



# The concept of $B$ -efficient solution in fair multiobjective optimization problems

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## Abstract

A problem that sometimes occurs in multiobjective optimization is the existence of a large set of fairly efficient solutions. Hence, the decision making based on selecting a unique preferred solution is difficult. Considering models with fair  $B$ -efficiency relieves some of the burden from the decision maker by shrinking the solution set, since the set of fairly  $B$ -efficient solutions is contained within the set of fairly efficient solutions for the same problem. In this paper, first some theoretical and practical aspects of fairly  $B$ -efficient solutions are discussed. Then, some scalarization techniques are developed to generate fairly  $B$ -efficient solutions.

**Keywords:** Fair optimization; Nondominated; Equitability;  $B$ -efficiency; Scalarization.

## 1 Introduction

The Multiobjective programming has been studied for many years and multiobjective methods have found applications in diverse areas of human life. It is well-known that any multiobjective optimization problem starts usually with an assumption that the criteria are incomparable, i.e., different criteria may have different units and physical interpretations. Many applications, however, arise from situations which present equitable criteria. Equitability is based on the assumption that the criteria are not only comparable (measured on a common scale) but also anonymous (impartial). The latter makes

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the distribution of outcomes among the criteria more important than the assignment of outcomes to specific criteria, and therefore models fair allocation of resources.

The fair preference was first known as the generalized Lorenz dominance [5, 7]. Kostreva and Ogryczak [3] are the first ones who introduced the concept of equitability into multiobjective programming. They have shown fair efficiency to be a refinement of Pareto efficiency by adding, to the reflexivity, strict monotonicity and transitivity of the Pareto preference order, the requirements of impartiality and satisfaction of the principle of transfers. Then, Kostreva et al. [4] presented the theory of equitable efficiency in greater generality. They have developed scalarization approaches to generate equitably efficient solutions of linear and nonlinear multiobjective programs. Ogryczak applied equitability to portfolio optimization [8], location problems [9, 12], and telecommunications [11]. Moreover Ogryczak et al. [10], applied fair (equitable) optimization methods for the resource allocation problems in communication networks.

It should be noted that some authors have used the term “equitable” rather than “fair”. In this paper, we investigate some theoretical and practical aspects of the fairly  $B$ -efficient solutions and propose some approaches to generate equitably  $B$ -efficient solutions. The results are generalizations of the results of Kostreva and Ogryczak [3] and Kostreva et al. [4].

## 2 Terminology

Throughout this article, the following notation is used. Let  $R^m$  be the Euclidean vector space and  $y', y'' \in R^m$ .  $y' \leq y''$  denotes  $y'_i \leq y''_i$  for all  $i = 1, \dots, m$ .  $y' < y''$  denotes  $y'_i < y''_i$  for all  $i = 1, \dots, m$ .  $y' \leq y''$  denotes  $y' \leq y''$  but  $y' \neq y''$ .

Consider a decision problem defined as an optimization problem with  $m$  objective functions. For simplification we assume, without loss of generality, that the objective functions are to be minimized. The problem can be formulated as follows:

$$\begin{cases} \min(f_1(x), f_2(x), \dots, f_m(x)), \\ \text{subject to } x \in X \end{cases} \quad (1)$$

where  $x$  denotes a vector of decision variables selected from the feasible set  $X$  and  $f(x) = (f_1(x), f_2(x), \dots, f_m(x))$  is a vector function that maps the feasible set  $X$  into the objective (criterion) space  $R^m$ . We refer to the elements of the objective space as outcome vectors. An outcome vector  $y$  is attainable if it expresses outcomes of a feasible solution, i.e.,  $y = f(x)$  for some  $x \in X$ . The set of all attainable outcome vectors will be denoted by  $Y = f(X)$ .

In single objective minimization problems, we compare the objective values at different feasible decisions to select the best decision. Decisions are ranked according to the objective values at those decisions and any decision with smallest objective value is called an optimal solution. Similarly, to make the multiobjective optimization model operational, one needs to assume some solution concept specifying what it means to minimize multiobjective functions. The solution concepts are defined by the properties of the corresponding preference model. We assume that solution concepts depend only on the evaluation of the outcome vectors while not taking into account any other solution properties not represented within the outcome vectors. Thus, we can limit our considerations to the preference model in the objective space  $Y$ .

In the following, some basic concepts and definitions of preference relations are reviewed from [3]. Preferences are represented by a weak preference relation by  $\preceq$ , which allows us to compare pairs of outcome vectors  $y', y''$  in the objective space  $Y$ . We say  $y' \preceq y''$  if and only if “ $y'$  is at least as good as  $y''$ ” or “ $y'$  is weakly preferred to  $y''$ ”. In words,  $y' \preceq y''$  means that the decision maker thinks that the outcome vector  $y'$  is at least as good as the outcome vector  $y''$ . It should be noticed that the weak preference relation, the “at least as good as” relation, is given by the decision maker. From  $\preceq$ , we can derive two other important relations on  $Y$ .

**Definition 1.** Let  $y', y'' \in R^m$  and let  $\preceq$  be a relation of weak preference defined on  $R^m \times R^m$ . The strict preference relation,  $\prec$ , defined by

$$y' \prec y'' \Leftrightarrow (y' \preceq y'' \text{ and not } y'' \preceq y'), \quad (2)$$

and read  $y'$  is strictly preferred to  $y''$ . Also the indifference relation,  $\simeq$ , defined by

$$y' \simeq y'' \Leftrightarrow (y' \preceq y'' \text{ and } y'' \preceq y'), \quad (3)$$

and read  $y'$  is indifferent to  $y''$ .

Hence “ $y'$  is at least as good as  $y''$ ” means that “ $y'$  is strictly preferred to  $y''$ ” or “ $y'$  is indifferent to  $y''$ ”.

**Definition 2.** Preference relations satisfying the following axioms are called rational preference relations:

1. Reflexivity: for all  $y \in R^m$ ,  $y \preceq y$ .
2. Transitivity: for all  $y', y'', y''' \in R^m$ ,  $y' \preceq y''$  and  $y'' \preceq y''' \Rightarrow y' \preceq y'''$ .
3. Strict monotonicity: for all  $y \in R^m$ ,  $y - \epsilon e_i \prec y$  for all  $\epsilon > 0$  where  $e_i$  denotes the  $i^{\text{th}}$  unit vector in  $R^m$ , for all  $i \in \{1, 2, \dots, m\}$ .

The rational preference relations allow us to formalize the Pareto-optimal solution concept with the following definitions.

**Definition 3.** The outcome vector  $y' \in Y$  rationally dominates  $y'' \in Y$  iff  $y' \prec y''$  for all rational preference relations  $\preceq$ .

An outcome vector  $y$  is rationally nondominated if and only if there does not exist another outcome vector  $y'$  such that  $y'$  rationally dominates  $y$ . Analogously, a feasible solution  $x \in X$  is an efficient or Pareto-optimal solution of the multiobjective problem (1) if and only if  $y = f(x)$  is rationally nondominated.

It has been shown in [3] that the outcome vector  $y' \in Y$  rationally dominates  $y'' \in Y$  if and only if  $y' \leq y''$ . As a consequence, we can state that a feasible solution  $x \in X$  is a Pareto-optimal solution of the multiobjective problem (1), if and only if, there does not exist  $x' \in X$  such that  $f_i(x') \leq f_i(x)$  for  $i = 1, 2, \dots, m$ , where at least one strict inequality holds.

Let  $\preceq$  be a preference relation defined on  $\mathbb{R}^m$ .

**Definition 4.**  $\preceq$  is said to be impartial if

$$(y_1, y_2, \dots, y_m) \simeq (y_{\tau(1)}, y_{\tau(2)}, \dots, y_{\tau(m)})$$

for all  $y \in R^m$ , where  $\tau$  stands for an arbitrary permutation of components of  $y$ .

**Definition 5.**  $\preceq$  is said to satisfy the principle of transfers if  $y_i > y_j \Rightarrow y - \epsilon e_i + \epsilon e_j \preceq y$ , for all  $y \in R^m$  and all  $\epsilon \in [0, y_i - y_j]$ .

**Definition 6.** A preference relation  $\preceq$  defined on  $R^m$  is called a fair rational preference relation if it is reflexive, transitive, strictly monotonic, impartial and satisfies the principle of transfers.

The fair rational preference relations allow us to define the concept of fairly efficient solution.

**Definition 7.** Let  $y', y'' \in Y$ . We say that  $y'$  fairly dominates  $y''$ , and denote by  $y' \prec_e y''$  iff  $y' \prec y''$  for all fair rational preference relations  $\preceq$ .

An outcome vector  $y$  is fairly nondominated if and only if there does not exist another outcome vector  $y'$  such that  $y'$  fairly dominates  $y$ . Analogously, a feasible solution  $x$  is called a fairly efficient solution of the multiobjective problem (1) if and only if  $y = f(x)$  is fairly nondominated.

**Definition 8.** Let  $y \in R^m$ .

1. The function  $\Theta : R^m \rightarrow R^m$  is called an ordering map iff  $\Theta(y) = (\theta_1(y), \theta_2(y), \dots, \theta_m(y))$ , where  $\theta_1(y) \geq \theta_2(y) \geq \dots \geq \theta_m(y)$  in which  $\theta_i(y) = y_{\tau(i)}$  for  $i = 1, 2, \dots, m$ , and  $\tau$  is a permutation of the set  $\{1, 2, \dots, m\}$ .

2. The function  $\bar{\Theta} : R^m \rightarrow R^m$  is called a cumulative ordering map iff  $\bar{\Theta}(y) = (\bar{\theta}_1(y), \bar{\theta}_2(y), \dots, \bar{\theta}_m(y))$ , where  $\bar{\theta}_i(y) = \sum_{j=1}^i \theta_j(y)$  for  $i = 1, 2, \dots, m$ .

To make it practical, fair efficiency is defined in terms of vector inequalities.

**Proposition 1.** ([3], Proposition 2.3) For any two vectors  $y', y'' \in Y$

$$y' \preceq_e y'' \Leftrightarrow \bar{\Theta}(y') \leq \bar{\Theta}(y''),$$

where  $\preceq_e$  is given by Definition 7.

### 3 Fair $B$ -efficient solutions

In this section, we first introduce the concept of fairly  $B$ -efficient solution by fair rational preference relations. Then, similar to the fair dominance relation, we can express the fair  $B$ -dominance relation in terms of vector inequality on the ordered outcome vectors.

In fair multiobjective optimization, the focus is on the distribution of outcome values while ignoring their ordering. This means that in the multiobjective optimization problem (1) we are interested in a set of values of the objectives without taking into account which objective is taking a specific value. In this respect, let  $\{1, 2, \dots, m\}$  be the index of the vector  $\Theta(y) = (\theta_1(y), \theta_2(y), \dots, \theta_m(y))$  and  $n$  be a positive integer such that  $n \leq m$ . Throughout this article, we suppose that  $B = \{B_1, B_2, \dots, B_n\}$  is a partition of the set  $\{1, 2, \dots, m\}$  such that

$$\max B_i < \min B_{i+1}, \quad i = 1, 2, \dots, n - 1. \quad (4)$$

The following definition is a necessary notion for the solution concepts of interest in this paper.

**Definition 9.** Suppose that  $n \leq m$  and  $B = \{B_1, B_2, \dots, B_n\}$  is a partition of  $\{1, 2, \dots, m\}$  such that condition (4) is satisfied. The cumulative map  $\Theta_B : R^m \rightarrow R^n$  is defined by

$$\Theta_B(y) = \left( \sum_{j \in B_1} \theta_j(y), \sum_{j \in B_2} \theta_j(y), \dots, \sum_{j \in B_n} \theta_j(y) \right),$$

where  $\theta_j$ 's are as defined in Definition 8.

**Definition 10.** Suppose that  $y', y'' \in Y$  are two outcome vectors. We say that  $y'$  fairly  $B$ -dominates  $y''$  and write  $y' \prec_{Be} y''$  iff  $\Theta_B(y') \prec \Theta_B(y'')$  for all fair rational preference relations  $\preceq$ .

**Definition 11.** We say that outcome vector  $y \in Y$  is fairly  $B$ -nondominated

iff there does not exist  $y' \in Y$  such that  $y' \prec_{Be} y$ .

**Definition 12.** We say that feasible solution  $x \in X$  is a fairly  $B$ -efficient solution of the multiobjective problem (1), iff  $y = f(x)$  is fairly  $B$ -nondominated.

Similar to the relation of fair  $B$ -dominance, we can define the relation of fair  $B$ -indifference (indifference for all fair rational preference relations) and the relation of fair weak  $B$ -dominance (weak preference for all fair rational preference relations). The relations of fair  $B$ -dominance  $\prec_{Be}$ ,  $B$ -indifference  $\simeq_{Be}$ , and weak  $B$ -dominance  $\preceq_{Be}$  satisfy conditions (2-3).

To make it practical, fair  $B$ -efficiency can be defined in terms of vector inequalities. In order to do that, we define certain mapping.

**Definition 13.** Suppose that  $n \leq m$  and  $B = \{B_1, B_2, \dots, B_n\}$  is a partition of  $\{1, 2, \dots, m\}$  such that condition (4) is satisfied. The cumulative ordering map  $\bar{\Theta}_B : R^m \rightarrow R^n$  is defined by

$$\bar{\Theta}_B(y) = \left( \sum_{j \in B_1} \theta_j(y), \sum_{j \in B_1 \cup B_2} \theta_j(y), \dots, \sum_{j \in B_1 \cup B_2 \cup \dots \cup B_n} \theta_j(y) \right),$$

where  $\theta_j$ 's are as defined in Definition 8.

**Definition 14.** Suppose that  $y', y'' \in Y$  are two outcome vectors. The relation  $\leq_{Bie}$  is defined as follows.

$$y' \leq_{Bie} y'' \Leftrightarrow \bar{\Theta}_B(y') \leq \bar{\Theta}_B(y'').$$

Note that if  $B_j = \{j\}$  for  $j = 1, 2, \dots, m$ , then  $\Theta_B(y) = \Theta(y)$  and  $\bar{\Theta}_B(y) = \bar{\Theta}(y)$ , so the relation  $\preceq_{Be}$  becomes the relation  $\preceq_e$ .

The relations  $<_{Bie}$  and  $=_{Bie}$  are defined by

$$\begin{aligned} y' <_{Bie} y'' &\Leftrightarrow (y' \leq_{Bie} y'' \text{ and not } y'' \leq_{Bie} y'), \\ y' =_{Bie} y'' &\Leftrightarrow (y' \leq_{Bie} y'' \text{ and } y'' \leq_{Bie} y'). \end{aligned}$$

Below, we will discuss the relationship between two preference relations  $\preceq_{Be}$  and  $\leq_{Bie}$ . In order to do that, we need the following proposition.

**Proposition 2.** ([3], Proposition 2.2) *If  $\bar{\Theta}(y') \leq \bar{\Theta}(y'')$ , then  $\Theta(y') \leq \Theta(y'')$  or there exists a finite sequence of vectors  $y^0 = y'', y^1, \dots, y^t$  such that  $y^k = y^{k-1} - \epsilon_k e_{i'} + \epsilon_k e_{i''}$ ,  $i', i'' \in \{1, 2, \dots, m\}$ ,  $0 < \epsilon_k < y_{i'}^{k-1} - y_{i''}^{k-1}$  for  $k = 1, 2, \dots, t$  and  $\Theta(y') \leq \Theta(y^t)$ .*

Since each fair rational preference relation  $\preceq$  satisfies the principle of transfers, by using Proposition 2, we have

$$\bar{\Theta}(y') \leq \bar{\Theta}(y'') \implies \Theta(y') \preceq \Theta(y'').$$

**Theorem 1.** Let  $|B_1| \geq |B_2| \geq \dots \geq |B_n|$  and  $y', y'' \in Y$  be two outcome vectors. We have

$$\begin{aligned} y' \preceq_{Be} y'' &\iff y' \leq_{Bie} y'', \\ y' \prec_{Be} y'' &\iff y' <_{Bie} y''. \end{aligned}$$

*Proof.* By assumption  $|B_1| \geq |B_2| \geq \dots \geq |B_n|$ , we have

$$\sum_{j \in B_1} \theta_j(y) \geq \sum_{j \in B_2} \theta_j(y) \geq \dots \geq \sum_{j \in B_n} \theta_j(y),$$

for all  $y \in Y$ . This means that the vector  $\Theta_B(y)$  is decreasing, so  $\bar{\Theta}(\Theta_B(y)) = \bar{\Theta}_B(y)$ . It is clear that

$$y' \preceq_{Be} y'' \implies y' \leq_{Bie} y'',$$

since the relation  $\preceq_e$  is a fair rational preference relation. Conversely suppose that  $y' \leq_{Bie} y''$ , we deduce  $\bar{\Theta}_B(y') \leq \bar{\Theta}_B(y'')$ . So

$$\bar{\Theta}(\Theta_B(y')) \leq \bar{\Theta}(\Theta_B(y'')).$$

Due to Proposition 2, we have  $\Theta_B(y') \preceq \Theta_B(y'')$  for any fair rational preference relation  $\preceq$ . Thus  $y' \preceq_{Be} y''$ . By the same reasoning, remaining cases are satisfied.  $\square$

Due to Theorem 1, we deduce that the relation  $\preceq_{Be}$  is a fair rational preference relation, when  $|B_1| \geq |B_2| \geq \dots \geq |B_n|$ . Preference relation  $\preceq_{Be}$ , in general, does not satisfy the principle of transfers axiom. The truth of this statement is examined by the following example.

**Example 1.** Let  $B_1 = \{1\}$ ,  $B_2 = \{2, 3\}$ ,  $y = (4, 2.5, 2)$ . If  $\epsilon = 1$  and  $y' = y - \epsilon e_1 + \epsilon e_3$ , then  $y' = (3, 2.5, 3)$ . Since  $\bar{\Theta}(\Theta_B(y)) = (4.5, 8.5)$  and  $\bar{\Theta}(\Theta_B(y')) = (5.5, 8.5)$ . We have  $\bar{\Theta}(\Theta_B(y')) \not\leq \bar{\Theta}(\Theta_B(y))$ , so  $y' \not\preceq_{Be} y$ .

It should be noted that in this example  $\bar{\Theta}_B(y) = (4, 8.5)$  and  $\bar{\Theta}_B(y') = (3, 8.5)$ . So  $y' \leq_{Bie} y$ . Also if  $y' = (4, 2, 1)$  and  $y'' = (3.5, 3, 2)$ , it is obvious that  $y' \preceq_{Be} y''$  while the  $y' \not\leq_{Bie} y''$ . This reflects the fact that Theorem 1, in general, is not true for any partition  $B$  and condition  $|B_1| \geq |B_2| \geq \dots \geq |B_n|$  is necessary.

By applying Theorem 1 and Definition 14, we have the following statement.

**Corollary 1.** Suppose that  $|B_1| \geq |B_2| \geq \dots \geq |B_n|$  and  $y', y'' \in Y$ . We have

$$\begin{aligned} y' \preceq_{Be} y'' &\Leftrightarrow \overline{\Theta}_B(y') \leq \overline{\Theta}_B(y''), \\ y' \prec_{Be} y'' &\Leftrightarrow \overline{\Theta}_B(y') < \overline{\Theta}_B(y''). \end{aligned}$$

**Remark 1.** If  $B_j = \{j\}$  for  $j = 1, 2, \dots, m$ , we have Proposition 2.3 from [3].

Note that Corollary 1 permits one to express fair  $B$ -efficiency for problem (1) in terms of the standard efficiency for the multiobjective problem with objectives  $\overline{\Theta}_B(f(x))$ :

$$\min\{\overline{\Theta}_B(f(x)) : x \in X\}. \quad (5)$$

**Theorem 2.** Let  $|B_1| \geq |B_2| \geq \dots \geq |B_n|$ . A feasible solution  $x \in X$  is a fairly  $B$ -efficient solution of the multiobjective problem (1) if and only if it is an efficient solution of the multiobjective problem (5).

*Proof.* By applying Corollary 1, we obtain the desired result.  $\square$

**Remark 2.** If  $B_j = \{j\}$  for  $j = 1, 2, \dots, m$ , then we have Corollary 2.2 from [3].

The following theorem provides the relationship between fairly  $B$ -efficient solutions and fairly efficient solutions.

**Theorem 3.** Let  $|B_1| \geq |B_2| \geq \dots \geq |B_n|$  and  $y \in Y$  be a outcome vector. If  $y$  is fairly  $B$ -nondominated, then it is fairly nondominated.

*Proof.* Suppose that  $y$  is not fairly nondominated. Then there exists a vector  $y' \in Y$  such that  $y' \prec_e y$ . Due to Proposition 1,  $\overline{\Theta}(y') < \overline{\Theta}(y)$ , so  $y' \prec_{Be} y$ . Since  $|B_1| \geq |B_2| \geq \dots \geq |B_n|$ , by using Corollary 1, we deduce that  $y' \prec_{Be} y$ .  $\square$

**Corollary 2.** Let  $|B_1| \geq |B_2| \geq \dots \geq |B_n|$  and  $x \in X$  be a feasible solution. If  $x$  is a fairly  $B$ -efficient solution of multiobjective problem (1), then it is an fairly efficient solution of (1).

This result suggests that the set of fairly  $B$ -efficient solutions is contained within the set of fairly efficient solutions, but the reverse inclusion doesn't hold in general. It should be noted the structure of fair dominance is discussed in Example 2.1 of [3] by Kostreva and Ogryczak. They showed that the set of fairly efficient solutions is contained within the set of efficient solutions. In the following examples, we investigate the effectiveness of fair  $B$ -dominance relation with respect to fair dominance.

**Example 2.** Let's consider the problem

$$\min \{(x_1, x_2) : 4x_1 + 5x_2 \geq 72, x_1 \geq 0, 0 \leq x_2 \leq 72/5\},$$

and  $B = \{B_1\}$  with  $B_1 = \{1, 2\}$ . It is obvious that

$$E = \{(x_1, x_2) : 4x_1 + 5x_2 = 72, x_1 \geq 0, x_2 \geq 0\},$$

$$F = \{(x_1, x_2) : 4x_1 + 5x_2 = 72, 0 \leq x_1 \leq 8, 8 \leq x_2 \leq 72/5\},$$

and  $G = \{(0, 72/5)\}$  are the set of efficient solutions, the set of fairly efficient solutions and the set of fairly  $B$ -efficient solutions, respectively. Thus  $G \subsetneq F \subsetneq E$ .

In the next example, a large number of random solutions are generated for scalable test functions. From this large set of solutions, the nondominated solutions with respect to rational dominance (Pareto dominance), fair dominance and fair  $B$ -dominance are calculated.

**Example 3.** The test problem considered is the VFM1 [14],

$$\min_{x \in \mathbb{R}^2} y = \{f_1(x), f_2(x), f_3(x)\}$$

$$f_1(x) = x_1^2 + (x_2 - 1)^2$$

$$f_2(x) = x_1^2 + (x_2 + 1)^2 + 1$$

$$f_3(x) = (x_1 - 1)^2 + x_2^2 + 2$$

$$x_1, x_2 \in [-2, 2].$$

Figure 1 shows the Pareto, fair and fair  $B$ -dominance fronts (objective space).

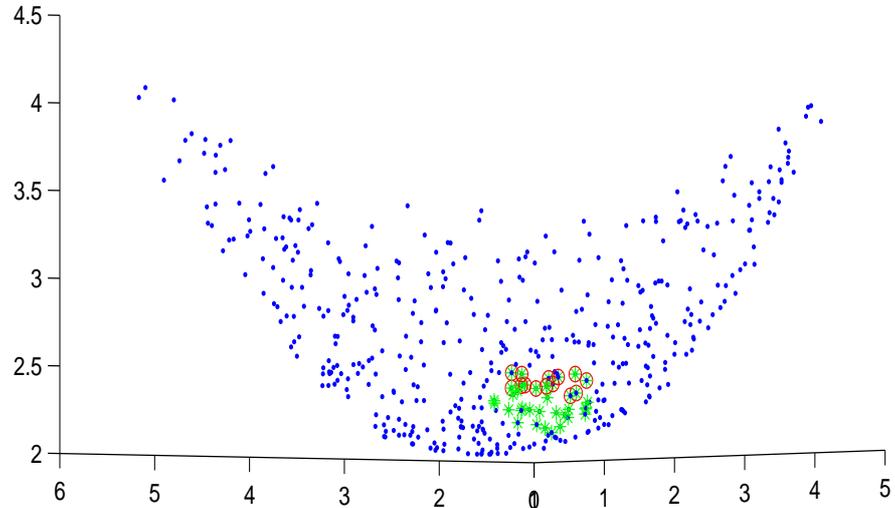


Figure 1: The Pareto, fair and fair  $B$ -dominance fronts (objective space) for the VFM1 problem (2 variables and 3 objectives)

From 5000 random solutions, 496 solutions (point) are rationally non-dominated, 37 solutions (star) are fairly nondominated and 14 solutions (circle) are fairly  $B$ -nondominated, which are obtained by assuming  $B_1 = \{1, 2\}$ ,  $B_2 = \{3\}$ .

In the rest of this section, we will investigate the structure of fairly  $B$ -efficient set. For this purpose, problem (1) is considered as a multiobjective linear programming problem. It is assumed that,  $X \subset R^n$  denotes the feasible set defined by a system of linear equations and inequalities, and the objective functions are

$$f_i(x) = c_i x \quad (i = 1, 2, \dots, m), \quad (6)$$

where  $c_i^T \in R^n$ . Suppose that  $n \leq m$  and  $B = \{B_1, B_2, \dots, B_n\}$  is a partition of  $\{1, 2, \dots, m\}$  such that condition (4) is satisfied. We put  $b_i = \max B_i$  for  $i = 1, 2, \dots, n$ , so

$$\begin{aligned} \bar{\Theta}_B(y) &= \left( \sum_{j=1}^{b_1} \theta_j(y), \sum_{j=1}^{b_2} \theta_j(y), \dots, \sum_{j=1}^{b_n} \theta_j(y) \right) \\ &= (\bar{\theta}_{b_1}(y), \bar{\theta}_{b_2}(y), \dots, \bar{\theta}_{b_n}(y)), \end{aligned}$$

where  $\bar{\theta}_{b_i}(y) = \sum_{j=1}^{b_i} \theta_j(y)$  for  $i = 1, 2, \dots, n$ , and  $b_n = m$ . In Theorem 2, we established the equivalence between the fairly  $B$ -efficient solutions of (1) and the efficient solutions of problem (5). Now, we further use this result to explore the structure of the fairly  $B$ -efficient set. Note that the individual objective functions of problem (5) are convex piecewise linear functions of  $y = f(x)$ . They can be written in the form

$$\bar{\theta}_{b_i}(y) = \max_{\tau \in \Pi} \left( \sum_{j=1}^{b_i} y_{\tau(j)} \right) \quad (i = 1, 2, \dots, n), \quad (7)$$

where  $\Pi$  denotes the set of all permutations  $\tau$  of the set  $\{1, 2, \dots, m\}$ . Thus, the corresponding problem (5) can be expressed in the form of multiobjective linear program

$$\min (z_{b_1}, z_{b_2}, \dots, z_{b_n}) \quad (8)$$

subject to

$$x \in X, \quad (9)$$

$$y_i = c_i x \quad \text{for } i = 1, 2, \dots, m, \quad (10)$$

$$z_{b_i} \geq \sum_{j=1}^{b_i} y_{\tau(j)} \quad \text{for } \tau \in \Pi; i = 1, 2, \dots, n. \quad (11)$$

Put  $z_B = (z_{b_1}, z_{b_2}, \dots, z_{b_n})$ . According to (7), the problem above is equivalent to problem (5) as stated in the following theorem.

**Theorem 4.** *A triple  $(x, y, z_B)$  is an efficient solution of (8)-(11), if and only if  $y = Cx$ ,  $z_B = \overline{\Theta}_B(y)$  and  $x$  is an efficient solution of the multiobjective problem (5).*

**Remark 3.** If  $B_j = \{j\}$  for  $j = 1, 2, \dots, m$ , then we have Proposition 4.1 from [3].

The question that arises is whether there exist fairly  $B$ -efficient solutions. The next result provides some sufficient conditions to answer this question. For this purpose, we need the following theorem.

**Theorem 5.** (*[1], Theorem 6.5*) *If  $X \neq \emptyset$  and there exists  $y^0 \in Y$  such that  $y^0 \leq Cx$  for all  $x \in X$ , then there exists a efficient solution of problem (1).*

**Theorem 6.** *Let  $|B_1| \geq |B_2| \geq \dots \geq |B_n|$ . If  $X \neq \emptyset$  and there exists  $y^0 \in Y$  such that  $y^0 \preceq_{B^e} Cx$  for all  $x \in X$ , then there exists a fairly  $B$ -efficient solution of problem (1).*

*Proof.* Due to Corollary 1,  $z_B^0 = \overline{\Theta}_B(y^0) \leq \overline{\Theta}_B(Cx)$  for all  $x \in X$ . Thus,  $z_B^0 \leq z_B$  for any attainable achievement vector  $z_B$  of the multiobjective linear programming (8)-(11), which is by Theorem 4 equivalent to the problem (5). Hence, by using Theorem 5, there exists an efficient solution  $x^0$  of (8)-(11). Due to Corollary 1,  $x^0$  is a fairly  $B$ -efficient solution of problem (1).  $\square$

**Remark 4.** If  $B_j = \{j\}$  for  $j = 1, 2, \dots, m$ , then we have Proposition 4.3 from [3].

Kostreva and Ogryczak in [3] have shown that the set of fairly efficient solutions is nonempty provided that the set of efficient solutions is nonempty.

**Proposition 3.** (*[3], Proposition 4.4*) *If there exists an efficient solution of problem (1), then there exists a fairly efficient solution of problem (1).*

By applying Corollary 2 and Proposition 3, we obtain the following statement.

**Corollary 3.** Let  $|B_1| \geq |B_2| \geq \dots \geq |B_n|$ . If there exists an efficient solution of problem (1), then there exists a fairly  $B$ -efficient solution of problem (1).

In addition to the existence of the fairly  $B$ -efficient set and its relationship to the efficient set, we investigate the fact that the fairly  $B$ -efficient set is connected. To do so, we need the following theorem.

**Theorem 7.** (*[6], Theorem 2.3*) *If the set of efficient solutions of a multiobjective linear programming problem is nonempty, then it is connected.*

**Theorem 8.** *If  $|B_1| \geq |B_2| \geq \dots \geq |B_n|$  and the set of efficient solutions of multiobjective linear programming problem (8)-(11) is nonempty. Then the set of fairly  $B$ -efficient solutions of problem (1) is connected.*

*Proof.* The set of fairly  $B$ -efficient solutions of problem (1) is the same as the set of efficient solutions of problem (5). By using Theorem 4, the efficient solutions of problem (5) are in one-to-one correspondence to the efficient solutions of (8)-(11). Since (8)-(11) is a linear multiobjective optimization problem, its efficient set is a connected set according to Theorem 7.  $\square$

**Remark 5.** If  $B_j = \{j\}$  for  $j = 1, 2, \dots, m$ , then we have Proposition 4.5 from [3].

## 4 Generation techniques

In this section, we develop some scalarization-based methods to generate fairly  $B$ -efficient solutions. Scalarization is one of the most common approaches used to solve a multiobjective problem. Scalarizing functions are used to transform a given multiobjective problem into a single objective optimization problem, by aggregating the objectives of a multiobjective problem into a single objective. Typical solution concepts for multiobjective problems are defined by scalarizing functions  $s : Y \rightarrow \mathbb{R}$  to be minimized. Thus the multiobjective problem (1) is replaced with the minimization problem

$$\min\{s(f(x)) : x \in X\}. \quad (12)$$

The preference relation corresponding to the problem (12) is defined as follows:

$$y' \preceq y'' \Leftrightarrow s(y') \leq s(y'').$$

For any strictly convex, increasing function  $g : R \rightarrow R$ , the scalarizing function defined by

$$s(y) = \sum_{i=1}^m g(y_i)$$

is a strictly monotonic and strictly Schur-convex function [7]. It has been shown in Proposition 3.1 from [3] that the preference relation corresponding to this scalarizing function is a fair rational preference relation. Also, every optimal solution of problem (12) is a fairly efficient solution of the original multiobjective problem.

In the following, we will generate the fairly  $B$ -efficient solutions by introducing certain scalarizing functions.

**Theorem 9.** *Let  $g : R \rightarrow R$  be a strictly convex and increasing function. The optimal solution of the problem*

$$\min \left\{ \sum_{i=1}^n g \left( \sum_{j \in B_i} \theta_j(f(x)) \right) : x \in X \right\}, \quad (13)$$

is a fairly  $B$ -efficient solution of the multiobjective problem (1).

*Proof.* Suppose that  $x$  is not a fairly  $B$ -efficient solution of the multiobjective problem (1). Then a feasible vector  $x'$  must exist such that the vectors  $f(x)$  and  $f(x')$  satisfy  $f(x') \prec_{Be} f(x)$ . Namely for all fair rational preference relations  $\prec$ , we have  $\Theta_B(f(x')) \prec \Theta_B(f(x))$ . Since the preference relation corresponding to scalarizing function is a fair rational preference relation, we deduce that

$$\sum_{i=1}^n g \left( \sum_{j \in B_i} \theta_j(f(x')) \right) < \sum_{i=1}^n g \left( \sum_{j \in B_i} \theta_j(f(x)) \right),$$

which contradicts the optimal solution of  $x$  for (13).  $\square$

**Remark 6.** If  $B_j = \{j\}$  for  $j = 1, 2, \dots, m$ , we have Corollary 3.1 from [3].

The weighted sum method is one of the most common ways of finding efficient solutions of multiobjective problem. Details of the method can be found in [2]. Further, Kostreva et al. [4] have proven every optimal solution of the weighted sum problem with strictly decreasing positive weights and ordering map  $\Theta(f(x))$ , is a fairly efficient solution of the original multiobjective optimization problem. If the weighted sum method is applied to problem (5), due to the definition of map  $\Theta_B$ , we have

$$\min \left\{ \sum_{i=1}^n w_i \left( \sum_{j \in B_1 \cup B_2 \cup \dots \cup B_i} \theta_j(f(x)) \right) : x \in X \right\} \quad (14)$$

where  $w \in R^n$  is any positive vector. The above problem is equivalent to

$$\min \left\{ \sum_{i=1}^n \lambda_i \left( \sum_{j \in B_i} \theta_j(f(x)) \right) : x \in X \right\} \quad (15)$$

where  $\lambda_i = \sum_{j=i}^n w_j$  for  $i = 1, \dots, n$ .

**Theorem 10.** Let  $\lambda_1 > \lambda_2 > \dots > \lambda_n > 0$ . If  $|B_1| \geq |B_2| \geq \dots \geq |B_n|$ , then the optimal solution of problem (15) is a fairly  $B$ -efficient solution of the multiobjective problem (1).

*Proof.* Suppose that  $x$  is not a fairly  $B$ -efficient solution of the multiobjective problem (1). Then a feasible vector  $x'$  must exist such that the vectors  $f(x)$  and  $f(x')$  satisfy  $f(x') \prec_{Be} f(x)$ . By Corollary 1, we deduce that

$$\sum_{B_1 \cup \dots \cup B_k} \theta_j(f(x')) \leq \sum_{B_1 \cup \dots \cup B_k} \theta_j(f(x)) \quad k = 1, 2, \dots, n,$$

where strict inequality holds at least once. If  $w_i = \lambda_i - \lambda_{i+1}$ , then  $w_i > 0$  for  $i = 1, 2, \dots, n$ . So,

$$\sum_{k=1}^n w_k \sum_{B_1 \cup \dots \cup B_k} \theta_j(f(x')) < \sum_{k=1}^n w_k \sum_{B_1 \cup \dots \cup B_k} \theta_j(f(x)).$$

This means that  $x$  cannot be the optimal solution of problem (14). Since problems (14) and (15) are equivalent, the desired result is obtained.  $\square$

**Remark 7.** If  $B_j = \{j\}$  for  $j = 1, 2, \dots, m$ , then we have Theorem 2 from [4].

A very important result in multiojective linear optimization is the association of efficient solutions with optimal solutions of the scalar weighting problem using positive weights [13]. The following result is the analogue for our fairly  $B$ -efficient solution set. It seems that such a result should thus play an important role in analysis of fair  $B$ -efficiency.

**Theorem 11.** *Suppose that  $|B_1| \geq |B_2| \geq \dots \geq |B_n|$  and the function  $f$  is defined as in (6). A feasible solution  $x^0$  is a fairly  $B$ -efficient solution of problem (1), if and only if, there exists a weight vector  $\lambda \in R^n$  with  $\lambda_1 > \lambda_2 > \dots > \lambda_n > 0$  such that  $x^0$  is an optimal solution of problem (15).*

*Proof.* Sufficiency of the condition follows from Theorem 10. Thus we only need to show that for each fairly  $B$ -efficient solution  $x^0$  there exists a weight vector  $\lambda \in R^n$  with  $\lambda_1 > \lambda_2 > \dots > \lambda_n > 0$  such that  $x^0$  is an optimal solution of the problem (15). Due to Theorems 2 and 4, if  $x^0$  is a fairly  $B$ -efficient solution of (1), then  $(x^0, Cx^0, \bar{\Theta}_B(Cx^0))$  is an efficient solution of multiobjective linear program (8)-(11). Thus, from the theory of multiobjective linear optimization [13], there exist positive weights  $w_1, w_2, \dots, w_n$  such that  $(x^0, Cx^0, \bar{\Theta}_B(Cx^0))$  is an optimal solution of the problem

$$\min \left\{ \sum_{i=1}^n w_i z_{b_i} : (8) - (11) \right\} \quad (16)$$

Due to positive weights  $w_i$ , the above problem is equivalent to the problem

$$\min \left\{ \sum_{i=1}^n w_i \bar{\theta}_{b_i}(Cx) : (8) - (11) \right\} \quad (17)$$

where  $\bar{\theta}_{b_i}(y) = \sum_{j=1}^{b_i} \theta_j(y)$  for  $i = 1, 2, \dots, n$ . Problem (17) is equivalent to problem (15) where  $\lambda_i = \sum_{j=i}^n w_j$  for  $i = 1, \dots, n$ . This completes the proof of the theorem.  $\square$

**Remark 8.** If  $B_j = \{j\}$  for  $j = 1, 2, \dots, m$ , then we have Proposition 4.6 from [3].

Another way to generate fairly  $B$ -efficient solutions is lexicographic minimax approach. It is shown that any optimal solution of the lexicographic minimax problem is fairly efficient for the original multiobjective problem [4]. When applying the lexicographic optimization to the problem (5), we get the lexicographic problem

$$\text{lexmin}\{\bar{\Theta}_B(f(x)) : x \in X\}. \quad (18)$$

Due to Definition 13, problem (18) is equivalent to the problem

$$\text{lexmin}\{\Theta_B(f(x)) : x \in X\}. \quad (19)$$

We recall that, if  $y' \neq y''$  and  $s = \min\{i : y'_i \neq y''_i\}$ , then  $y' <_{lex} y''$  if and only if  $y'_s < y''_s$ . Also  $y' \leq_{lex} y''$  if and only if  $y'_s < y''_s$  or  $y' = y''$ .

**Definition 15.** A feasible solution  $x \in X$  is lexicographically optimal or a lexicographic solution of the multiobjective problem (18), if there is no  $x' \in X$  such that  $\bar{\Theta}_B(f(x')) <_{lex} \bar{\Theta}_B(f(x))$ .

**Theorem 12.** Let  $x \in X$  be a lexicographically optimal solution of the multiobjective problem (18). Then  $x$  is a fairly  $B$ -efficient solution of the multiobjective problem (1).

*Proof.* Suppose that  $x$  is not a fairly  $B$ -efficient solution of the multiobjective problem (1). Then there exists a feasible vector  $x'$  such that  $f(x') \prec_{Be} f(x)$ . Due to Corollary 1,  $\bar{\Theta}_B(f(x')) \leq \bar{\Theta}_B(f(x))$ . So

$$\sum_{j \in B_1 \cup B_2 \cup \dots \cup B_k} \theta_j(f(x')) < \sum_{j \in B_1 \cup B_2 \cup \dots \cup B_k} \theta_j(f(x)),$$

for some  $k \in \{1, \dots, r\}$ . Defining

$$s = \min \left\{ k : \sum_{j \in B_1 \cup B_2 \cup \dots \cup B_k} \theta_j(f(x')) < \sum_{j \in B_1 \cup B_2 \cup \dots \cup B_k} \theta_j(f(x)) \right\}.$$

we get

$$\sum_{j \in B_1 \cup B_2 \cup \dots \cup B_k} \theta_j(f(x')) = \sum_{j \in B_1 \cup B_2 \cup \dots \cup B_k} \theta_j(f(x)),$$

for  $k = 1, 2, \dots, s - 1$  and

$$\sum_{j \in B_1 \cup B_2 \cup \dots \cup B_s} \theta_j(f(x')) < \sum_{j \in B_1 \cup B_2 \cup \dots \cup B_s} \theta_j(f(x)).$$

Therefore  $\bar{\Theta}_B(f(x')) <_{lex} \bar{\Theta}_B(f(x))$  contradicting the lexicographic optimality of  $x$ .  $\square$

**Remark 9.** If  $B_j = \{j\}$  for  $j = 1, 2, \dots, m$ , then we have Corollary 3.3 from [3].

Since an efficient solution with equal outcomes is a lexicographic minimax solution. By Theorem 12, such a solution is fairly  $B$ -efficient.

**Corollary 4.** If there exists any efficient solution  $x^0$  of problem (1) with equal outcomes  $f_1(x^0) = f_2(x^0) = \dots = f_m(x^0)$ , then it is a fairly  $B$ -efficient solution.

## 5 Conclusion

In this paper, we introduced a theoretical development of a new concept of solution of a multiobjective optimization problem. The concept of fair  $B$ -efficiency is obtained from fair rational preference relations on a certain cumulative ordered vector. We introduced a new multiobjective optimization problem and by seeking efficient solutions of this new problem, we found fairly  $B$ -efficient solutions of the original problem. Furthermore, we examined some properties of the set of fairly  $B$ -efficient solutions. These include sufficient conditions for existence, connectivity of the fairly  $B$ -efficient set, scalarization methods, and characterizations related to weighting problems.

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## مفهوم جواب $B$ -موثر در مسائل بهینه‌سازی چندهدفه منصف

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**چکیده:** مسئله‌ای که گاهی مواقع در بهینه‌سازی چندهدفه رخ می‌دهد وجود مجموعه‌ای بزرگ از جواب‌های موثر منصف است. از اینرو تصمیم‌گیری مبنی بر انتخاب جواب ترجیح داده شده منحصریفرده مشکل است. مدل‌های در نظر گرفته شده با  $B$ -موثری منصف، کمی از بار تصمیم‌گیری را به وسیله کوچک کردن مجموعه جواب کم می‌کند، چون برای هر مسئله مجموعه جواب‌های  $B$ -موثر منصف زیرمجموعه‌ای از جواب‌های موثر منصف است. در این مقاله، ابتدا بعضی از مفاهیم تئوری و عملی جواب‌های  $B$ -موثر منصف بحث شده است. سپس، بعضی از تکنیک‌های عددی‌سازی برای تولید جواب‌های  $B$ -موثر منصف توسعه داده شده است.

**کلمات کلیدی:** بهینه‌سازی منصف؛ غیر تسلطی؛ منصفانه؛  $B$ -موثری؛ عددی‌سازی.