$	$\frac{107}{210}$	$-\frac{19}{210}$	$\frac{1}{105}$		
4		$\frac{1}{630}$	$-\frac{41}{2520}$	$\frac{199}{2520}$	$-\frac{641}{2520}$	$\frac{1879}{2520}$	$\frac{275}{504}$	$-\frac{61}{504}$	$\frac{11}{504}$	$-\frac{1}{504}$	
5	$-\frac{1}{2772}$	$\frac{61}{13860}$	$-\frac{703}{27720}$	$\frac{371}{3960}$	$-\frac{7303}{27720}$	$\frac{20417}{27720}$	$\frac{15797}{27720}$	$-\frac{4003}{27720}$	$\frac{947}{27720}$	$-\frac{17}{3080}$	$\frac{1}{2310}$

to distinguish between extremum and discontinuity points. Therefore, one choice is to consider the original numerical flux of order $\mathcal{O}(\Delta x^5)$ as follows:

$$\hat{f}_{j+\frac{1}{2}} = \frac{1}{60}\Big(2g_{j-2} - 13g_{j-1} + 47g_j + 27g_{j+1} - 3g_{j+2}\Big). \tag{7}$$

Furthermore, to increase spatial accuracy, the original numerical fluxes of the $(2k+1)$-th order in the following form

$$\hat{f}_{j+\frac{1}{2}} = \sum_{l=-k}^{k} \beta_l g_{j+l},\ k = 3, 4, 5,$$

can be used. The constant coefficients β_l are shown in Table 1. Moreover, compact numerical schemes are relatively high-order with a small stencil. However, these numerical schemes can be applied as another choice [15]. This process produces some oscillations in the numerical solution as the above-mentioned numerical fluxes are assembled. To overcome this difficulty, we will discuss the monotonicity property for homogeneous conservation laws in the next section to obtain accurate numerical schemes inspired by the process of MP as proposed in [5].

3 Construction of the MP process for conservation laws

In this section, we will study the construction of the MP process for conservation laws with numerical flux (7). To this end, the numerical flux (7) is considered to be the original numerical flux $\hat{f}_{j+\frac{1}{2}}$. Because of this, some oscillations near the discontinuous points are produced as we utilize the numerical flux $\hat{f}_{j+\frac{1}{2}}$. Therefore, we limit the numerical flux endowed with MP process in such a way that the oscillations near the discontinuous points are damped. To construct an MP process for the numerical flux (7), some definitions are required. The first definition is the minmod function for n arguments as follows:

$$\text{minmod}\big(z_1, z_2, ..., z_n\big) = s.\min\big(\mid z_1 \mid, \mid z_2 \mid, ..., \mid z_n \mid\big),$$

where

$$s = \frac{1}{2}\Big(\text{sign}(z_1)+\text{sign}(z_2)\Big)\times\left|\frac{1}{2}\Big(\text{sign}(z_1)+\text{sign}(z_3)\Big)...\frac{1}{2}\Big(\text{sign}(z_1)+\text{sign}(z_n)\Big)\right|.$$

The second one is the interval $\big[\min(z_1, z_2, ..., z_n), \max(z_1, z_2, ..., z_n)\big]$ denoted by $I\big[z_1, z_2, ..., z_n\big]$. The third one is the local curvature

$$d_{j+\frac{1}{2}}^{MM} = \text{minmod}\big(d_j, d_{j+1}\big),$$

where

$$d_j = u_{j+1} - 2u_j + u_{j-1}.$$

For simplicity of notation, we omitted the superscript n, e.g., u_j^n denoted by u_j. We consider the original numerical flux of five-order in the formula (7). Then, this original numerical flux is maintained or replaced according to the following limiting procedures. More precisely, if we suppose $u_j \leq \hat{f}_{j+\frac{1}{2}} \leq u_{j+1}$ some simple calculations will show that the derived solution, u_j^{n+1}, lies somewhere between u_{j-1} and u_j, which yields MP-property. Therefore, these ensure that $\hat{f}_{j+\frac{1}{2}}$ lies between u_j and u^{UL} as follows:

$$u^{UL} = u_j + \alpha\big(u_j - u_{j-1}\big),$$

where $\alpha \geq 2$. According to the above-mentioned assumptions, we derive a first-order MP interval $I[u_j, u^{MP}]$, where u^{MP} is defined in the following form

$$u^{MP} = u_j + \text{minmod}\Big(u_{j+1} - u_j, \alpha\big(u_j - u_{j-1}\big)\Big).$$

For simulation, the condition for which the original numerical flux $\hat{f}_{j+\frac{1}{2}}$ lies in $I[u_j, u^{MP}]$, is equivalent to

$$\big(\hat{f}_{j+\frac{1}{2}} - u_j\big)\big(\hat{f}_{j+\frac{1}{2}} - u^{MP}\big) \leq \mu,$$

with a tolerance $\mu = 10^{-10}$. For $\hat{f}_{j+\frac{1}{2}} \in I[u_j, u^{MP}]$ the accuracy of the MP method decreases to the first order in the vicinity of the extremum points. To increase accuracy in the vicinity of the extremum points, the intervals $I[u_j, u_{j+1}]$ and $I[u_j, u^{UL}]$ were enlarged as $I\big[u_j, u_{j+1}, u^{MD}\big]$ and $I\big[u_j, u^{UL}, u^{LC}\big]$, respectively, where

$$u^{MD} = \frac{1}{2}\big(u_j + u_{j+1}\big) - \frac{1}{2}d_{j+\frac{1}{2}}^{MM},$$

and

$$u^{LC} = \frac{1}{2}\big(3u_j - u_{j-1}\big) + \frac{4}{3}d_{j-\frac{1}{2}}^{MM}.$$

Two intervals $I\big[u_j, u^{UL}, u^{LC}\big]$ and $I\big[u_j, u_{j+1}, u^{MD}\big]$ enlarge only for the local non-monotonicity numerical data, and degenerate to $I[u_j, u^{UL}]$ and $I[u_j, u_{j+1}]$ for the local monotonicity numerical data [5]. In practice, we replace $d^{MM}_{j+\frac{1}{2}}$ with $d^{M4}_{j+\frac{1}{2}}$ and define

$$d^{M4}_{j+\frac{1}{2}} = \text{minmod}\Big(4d_j - d_{j+1}, 4d_{j+1} - d_j, d_j, d_{j+1}\Big).$$

Intersection $I[u^{\min}, u^{\max}]$ of the two intervals $I[u_j, u^{UL}, u^{LC}]$ and $I[u_j, u_{j+1}, u^{MD}]$ can be computed by

$$\begin{aligned} u^{\min} &= \max\Big[\min\big(u_j, u_{j+1}, u^{MD}\big), \min\big(u_j, u^{UL}, u^{LC}\big)\Big], \\ u^{\max} &= \min\Big[\max\big(u_j, u_{j+1}, u^{MD}\big), \max\big(u_j, u^{UL}, u^{LC}\big)\Big], \end{aligned}$$

which indicate the local interval $I[u^{\min}, u^{\max}]$ for the MP process. The numerical original flux is replaced by the nearest bound of the interval if the original numerical flux is outside of the interval. Therefore, we have

$$f_{j+\frac{1}{2}} = \hat{f}_{j+\frac{1}{2}} + \text{minmod}\Big(u^{\max} - \hat{f}_{j+\frac{1}{2}}, u^{\min} - \hat{f}_{j+\frac{1}{2}}\Big).$$

Summarizing the above scenario, accuracy and monotonicity are preserved, which means the MP process is of high-order and high-resolution in the case of the smooth and non-smooth initial profiles, respectively. In comparison with the other existing methods such as the fifth-order weighted-ENO (WENO5) and the third-order essentially nonoscillatory (ENO3), the speed of the MP algorithm is of high rate, which makes the MP algorithm an efficient and effective procedure to solve the homogeneous conservation laws [5]. It is well-known that the MP scheme has higher accuracy and less dissipation compared with the mentioned schemes [1, 6, 5]. Moreover, the high-order WENO methods may not have monotonicity property but the MP scheme preserves monotonicity property [1]. Therefore, due to the mentioned reasons, we consider the MP method to discretize the homogeneous conservation laws part in the splitting process.

4 Main results

4.1 Monotonicity preserving

In this section, we determine results for the basic splitting (2), (3) and the Strang splitting (4)−(6) with the above mentioned MP process for the homogeneous conservation laws part. These schemes are called the MP basic splitting scheme and the MP Strang splitting scheme, respectively. We state the

following theorem which determines the condition of monotonicity-preserving on the MP basic splitting scheme.

Theorem 1. The MP basic splitting scheme is monotonicity-preserving under the CFL condition $\nu \leq \frac{1}{1+\alpha}$, where ν is CFL number.

Proof. Suppose that the sub-problem (2) is computed with the TVD-RK3 method in time and the MP scheme of five-order in space. Therefore, we suppose that $u_{j-1} \leq \hat{f}_{j-\frac{1}{2}} \leq u_j$, and $u_j \leq \hat{f}_{j+\frac{1}{2}} \leq u^{UL}$. For increasing data, these two assumptions imply that $(u^{UL} - u_{j-1}) \leq (\alpha+1)(u_j - u_{j-1})$. After applying the relation $(u^{UL} - u_{j-1}) \leq (\alpha+1)(u_j - u_{j-1})$, we will have

$$0 \leq (\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}) \leq (u^{UL} - u_{j-1}) \leq (\alpha+1)(u_j - u_{j-1}). \tag{8}$$

Multiplying $-\nu$ and then adding u_j to the relation (8), we will obtain

$$(1 - \nu(1+\alpha))u_j + \nu(\alpha+1)u_{j-1} \leq u^* \leq u_j,$$

where $u^* = u_j - \nu(\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}})$ is the first step of the TVD-RK3 method. To hold $u_{j-1} \leq u^* \leq u_j$, the inequality $u_{j-1} \leq (1 - \nu(1+\alpha))u_j + \nu(\alpha+1)u_{j-1}$ must be satisfied, which in turn yields

$$\nu(\alpha+1)(u_j - u_{j-1}) \leq (u_j - u_{j-1}).$$

Therefore, we have $\nu(\alpha+1) \leq 1$. Since the second and third steps of the TVD-RK3 method are combinations of the first step and u_j with positive weights, the final step $u^*(t_{n+1})$ is also monotonic. Consequently, the first step of the basic splitting scheme under the CFL condition $\nu(\alpha+1) \leq 1$ is monotonicity-preserving. The sub-problem (3) is computed with the TVD-RK3 method with initial condition $u^{**}(t_n) = u^*(t_{n+1})$ and also the TVD-RK3 method is monotonic; It is well known that if sub-problems is solved by a high resolution or a monotonicity-preserving method, then the splitting methods (such as the basic splitting, Strang splitting and dimension splitting schemes) are monotonicity-preserving and convergent to the correct solution of original equation [4, 1, 29]. however, the MP basic splitting is monotonicity-preserving under the CFL condition $\nu \leq \frac{1}{1+\alpha}$. □

Here, we present the following theorem to estimate the error of the MP basic splitting method.

Theorem 2. Suppose (2), (3) are computed with TVD-RK3 in time and the MP process of five-order in space. Then, in the absence of the splitting error, the global order of the error can be introduced as $\mathcal{O}(\Delta t^3) + \mathcal{O}(\Delta x^5)$.

Proof. The computation of equation (2) with initial data u_n yields approximations $u^*(t_{n+1})$ with an order error of the form $\mathcal{O}(\Delta t^3) + \mathcal{O}(\Delta x^5)$ [5]. More pointedly, the evaluation of equation (3) with TVD-RK3 method would

result in local error of $\mathcal{O}(\Delta t^3)$ in time [36]. Consequently, Due to the stability of sub-schemes the global order of the error can be obtained as $\mathcal{O}(\Delta t^3) + \mathcal{O}(\Delta x^5)$, when the splitting error is neglected. □

Also, to make sure that the convergency of the MP basic splitting method is held, we state the following lemma.

Lemma 1. The MP basic splitting scheme is convergence to the true solution of original equation (1).

Proof. As was mentioned above, subproblem (2) is computed with a high-resolution method in space (i.e., the MP process of five-order) and the method of TVD-RK3 in time with initial condition $u^*(t_n) = u_n$ in interval $[t_n, t_{n+1}]$, while the subproblem is solved in interval $[t_n, t_{n+1}]$ only by the TVD-RK3 method with monotone initial data $u^{**}(t_n) = u^*(t_{n+1})$. Therefore, because sub-problems are solved using high-resolution methods (we know that high-resolution methods converge to the correct solution of the problem [29]) and the use of the well-known basic splitting method, also, because there is consistency between the approximation of the splitting method (2), (3) with the solution of original equation (1), the convergence of basic MP splitting method to the true solution of original equation (1) is guaranteed [4, 1, 29, 4]. □

4.2 Positivity preserving

Next, we will investigate positivity property for the basic splitting (2), (3) sophistically in detail. We consider the case that the solutions $u(x,t)$ of the basic splitting (2), (3) are nonnegative for all times $t \geq 0$.

In the following, we state a lemma for positivity property of the basic splitting (2), (3) obtained using the MP method and the TVD-RK3 method.

Lemma 2. The MP basic splitting scheme will be positivity-preserving under the CFL condition $\nu \leq \frac{1}{1+\alpha}$.

Proof. Suppose that for $t > 0$ the problem (2) is solved with initial condition $u^*(t_n) = u_n \geq 0$ in interval $[t_n, t_{n+1}]$ using the MP process in space and the TVD-RK3 method in time. Therefore, the solution of the MP method is monotonicity-preserving under the CFL condition $\nu \leq \frac{1}{1+\alpha}$. However, the problem (3) is solved in interval $[t_n, t_{n+1}]$ by the TVD-RK3 method with monotone initial data $u^{**}(t_n) = u^*(t_{n+1})$. Since the TVD-RK3 method with positive weights is positivity-preserving, therefore, $u^{n+1} := u^{**}(t_{n+1}) \geq 0$. As a result, the MP basic splitting is positivity-perserving under the CFL condition $\nu \leq \frac{1}{1+\alpha}$. □

It is interesting to note that, the above-mentioned results are also valid for the MP Strang splitting method, i.e., the MP Strang splitting method

is convergent to the true solution of original equation (1) and also preserves monotonicity and positivity under the CFL condition $\nu \leq \frac{1}{1+\alpha}$.

4.3 The case for diffusion source term

In this section, we study the MP splitting process for balance laws with diffusion source term in the following form

$$u_t + f(u)_x = \eta\, u_{xx}, \tag{9}$$

where $f(u)$ is the flux function, and η denoting the diffusion coefficient. By applying the basic splitting method, the equation (9) is divided into two sub-problems as follows:

$$u_t^* + f(u^*)_x = 0, \qquad \text{for} \quad t_n \leq t \leq t_{n+1}, \;\; u^*(x,t_n) = u(x,t_n), \tag{10}$$

$$u_t^{**} = \eta\, u_{xx}^{**}, \quad \text{for} \;\; t_n \leq t \leq t_{n+1}, \;\; u^{**}(x,t_n) = u^*(x,t_n), \tag{11}$$

where u^* and u^{**} stand for concentration in the case of homogeneous conservation law and the diffusion process, respectively. Executing the MP process, equation (10) can simply be solved by using the initial condition of equation (9). The solution of this problem can be regarded as an initial condition for equation (11). Using the Crank-Nicolson method [8, 23], equation (11) can be numerically solved through an initial condition derived from the previous step. Consequently, the solution to the original problem (9) is obtained. It turns out that the proposed process can stably and consistently estimate the true solution of the problem (9). This process can, however, be applied to solving a wide variety of balance laws with possibly nonlinear source terms. As it is well-known, the effective and applicable strategy for solving the considered problem turns out to be implemented by the MP process. This method is a robust and powerful technique for treating balance laws. Moreover, it may be used in investigating problems involving diffusion source terms. To shed light on this issue, we will present the following remark which is, in essence, similar to the one given by Lemma 1.

Remark 1. It is notable that the proposed method is not only able to treat problems in the presence of reaction source terms, but also robustly and rigorously reconstructs the true solution as some diffusion source terms are imposed. Indeed, the proposed method can therefore retrieve the true solution of the original equation (9) by means of splitting method like what is mentioned in Lemma 1. This consideration makes us assert that this method can actually guarantee the convergency of the solution to the original equation (9).

5 Numerical results

In this section, some test problems are provided to show the efficiency and effectiveness of the MP splitting schemes. In all tests, $\alpha = 4$ and $\mu = 10^{-10}$ are considered. Meanwhile, all tests were performed by MATLAB routine codes.

Test 1. Consider the following equation

$$u_t + \big(au\big)_x = -\beta u, \quad a = 1 \text{ and } \beta = 1,\ 0 \le x \le 1,$$

with initial condition

$$u_0(x) = \sin^4\big(\pi x\big),$$

and the periodic boundary conditions, $u(0,t) = u(1,t)$, at final time $t_f = 1$. The results of the MP basic splitting scheme and the MP Strang splitting scheme are demonstrated with $\frac{\Delta t}{\Delta x} = 0.05$ and $\frac{\Delta t}{\Delta x} = 0.4$ in tables 2 and 3, respectively. The numerical results confirm the good efficiency and accuracy of the MP splitting schemes for this test. It is worth emphasizing that the numerical results for large CFL number $\frac{\Delta t}{\Delta x} = 0.4$ are also satisfactory.

Test 2. We solve

$$u_t + \big(au\big)_x = -\beta u, \quad a = 1 \text{ and } \beta = 10, \quad 0 \le t \le \frac{1}{2},$$

with initial condition given by

$$u_0(x) = 1000\cos\big(\pi x\big)^2,$$

and the periodicity conditions in the boundaries on the interval [0, 1], at final time $t_f = \frac{1}{2}$ and discrete points $N = 40$. Numerical solution of the advection-reaction equation with the MP basic splitting method for $\frac{\Delta t}{\Delta x} = 0.4$ is depicted in Figure 2. From this figure, one can deduce that the proposed method is successful in retrieving the exact solution in a careful manner. Moreover, in comparison with the third order TVD basic splitting scheme [1], our proposed method is not only able to recover the true solution of the problem accurately, but also it preserves the accuracy of the numerical solution at extremum points. This situation also happens for other test problems mainly because the spatial part of the homogeneous problem is discretized by the MP-process. Yet the lack of such property in the third order TVD-scheme makes its efficiency unsatisfactory. Furthermore, we consider the advection-reaction

equation as follows:

$$u_t + \big(au\big)_x = -\beta u, \quad a = 1 \text{ and } \beta = 1, \tag{12}$$

with initial condition

$$u_0(x) = \Big(\sin\big(\pi x\big)\Big)^{100}, \tag{13}$$

and the periodicity conditions in the interval [0, 1], at final time $t_f = 2$ and discrete points $N = 64$. Numerical solution of the advection-reaction equation (12)-(13) with the MP basic splitting method for $\frac{\Delta t}{\Delta x} = 0.2$ is illustrated in Figure 1. In this test, we can deduce that the numerical solution of the MP basic splitting method, which preserves nonlinear stability such as monotonicity and positivity. To be more precise, this test is given specifically to confirm the positivity property of the MP basic splitting for CFL condition $\nu \leq \frac{1}{1+\alpha}$ (as shown in Lemma 2) and it shows the MP basic splitting method is monotonicity-preserving for this CFL condition (as shown in Theorem 1).

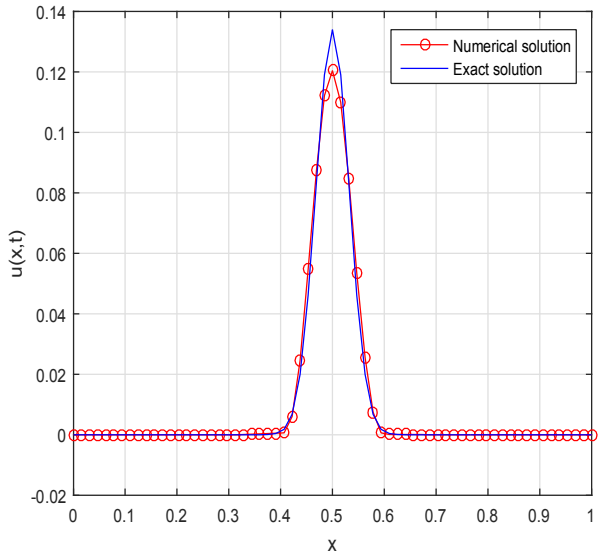

Figure 1: Numerical solution of the advection-reaction equation (12)-(13) with the MP basic splitting scheme at final time $t_f = 2$ and $N = 64$, $\frac{\Delta t}{\Delta x} = 0.2$.

Test 3. Consider the following equation governed by

$$u_t + \big(au\big)_x = -\beta u, \quad a = 1 \text{ and } \beta = 1,$$

with the initial condition

$$u_0(x) = \text{sign}(x),$$

and the periodic boundary conditions in the interval $[0, 2]$. The results are shown at final time $t_f = 1$ in Figure 6. We get a good behavior for the MP Strang splitting scheme (even for CFL$= 0.4$). It is obvious that the MP Strang splitting scheme produces sharp results in the vicinity of the shock.

Test 4. Let us take the equation

$$u_t + \big(au\big)_x = u\big(1 - u\big), \quad a = 1,$$

with the initial condition

$$u_0(x) = \exp\big(-0.01x^2\big),$$

and the periodic boundary conditions in the interval $[0, 2]$. The results of the MP Strang splitting scheme are illustrated in Figure 4 with $\frac{\Delta t}{\Delta x} = 0.4$, at final time $t_f = 1$ and $N = 100$. However, for the final time $t_f = 1$ and discrete points $N = 100$, the MP Strang splitting scheme is compared to the nonstandard finite difference (NSFD) scheme [33], which is stable and also is of first-order in time and space as follows:

$$\frac{u_j^{n+1} - u_j^n}{\phi(\Delta t)} + \frac{u_j^n - u_{j-1}^n}{\phi(\Delta x)} = u_{j-1}^n\big(1 - u_j^{n+1}\big),$$

where $\phi(h) = \exp(h) - 1$, and $\Delta t = \Delta x$ is an imperative condition to hold this scheme. We find that the MP Strang splitting scheme has a good resolution and convergence. Moreover, the result obtained by the MP Strang splitting scheme is free of the spurious oscillations.

Test 5. We apply the MP Strang splitting scheme to the equation

$$u_t + \Big(\frac{1}{2}u^2\Big)_x = u\big(1 - u\big), \tag{14}$$

with the initial value

$$u_0(x) = \begin{cases} 0.1 + 0.1\sin\big(2\pi x\big), \text{ if } \ 0 \le x \le 1, \\ 0.1, \qquad\qquad\qquad\qquad \text{else}, \end{cases} \tag{15}$$

and the periodic boundary conditions in the interval $[0, 2]$. The numerical solution is depicted at $t_f = 1.5$ in Figure 5. We can see that the MP Strang splitting scheme can solve high-accuracy and resolution. Obviously, the dissipation of the MP Strang splitting scheme is much small.

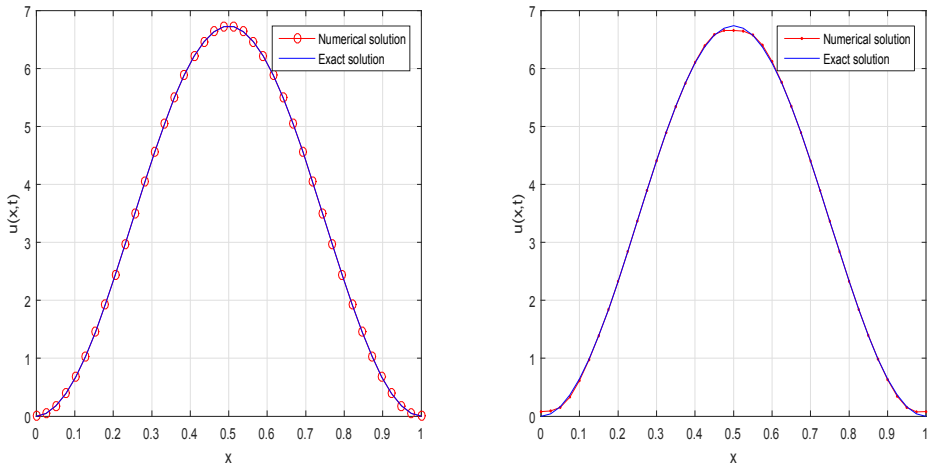

Figure 2: Numerical solution of the advection-reaction equation with the MP basic splitting scheme (left), the third order TVD basic splitting scheme (right) at final time $t_f = \frac{1}{2}$ and $N = 40$, $\frac{\Delta t}{\Delta x} = 0.4$ in Test 2.

Table 2: Estimated L_2-errors and L_2-orders for advection-reaction equation with the MP basic splitting scheme in Test 1.

N	$\frac{\Delta t}{\Delta x} = 0.05$		$\frac{\Delta t}{\Delta x} = 0.4$	
	L_2-error	L_2-order	L_2-error	L_2-order
20	6.13e−04	-	7.64e−04	-
40	2.06e−05	4.89	5.43e−05	3.96
80	6.58e−07	4.96	4.98e−06	3.45
160	2.15e−08	4.94	5.64e−07	3.14

Test 6. In this test, we apply the MP basic splitting scheme mentioned in subsection 4.3 to the equation

$$u_t + \left(\frac{1}{2}u^2\right)_x = \eta\, u_{xx}, \quad \eta = 1, \tag{16}$$

with the initial condition

$$u_0(x) = \sin(x), \tag{17}$$

Table 3: Estimated L_2-errors and L_2-orders for advection-reaction equation with the MP Strang splitting scheme in Test 1.

N	$\frac{\Delta t}{\Delta x} = 0.05$		$\frac{\Delta t}{\Delta x} = 0.4$	
	L_2-error	L_2-order	L_2-error	L_2-order
20	6.10e−04	-	6.42e−04	-
40	2.04e−05	4.90	2.46e−05	4.70
80	6.50e−07	4.98	1.18e−06	4.38
160	2.04e−08	4.99	8.81e−08	3.80

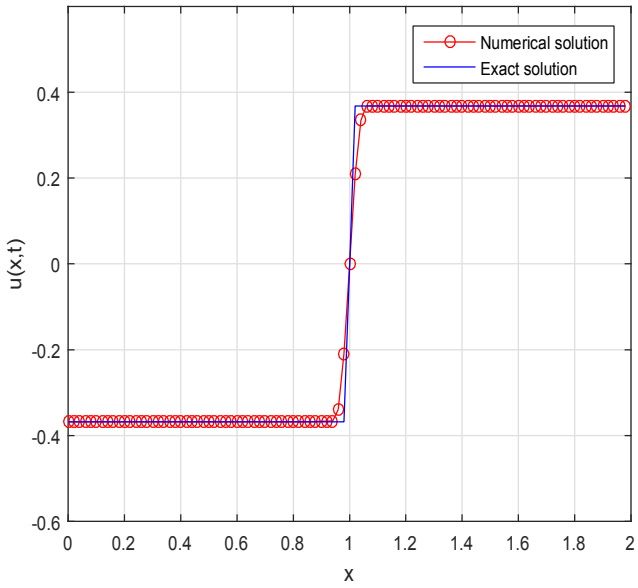

Figure 3: Numerical solution of the advection-reaction equation with the MP Strang splitting scheme at final time $t_f = 1$ and $N = 200$, $\frac{\Delta t}{\Delta x} = 0.4$ in Test 3.

and the boundary conditions $u(-4,t) = u(4,t) = 0$ in the interval $[-4,4]$. The results of the MP basic splitting scheme are illustrated in Figure 6 with $\frac{\Delta t}{\Delta x} = 0.4$, at final time $t_f = 4$ and $N = 100$. We can deduce that numerical solution given by the MP basic splitting scheme nicely converges to the true solution.

Test 7. Now we consider the one-dimensional modified Burger's equation (MBE) in the following form

$$u_t + u^\beta(u)_x - \eta u_{xx} = 0, \quad 0 \le x \le 1, \quad t \ge 1, \quad \beta = 2, \tag{18}$$

with the initial condition

$$u(x, t_0 = 1) = \frac{x}{1 + \frac{1}{c} e^{\frac{x^2}{4\eta}}}, \ t \geq 1, \ 0 \leq x \leq 1, \ 0 \leq c \leq 1, \qquad (19)$$

and the boundary conditions

$$\begin{cases} u(0,t) = u(1,t) = 0, \\ u_x(0,t) = u_x(1,t) = 0, \\ u_{xx}(0,t) = u_{xx}(1,t) = 0. \end{cases}$$

To apply the MP basic splitting, we split the original equation (18) into two the sub-problems P_1 and P_2 as follows:

$$\begin{cases} P_1 : \ u_t + \left(\frac{u^{\beta+1}}{\beta+1}\right)_x = 0, \\ \\ P_2 : \ u_t = \eta u_{xx}. \end{cases}$$

Then, we get the solution of the MP basic splitting scheme mentioned in subsection 4.3 in the interval $[0, 1]$. The results of the MP basic splitting scheme are illustrated in Table 4 with $\Delta t = 0.01$, $\Delta x = 0.0625$, $\eta = 0.01$, $c = 0.5$, at final time $t_f = 2$. From Table 4 with these parameters, we can see that the MP basic splitting scheme error norms L_∞ and L_2 are in general better than the proposed methods in references [34, 40]. MATLAB code of the MP process for this test is placed in the Appendix.

Table 4: A comparison of the MP basic splitting scheme error norms L_∞ and L_2 with the proposed methods in references [34, 40].

Schemes	L_∞-norm	L_2-norm
$S_{\Delta t}$[40]	$7.4741e-04$	$3.2033e-04$
[34]	$7.5978e-04$	$3.4748e-04$
MP basic splitting scheme	$2.2677e-04$	$1.4944e-04$

6 Conclusion

In this paper, the MP splitting schemes for solving balance laws are proposed. The MP splitting schemes are introduced to preserve the monotonicity and positivity properties of the numerical scheme for balance laws with linear and

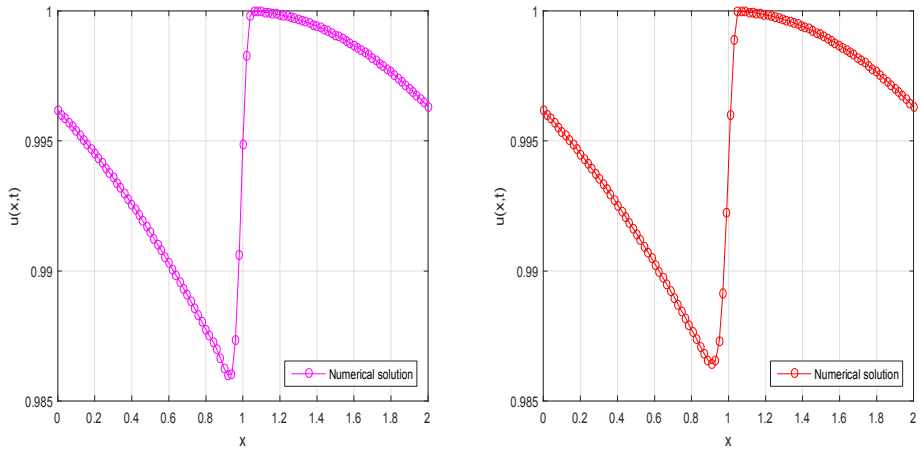

Figure 4: Numerical solution of the advection-reaction equation with the MP Strang splitting method (left), the NSFD method (right) at final time $t_f = 1$ and $N = 100$ in Test 4.

nonlinear source terms. In these methods, the homogeneous conservation laws step is discretized by the fifth order MP method, and the reaction step and time integration are discretized by the third-order TVD-RK3 method. The numerical results for balance laws as well as the case for diffusion source term are also provided results that verify the applicability and efficiency of the proposed schemes. It is worth studing the MP procedure to the interface problems that have a wide application in science and engineering [11].

Acknowledgements

Authors are grateful to there anonymous referees and editor for their constructive comments.

APPENDIX: MATLAB code of the MP process

```
function [fplus] = MP(u)
B1 = 1/60;
B2 = 4/3;
alpha =4.0;
eps = 10^-10;
```

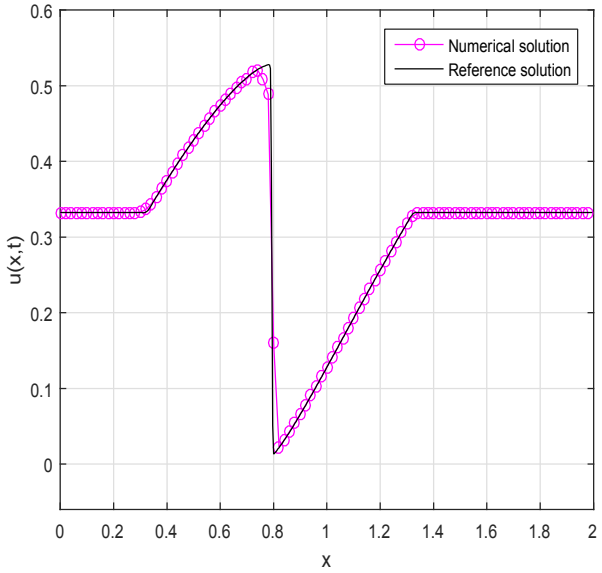

Figure 5: Numerical solution of the problem (14)-(15) with the MP Strang splitting method at final time $t_f = 1.5$ and $N = 100$, $\frac{\Delta t}{\Delta x} = 0.4$ in Test 5.

```
fhat =B1*(2 *u(1)-13*u(2)+47*u(3)+27*u(4)-3*u(5));
uMP = u(3)+DMM((u(4)-u(3)),alpha*(u(3)-u(2)));
if ((fhat-u(3))*(fhat-uMP)<=eps)
   fplus=fhat ;
else
DJM1 = (u(1)-2*u(2)+u(3));
DJ   = (u(2)-2*u(3)+u(4));
DJP1 = (u(3)-2*u(4)+u(5));
 DM4JPH = DM4(4*DJ-DJP1,4*DJP1-DJ,DJ,DJP1);
 DM4JMH = DM4(4*DJ-DJM1,4*DJM1-DJ,DJ,DJM1);
uUL = u(3)+alpha*(u(3)-u(2));
uAV = 0.5*(u(3)+u(4));
uMD = uAV-0.5*DM4JPH;
uLC = u(3)+0.5*(u(3)-u(2))+B2*DM4JMH;
umin =  max(min(min(u(3),u(4)),uMD),min(min(u(3),uUL),uLC));
umax =  min(max(max(u(3),u(4)),uMD),max(max(u(3),uUL),uLC));
fplus = fhat+DMM(umin-fhat,umax-fhat);
end
return
function A=DM4(W,X,Y,Z)
```

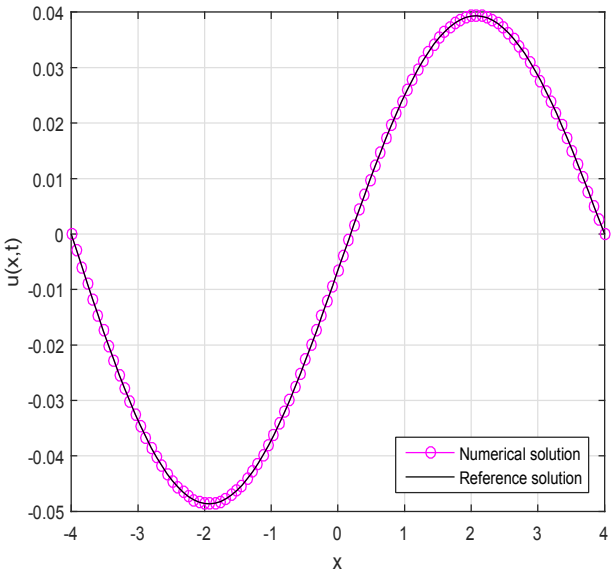

Figure 6: Numerical solution of the problem (16)-(17) with the MP basic splitting method at final time $t_f = 4$ and $N = 100$, $\frac{\Delta t}{\Delta x} = 0.4$ in Test 6.

```
 A=0.125*(sign(W)+sign(X))*abs((sign(W)+sign(Y))*(sign(W)+sign(Z)))*
min(min(abs(W),abs(X)),min(abs(Y),abs(Z)));
return
function B=DMM(X,Y)
 B = 0.5*(sign(X)+sign(Y))*min(abs(X),abs(Y));
return
```

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