Iranian Journal of Numerical Analysis and Optimization
Vol. 16, No. 2, 2026, pp 447–496
https://doi.org/10.22067/ijnao.2025.95757.1742
https://ijnao.um.ac.ir/

Research Article

Finite element analysis of cross-diffusion three-species Shigesada–Kawasaki–Teramoto model

G.A. Al-Musawi, H.A. Challoob and A.J. Harfash*

Abstract

In this study, a three-species Shigesada–Kawasaki–Teramoto (SKT) model was studied numerically using the finite element method (FEM). The first step is to introduce the regularized problem to the original problem. After that, some bounds of the solution to the regularized problem are found in various spaces, which is considered the basic step to start the study. The regularized problem is approximated using FEM, and then the existence of the solution to the approximate problem is proven. Also, the approximate solutions' bounds will be found, and their spaces can be determined using these bounds. After that, the convergence of the approximate solutions to the exact solutions will be studied, and these findings will be followed by an examination of how the weak form of the approximate problem converges to the continuous weak form. Thus, the existence of the solutions of the SKT model will be proven. Finally,

*Corresponding author

Received 3 October 2025; revised 18 November 2025; accepted 21 November 2025

Ghufran A. Al-Musawi
Department of Mathematics, College of Education for Pure Sciences, University of Thi-Qar, Thi-Qar, Iraq. e-mail: ghufranalmusawi@utq.edu.iq

Huda A. Challoob
Department of Mathematics, College of Sciences, University of Basrah, Basrah, Iraq. e-mail: huda.challoob@uobasrah.edu.iq

Akil J. Harfash
Department of Mathematics, College of Sciences, University of Basrah, Basrah, Iraq. e-mail: akil.harfash@uobasrah.edu.iq

How to cite this article
Al-Musawi, G.A., Challoob, H.A. and Harfash, A.J., Finite element analysis of cross-diffusion three-species Shigesada–Kawasaki–Teramoto model. *Iran. J. Numer. Anal. Optim.*, 2026; 16(2): 447-496. https://doi.org/10.22067/ijnao.2025.95757.1742

the algorithm for the approximate problem will be presented, and the numerical results will be introduced.

AMS subject classifications (2020): Primary 65N12, 65M60, 35D30.

Keywords: finite element, Shigesada–Kawasaki–Teramoto, convergence, weak solution.

1 Introduction

Partial differential equations (PDEs) play a fundamental role in modeling various physical phenomena in science and engineering. Solving PDEs analytically can be challenging or even impossible for many complex problems. In such cases, numerical methods come to the rescue. However, numerical solutions may not always exist or be unique due to the nature of the PDE and the boundary conditions involved. In such situations, a concept called "weak solutions" provides an alternative way to tackle the problem. A weak solution is a more relaxed notion of a solution that does not necessarily satisfy the PDE pointwise but instead fulfills it in a distributional or integral sense. The concept of weak solutions is particularly useful for dealing with PDEs with irregular data or complex geometries [37].

The existence of weak solutions can be proven by approximating the original PDE with a sequence of simpler problems and showing the convergence of the approximate solutions to a limiting solution that satisfies the PDE weakly. The first step is to convert the strong form of the PDE into its weak form. This typically involves multiplying the PDE by a suitable test function and integrating both sides over the domain. The next step involves choosing a numerical method to approximate the weak formulation. The finite element method (FEM) is commonly used for this purpose. In FEM, the domain is discretized into a finite number of elements, and the unknown functions are approximated using piecewise polynomial basis functions. Then, some bounds are found for the numerical solutions by performing a series of mathematical steps. Next, a sequence of numerical approximations is considered as the mesh size tends to zero. Then, the convergence analysis aims to show that as the mesh size tends to zero, the sequence of numerical approximations converges to a limiting function in a suitable norm. It can then be demonstrated that the limiting function satisfies the weak formulation of the original PDE. Finally, with the convergence and the weak formulation satisfied, it can be concluded that the limiting function is a weak solution to the original PDE. This method of proving the existence of a weak formulation of PDE has been used previously in several studies; see, for example, [10, 9, 8, 11, 13, 12, 27, 28, 32, 30, 33, 31, 2, 3, 4, 5, 6]. Also, the FEM has been used and analyzed recently to solve various problems; see, for example, [29, 15, 38, 40, 7, 36].

A self- and cross-diffusion population model refers to a computational or mathematical framework that incorporates interactions within and between multiple populations. It allows for the examination of both the dynamics within a single population and the interplay between different populations. In such a model, individuals within a population interact with each other, influencing their behavior, characteristics, or attributes. Interactions between different groups of people involve sharing information, resources, and other important factors. This kind of analysis is used in many areas like public health, ecology, sociology, and economics to understand complex systems with multiple interacting groups. By examining how these groups connect and interact, researchers can learn more about their overall behavior, notice emerging trends, and see how these dynamics between populations affect them all.

The Shigesada–Kawasaki–Teramoto (SKT) model is a mathematical tool created to help us understand how different species interact within ecosystems. Developed by Shigesada, Kawasaki, and Teramoto [39], this model uses a series of equations to track how the populations of predators and their prey change over time. One key feature of the model is its assumption that, when there are no predators around, prey populations can grow quickly. Meanwhile, the number of predators relies heavily on how many prey are available. The SKT model also looks at various factors, like how often predators catch their prey, how effectively they convert that prey into their own body mass, and the mortality rates among both groups. It helps researchers explore the relationship between these populations, including situations where populations stabilize or fluctuate dramatically.

This model has been a valuable part of ecological studies, shedding light on topics like predator-prey relationships, population cycles, and how environmental changes impact these dynamics. In this paper, we take a closer look at the SKT model as it applies to three species, incorporating cross-diffusion effects for one of them. The model has the following form: (Q) Find $\{u, v, w\} \in \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$ such that

$$\partial_t u = \Delta(\theta_1 u + \theta_{11} u^2) + (a_1 - b_1 u - c_1 v - d_1 w)u, \quad \text{in } \mathfrak{O}, \quad t > 0, \tag{1}$$

$$\partial_t v = \Delta(\theta_2 v + \theta_{21} uv + \theta_{22} v^2 + \theta_{23} vw) + (a_2 - b_2 u - c_2 v - d_2 w)v, \\ \text{in} \mathfrak{O},\ t > 0, \tag{2}$$

$$\partial_t w = \Delta(\theta_3 w + \theta_{33} w^2) + (a_3 - b_3 u - c_3 v - d_3 w)w, \quad \text{in } \mathfrak{O}, \quad t > 0, \tag{3}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \qquad \text{in } \partial\mathfrak{O}, \quad t > 0, \tag{4}$$

$$u(\mathbf{x},0) = u_0(\mathbf{x}), \quad v(\mathbf{x},0) = v_0(\mathbf{x}), \quad w(\mathbf{x},0) = w_0(\mathbf{x}), \quad \text{in } \mathfrak{O}. \tag{5}$$

In this model, we have a bounded and connected open area $\mathfrak{O}$ shaped like a polygon in three-dimensional space or less. The boundary of this area, referred to as the Lipschitz boundary, is denoted by the symbol $\partial\mathfrak{O}$. Additionally, ν represents the outward unit normal vector at the boundary. This setup helps in understanding the geometric and mathematical properties of the region we're studying. This model, proposed in [39], is used to describe spatial segregation processes in populations of interacting species. In this context, $u = u(\mathbf{x},t), v = v(\mathbf{x},t)$, and $w = w(\mathbf{x},t)$ represent the densities of three competing species. The parameters θ_1, θ_2, and θ_3 represent diffusion coefficients, while θ_{11}, θ_{22}, and θ_{33} correspond to self-diffusion coefficients. The coefficients θ_{21} and θ_{23} describe cross-diffusion, which reflects the population fluxes of one species influenced by the presence of other species. The cross-diffusion coefficient can assume positive, negative, or zero values. Additionally, a_1, a_2, and a_3 denote intrinsic growth rates, b_1, c_2, and d_3 account for intraspecific competition, and c_1, d_1, b_2, d_2, b_3, and c_3 represent inter-specific competition coefficients.

The existence and properties of weak global solutions for cross-diffusion population models are challenging and are a subject of ongoing research in mathematical biology and PDEs. The notion of weak solutions allows for the possibility of certain singularities or discontinuities in the solutions while satisfying the PDEs on a global scale. There is a wide range of research and literature on cross-diffusion population models, and several papers discuss the existence and behavior of weak global solutions; see, for example, [16, 17, 34, 21, 18].

Numerous research studies have delved into the numerical analysis of nonlinear cross-diffusion population models. Specifically, the attraction-repulsion chemotaxis model has been investigated using FEM in [25, 26]. This research provides a fully discrete FEM for the regularized problem and establishes the existence of approximate solutions. Convergence of this FEM, both in spatial and temporal domains, was examined separately. Furthermore, in [23, 24], finite element solutions for the Keller–Segel equations with additional cross-diffusion and logistic source terms were demonstrated to exist. The research also proved the convergence of the approximate solutions in both space and time. Another set of studies (see [32, 30, 33]) focused on a fully discrete FEM for different models: The Keller-Segel model, the two-species chemotaxis system with two chemicals, and the chemotaxis-growth model with indirect attractant production, respectively. Notably, these studies achieved simultaneous convergence of the numerical solutions concerning both time and space by modifying the regularized method. This marks a significant advancement in the study of the convergence of numerical solutions for such problems. The present work aims to develop and generalize the ideas discussed in the aforementioned articles, with a focus on addressing the treatment of self-cross-diffusion terms, which are prevalent in the current system. Moreover, this work investigates the existence of weak solutions for the three-species SKT model with self- and cross-diffusion terms using the FEM.

This paper focuses on the FEM of the three-species SKT system. A significant contribution of this work is the verification of the existence of a weak global solution to this system utilizing the FEM. The following section introduces the key definitions that will be utilized throughout the paper. In Section 3, the regularized problem of (Q) is introduced, and certain bounds for the solutions of the regularized problem are established. Next, in Section 4.1, FEM for spatial discretization and the forward finite difference method for time discretization are employed to approximate the regularized problem. The existence of the solution to the approximate problem is proven in Section 4.2 through the use of the fixed point theorem. To determine the spaces to which the approximate solutions belong, various bounds for the solutions are explored in Section 5. The uniqueness of the approximated solutions is then shown in Section 6. Section 7 deals with the examination of the convergence of the approximate solutions to the exact solutions, along with the convergence of the weak form of the approximate problem (Q). Finally, in Section 8, an algorithm for computing the numerical solutions is presented. This paper expands and develops upon the ideas studied in [10, 9, 8, 11, 13, 12].

2 Notation

Let us begin by presenting the following Sobolev theorem:

$$H^1(\mathfrak{D}) \overset{c}{\hookrightarrow} L^\iota(\mathfrak{D}) \hookrightarrow (H^1(\mathfrak{D}))' \text{ holds for } \iota \in \begin{cases} [1,\infty] & \text{if } d=1, \\ [1,\infty) & \text{if } d=2, \\ [1,6] & \text{if } d=3, \end{cases} \tag{6}$$

where we utilize the notation $\hookrightarrow$ to denote continuous embedding. Furthermore, it has been demonstrated through the Rellich–Kondrachov theorem, as seen in [14, p. 114] and [19, p. 8], that the embedding in (6) is compact. In the case when $d = 3$, the range for ι changes from $\iota \in [1, 6]$ to $\iota \in [1, 6)$. We will denote the compact embedding using the symbol $\overset{c}{\hookrightarrow}$. Also, the following relationship will be necessary:

$$\langle \vartheta, \eta \rangle = (\vartheta, \eta), \quad \text{for all } \vartheta \in L^2(\mathfrak{D}) \text{ and } \eta \in H^1(\mathfrak{D}). \tag{7}$$

We also recall the Gagliardo–Nirenberg inequality [1] as follows: Let $\lambda \in [1,\infty]$, $\mathfrak{r} \geq 1$, and $\vartheta \in W^{\mathfrak{r},\lambda}(\mathfrak{O})$. Then, there exist a constant C and $\gamma = \frac{d}{\mathfrak{r}}\left(\frac{1}{\lambda} - \frac{1}{\iota}\right)$ such that

$$\|\vartheta\|_{0,\iota} \leq C\|\vartheta\|_{0,\lambda}^{1-\gamma}\|\vartheta\|_{\mathfrak{r},\lambda}^{\gamma} \quad \text{holds for } \iota \in \begin{cases} [\lambda,\infty] & \text{if } \mathfrak{r} - \frac{d}{\lambda} > 0, \\ [\lambda,\infty) & \text{if } \mathfrak{r} - \frac{d}{\lambda} = 0, \\ [\lambda, -\frac{d}{\mathfrak{r}-d/\lambda}] & \text{if } \mathfrak{r} - \frac{d}{\lambda} < 0. \end{cases} \tag{8}$$

We require the Sobolev interpolation result [14]: For $\vartheta \in H^1(\mathfrak{O})$, there exist a constant C and $\gamma = \frac{2d(\iota-1)}{\iota(d+2)}$ such that the following inequality holds:

$$\|\vartheta\|_{0,\iota} \leq C\|\vartheta\|_{0,1}^{1-\gamma}\|\vartheta\|_{1}^{\gamma} \quad \text{holds for } \iota \in \begin{cases} [1,\infty] & \text{if } d = 1, \\ [1,\infty) & \text{if } d = 2, \\ [1,6] & \text{if } d = 3. \end{cases} \tag{9}$$

We also invoke the embedding compactness result stated in [35, p. 58]. According to this result, consider three Banach spaces: Y_0, Y_1, and Y_2, where Y_0 and Y_2 are reflexive and $Y_0 \overset{c}{\hookrightarrow} Y_1 \hookrightarrow Y_2$. Also, let

$$\mathcal{H} = \{\vartheta : \vartheta \in L^{\iota_1}(0,T;Y_0), \frac{\partial \vartheta}{\partial t} \in L^{\iota_2}(0,T;Y_2)\},$$

where $T < \infty$ and $1 < \iota_1, \iota_2 < \infty$. Then

$$\mathcal{H} \overset{c}{\hookrightarrow} L^{\iota_1}(0,T;Y_1). \tag{10}$$

The frequently used Hölder's inequality can be stated as follows: For $1 \leq r_1, r_2 \leq \infty$ with $\frac{1}{r_1} + \frac{1}{r_2} = 1$, if $\vartheta_1 \in L^{r_1}(\mathfrak{O})$ and $\vartheta_2 \in L^{r_2}(\mathfrak{O})$, then $\vartheta_1\vartheta_2 \in L^1(\mathfrak{O})$ and

$$\begin{aligned} \|\vartheta_1\vartheta_2\|_{0,1} &= \int_{\mathfrak{O}} |\vartheta_1\vartheta_2| d\mathbf{x} \leq \left(\int_{\mathfrak{O}} |\vartheta_1|^{r_1} d\mathbf{x}\right)^{\frac{1}{r_1}} \left(\int_{\mathfrak{O}} |\vartheta_2|^{r_2} d\mathbf{x}\right)^{\frac{1}{r_2}} \\ &= \|\vartheta_1\|_{0,r_1}\|\vartheta_2\|_{0,r_2}. \end{aligned} \tag{11}$$

Applying the above inequality twice leads to the generalization result in the form:

$$\begin{aligned} \|\vartheta_1\vartheta_2\vartheta_3\|_{0,1} &= \int_{\mathfrak{O}} |\vartheta_1\vartheta_2\vartheta_3| d\mathbf{x} \\ &\leq \left(\int_{\mathfrak{O}} |\vartheta_1|^{r_1} d\mathbf{x}\right)^{\frac{1}{r_1}} \left(\int_{\mathfrak{O}} |\vartheta_2|^{r_2} d\mathbf{x}\right)^{\frac{1}{r_2}} \left(\int_{\mathfrak{O}} |\vartheta_3|^{r_3} d\mathbf{x}\right)^{\frac{1}{r_3}} \\ &= \|\vartheta_1\|_{0,r_1}\|\vartheta_2\|_{0,r_2}\|\vartheta_3\|_{0,r_3}, \end{aligned} \tag{12}$$

for $1 \leq r_1, r_2, r_3 \leq \infty$ such that $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1$.

The following inequality is also needed:

$$(\mathfrak{r}_1 - \mathfrak{r}_2)^2 \geq \frac{\mathfrak{r}_1^2}{2} - \mathfrak{r}_2^2, \qquad \text{for all } \mathfrak{r}_1, \mathfrak{r}_2 \in \mathbb{R}. \tag{13}$$

We need the following essential inequalities, which are valid for all $\chi \in \mathbb{R}$, which can be expressed as follows:

$$(1-\chi) = [1-\chi]_- + [1-\chi]_+ \leq [1-\chi]_+ \leq 1 - [\chi]_-, \tag{14}$$

$$(\chi - 1) = [\chi - 1]_- + [\chi - 1]_+ \geq [\chi - 1]_- \geq [\chi]_- - 1, \tag{15}$$

where $[\chi]_+ = \max\{\chi, 0\}$ and $[\chi]_- = \min\{\chi, 0\}$.

3 A regularized problem

To assure a well-defined entropy inequality in the multidimensional existence proof, an entropy function must be established. This step is critical in our finite element approximation of problem (Q). To create a well-defined entropy inequality, we construct a function $\mathfrak{G} \in C^\infty(\mathbb{R}^{>0})$ with the conditions $\gamma\mathfrak{G}''(\gamma) = 1$ and $\mathfrak{G}(1) = 0$. This function $\mathfrak{G} : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ is defined as follows:

$$\mathfrak{G}(\gamma) := (\ln\gamma - 1)\gamma + 1, \tag{16}$$

and hence,

$$\mathfrak{G}'(\gamma) = \ln\gamma, \qquad \mathfrak{G}''(\gamma) = \frac{1}{\gamma}. \tag{17}$$

By multiplying (1), (2), and (3) by $\frac{(\theta_{21})^2}{\theta_{11}\theta_{22}}\mathfrak{G}'(u)$, $\mathfrak{G}'(v)$, and $\frac{(\theta_{23})^2}{\theta_{22}\theta_{33}}\mathfrak{G}'(w)$, respectively, integrating over $\mathfrak{O}$, considering (4), (5) and the fact that $\lambda\nabla\mathfrak{G}'(\lambda) = \nabla\lambda$, and utilizing Young's inequality, we obtain that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathfrak{O}} \mathfrak{G}(v)d\mathbf{x} + \int_{\mathfrak{O}} \theta_2|\nabla v|^2 v^{-1}d\mathbf{x} + \int_{\mathfrak{O}} \theta_{21}uv^{-1}|\nabla v|^2 d\mathbf{x}\\
&- \int_{\mathfrak{O}} \frac{(\theta_{21})^2}{2\theta_{22}}|\nabla u|^2 d\mathbf{x} + \int_{\mathfrak{O}} \theta_{22}|\nabla v|^2 d\mathbf{x} - \frac{(\theta_{23})^2}{2\theta_{22}}|\nabla w|^2 d\mathbf{x}\\
&+ \int_{\mathfrak{O}} \theta_{23}wv^{-1}|\nabla v|^2 d\mathbf{x} \leq \int_{\mathfrak{O}} a_2 v\mathfrak{G}'(v)d\mathbf{x} - \int_{\mathfrak{O}} b_2uv\mathfrak{G}'(v)d\mathbf{x}\\
&\qquad\qquad - \int_{\mathfrak{O}} c_2v^2\mathfrak{G}'(v)d\mathbf{x} - \int_{\mathfrak{O}} d_2wv\mathfrak{G}'(v)d\mathbf{x},
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathfrak{O}}\frac{(\theta_{23})^2}{\theta_{22}\theta_{33}}\mathfrak{G}(w)d\mathbf{x}+\int_{\mathfrak{O}}\frac{\theta_3(\theta_{23})^2}{\theta_{22}\theta_{33}}w^{-1}|\nabla w|^2d\mathbf{x}+\int_{\mathfrak{O}}\frac{2(\theta_{23})^2}{\theta_{22}}|\nabla w|^2d\mathbf{x}\\
&\quad=\int_{\mathfrak{O}}\frac{a_3(\theta_{23})^2}{\theta_{22}\theta_{33}}w\mathfrak{G}'(w)d\mathbf{x}-\int_{\mathfrak{O}}\frac{b_3(\theta_{23})^2}{\theta_{22}\theta_{33}}uw\mathfrak{G}'(w)d\mathbf{x}\\
&\qquad-\int_{\mathfrak{O}}\frac{c_3(\theta_{23})^2}{\theta_{22}\theta_{33}}vw\mathfrak{G}'(w)d\mathbf{x}-\int_{\mathfrak{O}}\frac{d_3(\theta_{23})^2}{\theta_{22}\theta_{33}}w^2\mathfrak{G}'(w)d\mathbf{x}. \qquad (19)
\end{aligned}$$

Combining the above equations and rearranging the terms results in

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathfrak{O}}\left[\frac{(\theta_{21})^2}{\theta_{11}\theta_{22}}\mathfrak{G}(u)+\mathfrak{G}(v)+\frac{(\theta_{23})^2}{\theta_{22}\theta_{33}}\mathfrak{G}(w)\right]d\mathbf{x}\\
&+\int_{\mathfrak{O}}\frac{\theta_1(\theta_{21})^2}{\theta_{11}\theta_{22}}|\nabla u|^2u^{-1}d\mathbf{x}+\int_{\mathfrak{O}}\frac{3(\theta_{21})^2}{2\theta_{22}}|\nabla u|^2d\mathbf{x}\\
&+\int_{\mathfrak{O}}\theta_2|\nabla v|^2v^{-1}d\mathbf{x}+\int_{\mathfrak{O}}\theta_{21}uv^{-1}|\nabla v|^2d\mathbf{x}+\int_{\mathfrak{O}}\theta_{22}|\nabla v|^2d\mathbf{x}\\
&+\int_{\mathfrak{O}}\theta_{23}wv^{-1}|\nabla v|^2d\mathbf{x}+\int_{\mathfrak{O}}\frac{\theta_3(\theta_{23})^2}{\theta_{22}\theta_{33}}w^{-1}|\nabla w|^2d\mathbf{x}+\int_{\mathfrak{O}}\frac{3(\theta_{23})^2}{2\theta_{22}}|\nabla w|^2d\mathbf{x}\\
&\quad\leq\int_{\mathfrak{O}}\frac{a_1(\theta_{21})^2}{\theta_{11}\theta_{22}}u\mathfrak{G}'(u)d\mathbf{x}-\int_{\mathfrak{O}}\frac{b_1(\theta_{21})^2}{\theta_{11}\theta_{22}}u\mathfrak{G}'(u)d\mathbf{x}\\
&\qquad-\int_{\mathfrak{O}}\frac{c_1(\theta_{21})^2}{\theta_{11}\theta_{22}}vu\mathfrak{G}'(u)d\mathbf{x}-\int_{\mathfrak{O}}\frac{d_1(\theta_{21})^2}{\theta_{11}\theta_{22}}wu\mathfrak{G}'(u)d\mathbf{x}+\int_{\mathfrak{O}}a_2v\mathfrak{G}'(v)d\mathbf{x}\\
&\qquad-\int_{\mathfrak{O}}b_2uv\mathfrak{G}'(v)d\mathbf{x}-\int_{\mathfrak{O}}c_2v^2\mathfrak{G}'(v)d\mathbf{x}-\int_{\mathfrak{O}}d_2wv\mathfrak{G}'(v)d\mathbf{x}\\
&\qquad+\int_{\mathfrak{O}}\frac{a_3(\theta_{23})^2}{\theta_{22}\theta_{33}}w\mathfrak{G}'(w)d\mathbf{x}-\int_{\mathfrak{O}}\frac{b_3(\theta_{23})^2}{\theta_{22}\theta_{33}}uw\mathfrak{G}'(w)d\mathbf{x}\\
&\qquad-\int_{\mathfrak{O}}\frac{c_3(\theta_{23})^2}{\theta_{22}\theta_{33}}vw\mathfrak{G}'(w)d\mathbf{x}-\int_{\mathfrak{O}}\frac{d_3(\theta_{23})^2}{\theta_{22}\theta_{33}}w^2\mathfrak{G}'(w)d\mathbf{x}. \qquad (20)
\end{aligned}$$

The bound (20) is strictly formal since we are unable to predict a priori that $u(\mathbf{x},t)\in\mathbb{R}^{>0}$ for $\mathfrak{G}$ to be well-defined and the fourth term on the left to be positive. A regularization approach is required to properly establish this constraint and build a numerical approximation for problem (Q). We thus use an alternate regularization approach, where the function $\mathfrak{G}\in C^2(\mathbb{R}^{>0})$ is replaced by a regularized function $\mathfrak{G}_\varrho:\mathbb{R}\to\mathbb{R}^{\geq 0}$, where $\varrho\in(0,e^{-1})$, such that

$$\mathfrak{G}_\varrho(\gamma):=\begin{cases}\frac{\gamma^2-\varrho^2}{2\varrho}+(\ln\varrho-1)\gamma+1, & \gamma\leq\varrho,\\ (\ln\gamma-1)\gamma+1, & \varrho\leq\gamma\leq\varrho^{-1},\\ \frac{\gamma^2-\varrho^2}{2\varrho^{-1}}+(\ln\varrho^{-1}-1)\gamma+1, & \varrho^{-1}\leq\gamma.\end{cases} \qquad (21)$$

Therefore, $\mathfrak{G}_\varrho\in C^{2,1}(\mathbb{R})$, where the first two derivatives of $\mathfrak{G}_\varrho$ are provided as follows:

$$\mathfrak{G}'_\varrho(\gamma) := \begin{cases} \varrho^{-1}\gamma + \ln\varrho - 1, & \gamma \le \varrho, \\ \ln\gamma, & \varrho \le \gamma \le \varrho^{-1}, \\ \varrho\gamma + \ln\varrho^{-1} - 1, & \varrho^{-1} \le \gamma, \end{cases} \tag{22}$$

and

$$\mathfrak{G}''_\varrho(\gamma) := \begin{cases} \frac{1}{\varrho}, & \gamma \le \varrho, \\ \frac{1}{\gamma}, & \varrho \le \gamma \le \varrho^{-1}, \\ \varrho, & \varrho^{-1} \le \gamma, \end{cases} \tag{23}$$

respectively. The function $\xi_\varrho : \mathbb{R} \to [\varrho, \varrho^{-1}]$ is presented in the following form:

$$\xi_\varrho(\gamma) := [\mathfrak{G}''_\varrho(\gamma)]^{-1} := \begin{cases} \varrho, & \gamma \le \varrho, \\ \gamma, & \varrho \le \gamma \le \varrho^{-1}, \\ \frac{1}{\varrho}, & \varrho^{-1} \le \gamma. \end{cases} \tag{24}$$

In addition, we take into account the following features that are associated with the function $\mathfrak{G}_\varrho(\gamma)$ and $\xi_\varrho(\gamma)$:

- for all $\varrho \in (0, e^{-1})$ and for all $\gamma \le 0$, it follows that

$$\mathfrak{G}_\varrho(\gamma) \ge \frac{\gamma^2}{2\varrho}. \tag{25}$$

- for all $\varrho \in (0, e^{-1})$ and for all $\gamma \ge 0$, we have that

$$\mathfrak{G}_\varrho(\gamma) \ge \frac{\varrho\gamma^2}{2} - 1. \tag{26}$$

- Additionally, it is straightforward to show that for all $\varrho \in (0, e^{-1})$ and for all $\gamma \in \mathbb{R}$, we have the following results:

$$\max\{\xi_\varrho(\gamma), \gamma\mathfrak{G}'_\varrho(\gamma)\} \le 2\mathfrak{G}_\varrho(\gamma) + 1, \tag{27}$$

$$\xi_\varrho(\gamma)\,\mathfrak{G}'_\varrho(\gamma) \ge \gamma - 1. \tag{28}$$

Taylor's theorem implies that for any function $\mathfrak{G} \in C^2(\mathbb{R})$, we have the following result:

$$(\gamma - \hat{r})\mathfrak{G}'(\gamma) = \mathfrak{G}(\gamma) - \mathfrak{G}(\hat{r}) + \frac{(\gamma - \hat{r})^2}{2}\mathfrak{G}''(\upsilon), \tag{29}$$

for some υ between γ and $\hat{r}$. Here is the analogical regularized version of problem (Q) for any $\varrho \in (0, e^{-1})$:

(Q_ϱ) Find $\{u_\varrho, v_\varrho, w_\varrho\} \in \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$ such that

$$\begin{aligned}\partial_t u_\varrho =& \nabla \cdot (\theta_1 \nabla u_\varrho + 2\theta_{11}\xi_\varrho(u_\varrho)\nabla u_\varrho) + a_1 u_\varrho - b_1 \xi_\varrho(u_\varrho)\xi_\varrho(u_\varrho) \\ &- c_1 \xi_\varrho(v_\varrho)\xi_\varrho(u_\varrho) - d_1 \xi_\varrho(w_\varrho)\xi_\varrho(u_\varrho), \quad x \in \mathfrak{O}, \quad t > 0,\end{aligned} \tag{30}$$

$$\begin{aligned}\partial_t v_\varrho =& \nabla \cdot (\theta_2 \nabla v_\varrho + \theta_{21}\xi_\varrho(v_\varrho)\nabla u_\varrho + \theta_{21}\xi_\varrho(u_\varrho)\nabla v_\varrho \\ &+ 2\theta_{22}\xi_\varrho(v_\varrho)\nabla v_\varrho + \theta_{23}\xi_\varrho(v_\varrho)\nabla w_\varrho + \theta_{23}\xi_\varrho(w_\varrho)\nabla v_\varrho) \\ &+ (a_2 v_\varrho - b_2\xi_\varrho(u_\varrho)\xi_\varrho(v_\varrho) - c_2\xi_\varrho(v_\varrho)\xi_\varrho(v_\varrho) \\ &- d_2\xi_\varrho(w_\varrho)\xi_\varrho(v_\varrho), \quad x \in \mathfrak{O}, \; t > 0,\end{aligned} \tag{31}$$

$$\begin{aligned}\partial_t w_\varrho =& \nabla \cdot (\theta_3 \nabla w_\varrho + 2\theta_{33}\xi_\varrho(w_\varrho)\nabla w_\varrho) + (a_3 w_\varrho - b_3\xi_\varrho(u_\varrho)\xi_\varrho(w_\varrho) \\ &- c_3\xi_\varrho(v_\varrho)\xi_\varrho(w_\varrho) - d_3\xi_\varrho(w_\varrho)\xi_\varrho(w_\varrho)), \quad x \in \mathfrak{O}, \quad t > 0,\end{aligned} \tag{32}$$

$$\nabla u_\varrho \cdot \nu = \nabla v_\varrho \cdot \nu = \nabla w_\varrho \cdot \nu = 0 \qquad \text{on} \;\; \partial\mathfrak{O} \times [0,T], \tag{33}$$

$$u_\varrho(\cdot,0) = u_{\varrho_0}, \quad v_\varrho(\cdot,0) = v_{\varrho_0}, \quad w_\varrho(\cdot,0) = w_{\varrho_0}, \quad \text{in} \quad \mathfrak{O}. \tag{34}$$

Lemma 1. Given the initial conditions $u^0(\mathbf{x}), v^0(\mathbf{x}), w^0(\mathbf{x})$, where $\mathfrak{G}_\varrho(u^0(\mathbf{x})) \in L^1(\mathfrak{O})$ and $u^0(\mathbf{x}), v^0(\mathbf{x}) \in L^2(\mathfrak{O})$, and $w^0(\mathbf{x}) \in L^2(\mathfrak{O})$, we can establish the existence of a positive constant C (independent of the regularization parameter ϱ) such that the solution of (Q_ϱ) satisfies the following inequality:

$$\begin{aligned}&\sup_{0\leq t\leq T}\int_{\mathfrak{O}}\left[u_\varrho\frac{(\theta_{21})^2}{\theta_{11}\theta_{22}}\mathfrak{G}_\varrho(u_\varrho) + v_\varrho\mathfrak{G}_\varrho(v_\varrho) + w_\varrho\frac{(\theta_{23})^2}{\theta_{22}\theta_{33}}\mathfrak{G}_\varrho(w_\varrho)\right]d\mathbf{x} \\ &+ \int_0^T\int_{\mathfrak{O}}\frac{\theta_1(\theta_{21})^2}{\theta_{11}\theta_{22}}\frac{|\nabla u_\varrho|^2}{\xi_\varrho(u_\varrho)}d\mathbf{x}dt + \int_0^T\int_{\mathfrak{O}}\frac{3(\theta_{21})^2}{2\theta_{22}}|\nabla u_\varrho|^2 d\mathbf{x}dt \\ &+ \int_0^T\int_{\mathfrak{O}}\theta_2\frac{|\nabla v_\varrho|^2}{\xi_\varrho(v_\varrho)}d\mathbf{x}dt + \int_0^T\int_{\mathfrak{O}}\theta_{21}\xi_\varrho(u_\varrho)\frac{|\nabla v_\varrho|^2}{\xi_\varrho(v_\varrho)}d\mathbf{x}dt \\ &+ \int_0^T\int_{\mathfrak{O}}\theta_{22}|\nabla v_\varrho|^2 d\mathbf{x}dt + \int_0^T\int_{\mathfrak{O}}\theta_{23}\xi_\varrho(w_\varrho)\frac{|\nabla v_\varrho|^2}{\xi_\varrho(v_\varrho)}d\mathbf{x}dt \\ &+ \int_0^T\int_{\mathfrak{O}}\frac{\theta_3(\theta_{23})^2}{\theta_{22}\theta_{33}}\frac{|\nabla w_\varrho|^2}{\xi_\varrho(w_\varrho)}d\mathbf{x}dt + \int_0^T\int_{\mathfrak{O}}\frac{3(\theta_{23})^2}{2\theta_{22}}|\nabla w_\varrho|^2 d\mathbf{x}dt \leq C.\end{aligned} \tag{35}$$

Furthermore, we have that

$$\sup_{0\leq t\leq T}\int_{\mathfrak{O}}|[u_\varrho]_-|^2 d\mathbf{x} \leq C\varrho. \tag{36}$$

Proof. After multiplying (30) by $\mathfrak{G}'_\varrho(u_\varrho)$, integrating by parts over $\mathfrak{O}$, and rearranging the terms, the resulting expression is as follows:

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathfrak{O}} \mathfrak{G}_\varrho(u_\varrho)d\mathbf{x} + \int_{\mathfrak{O}} \theta_1 \frac{|\nabla u_\varrho|^2}{\xi_\varrho(u_\varrho)}d\mathbf{x} + \int_{\mathfrak{O}} 2\theta_{11}|\nabla u_\varrho|^2 d\mathbf{x}\\
&= \int_{\mathfrak{O}} a_1 u_\varrho \mathfrak{G}'_\varrho(u_\varrho)d\mathbf{x} - \int_{\mathfrak{O}} b_1 \xi_\varrho(u_\varrho)\xi_\varrho(u_\varrho)\mathfrak{G}'_\varrho(u_\varrho)d\mathbf{x}\\
&\quad - \int_{\mathfrak{O}} c_1 \xi_\varrho(v_\varrho)\xi_\varrho(u_\varrho)\mathfrak{G}'_\varrho(u_\varrho)d\mathbf{x} - \int_{\mathfrak{O}} d_1 \xi_\varrho(w_\varrho)\xi_\varrho(u_\varrho)\mathfrak{G}'_\varrho(u_\varrho)d\mathbf{x}, \qquad (37)
\end{aligned}$$

where we have used the fact that

$$\xi_\varrho(\lambda_\varrho)\nabla\mathfrak{G}'_\varrho(\lambda_\varrho) = \nabla\lambda_\varrho. \qquad (38)$$

By observing (24), (27), (28), (14), the Young's inequality, and (25), we can deduce that

$$\begin{aligned}
&\int_{\mathfrak{O}} a_1 u_\varrho \mathfrak{G}'_\varrho(u_\varrho)d\mathbf{x} - \int_{\mathfrak{O}} b_1 \xi_\varrho(u_\varrho)\xi_\varrho(u_\varrho)\mathfrak{G}'_\varrho(u_\varrho)d\mathbf{x}\\
&- \int_{\mathfrak{O}} c_1 \xi_\varrho(v_\varrho)\xi_\varrho(u_\varrho)\mathfrak{G}'_\varrho(u_\varrho)d\mathbf{x} - \int_{\mathfrak{O}} d_1 \xi_\varrho(w_\varrho)\xi_\varrho(u_\varrho)\mathfrak{G}'_\varrho(u_\varrho)d\mathbf{x}\\
&\quad \le \int_{\mathfrak{O}} [(2a_1 + 4b_1 + c_1 + d_1)\mathfrak{G}_\varrho(u_\varrho) + C + 3c_1\mathfrak{G}_\varrho(v_\varrho) + 3d_1\mathfrak{G}_\varrho(w_\varrho)]d\mathbf{x}.
\end{aligned} \qquad (39)$$

By substituting (39) into (37), we obtain that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathfrak{O}} \mathfrak{G}_\varrho(u_\varrho)d\mathbf{x} + \int_{\mathfrak{O}} \theta_1 \frac{|\nabla u_\varrho|^2}{\xi_\varrho(u_\varrho)}d\mathbf{x} + \int_{\mathfrak{O}} 2\theta_{11}|\nabla u_\varrho|^2 d\mathbf{x}\\
&\le \int_{\mathfrak{O}} [(2a_1 + 4b_1 + c_1 + d_1)\mathfrak{G}_\varrho(u_\varrho) + C + 3c_1\mathfrak{G}_\varrho(v_\varrho) + 3d_1\mathfrak{G}_\varrho(w_\varrho)]d\mathbf{x}. \qquad (40)
\end{aligned}$$

Similarly, by multiplying (31) by $\mathfrak{G}'_\varrho(v_\varrho)$ and (32) by $\mathfrak{G}'_\varrho(w_\varrho)$, integrating over $\mathfrak{O}$, and applying integration by parts, along with the utilization of (38) and Young's inequality, we can derive the following results:

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathfrak{O}} \mathfrak{G}_\varrho(v_\varrho)d\mathbf{x} + \int_{\mathfrak{O}} \theta_2 \frac{|\nabla v_\varrho|^2}{\xi_\varrho(v_\varrho)}d\mathbf{x} - \int_{\mathfrak{O}} \frac{(\theta_{21})^2}{2\theta_{22}}|\nabla u_\varrho|^2 d\mathbf{x} + \int_{\mathfrak{O}} \theta_{21}\xi_\varrho(u_\varrho)\frac{|\nabla v_\varrho|^2}{\xi_\varrho(v_\varrho)}d\mathbf{x}\\
&+ \int_{\mathfrak{O}} \theta_{22}|\nabla v_\varrho|^2 d\mathbf{x} - \int_{\mathfrak{O}} \frac{(\theta_{23})^2}{2\theta_{22}}|\nabla w_\varrho|^2 d\mathbf{x} + \int_{\mathfrak{O}} \theta_{23}\xi_\varrho(w_\varrho)\frac{|\nabla v_\varrho|^2}{\xi_\varrho(v_\varrho)}d\mathbf{x}\\
&\quad \le \int_{\mathfrak{O}} [(2a_2 + 4c_2 + b_2 + d_2)\mathfrak{G}_\varrho(v_\varrho) + C + 3b_2\mathfrak{G}_\varrho(u_\varrho) + 3d_2\mathfrak{G}_\varrho(w_\varrho)]d\mathbf{x}, \qquad (41)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathfrak{O}}\frac{(\theta_{23})^2}{\theta_{22}\theta_{33}}\mathfrak{G}_\varrho(w_\varrho)d\mathbf{x}+\int_{\mathfrak{O}}\theta_3\frac{|\nabla w_\varrho|^2}{\xi_\varrho(w_\varrho)}d\mathbf{x}+\int_{\mathfrak{O}}2\theta_{33}|\nabla w_\varrho|^2d\mathbf{x}\\
&\leq\int_{\mathfrak{O}}[(2a_3+4d_3+c_3+b_3)\mathfrak{G}_\varrho(w_\varrho)+C+3c_3\mathfrak{G}_\varrho(v_\varrho)+3b_3\mathfrak{G}_\varrho(u_\varrho)]d\mathbf{x}. \qquad (42)
\end{aligned}$$

By multiplying (40) and (43) by $\frac{(\theta_{21})^2}{\theta_{11}\theta_{22}}$ and $\frac{(\theta_{23})^2}{\theta_{22}\theta_{33}}$ respectively, and then summing the results with (41) and rearranging the terms, we obtain that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathfrak{O}}\left[u_\varrho\frac{(\theta_{21})^2}{\theta_{11}\theta_{22}}\mathfrak{G}_\varrho(u_\varrho)+v_\varrho\mathfrak{G}_\varrho(v_\varrho)+w_\varrho\frac{(\theta_{23})^2}{\theta_{22}\theta_{33}}\mathfrak{G}_\varrho(w_\varrho)\right]d\mathbf{x}\\
&+\int_{\mathfrak{O}}\frac{\theta_1(\theta_{21})^2}{\theta_{11}\theta_{22}}\frac{|\nabla u_\varrho|^2}{\xi_\varrho(u_\varrho)}d\mathbf{x}+\int_{\mathfrak{O}}\frac{3(\theta_{21})^2}{2\theta_{22}}|\nabla u_\varrho|^2d\mathbf{x}+\int_{\mathfrak{O}}\theta_2\frac{|\nabla v_\varrho|^2}{\xi_\varrho(v_\varrho)}d\mathbf{x}\\
&+\int_{\mathfrak{O}}\theta_{21}\xi_\varrho(u_\varrho)\frac{|\nabla v_\varrho|^2}{\xi_\varrho(v_\varrho)}d\mathbf{x}+\int_{\mathfrak{O}}\theta_{22}|\nabla v_\varrho|^2d\mathbf{x}+\int_{\mathfrak{O}}\theta_{23}\xi_\varrho(w_\varrho)\frac{|\nabla v_\varrho|^2}{\xi_\varrho(v_\varrho)}d\mathbf{x}\\
&+\int_{\mathfrak{O}}\frac{\theta_3(\theta_{23})^2}{\theta_{22}\theta_{33}}\frac{|\nabla w_\varrho|^2}{\xi_\varrho(w_\varrho)}d\mathbf{x}+\int_{\mathfrak{O}}\frac{3(\theta_{23})^2}{2\theta_{22}}|\nabla w_\varrho|^2d\mathbf{x}\\
&\leq C\left(\int_{\mathfrak{O}}\left\{\frac{(\theta_{21})^2}{\theta_{11}\theta_{22}}\mathfrak{G}_\varrho(u_\varrho)+\mathfrak{G}_\varrho(v_\varrho)+\frac{(\theta_{23})^2}{\theta_{22}\theta_{33}}\mathfrak{G}_\varrho(w_\varrho)\right\}d\mathbf{x}+1\right). \qquad (43)
\end{aligned}$$

By utilizing the Grönwall lemma on (43), we can obtain the desired outcome mentioned in (35). Moreover, by considering (35) and (25), we can directly derive the result stated in (36). $\square$

4 A finite element method

4.1 Notation

(A) Consider $\mathfrak{O}$ as a polygonal domain in $\mathbb{R}^2$ or a polyhedral domain in $\mathbb{R}^3$. Let $\mathcal{T}^h$ be a quasi-uniform and right-angled partition of $\mathfrak{O}$ into open simplices $\tilde{\tau}$ with diameters $h_{\tilde{\tau}}$ and h denoting the maximum diameter among the simplices. We have $\overline{\mathfrak{O}}=\cup_{\tilde{\tau}\in\mathcal{T}^h}\overline{\tilde{\tau}}$. Let $(0,T)$ be partitioned into time steps δt_n, where $0=t_0<t_1<\cdots<t_{N-1}<t_N=T$, and δt represents the maximum time step size. Assume X^h as a subset of $H^1(\mathfrak{O})$, the finite element space consisting of continuous piecewise linear functions.

$$X^h:=\{\omega\in C(\overline{\mathfrak{O}}):\omega|_{\tilde{\tau}}\text{ is linear,}\quad\text{for all }\tilde{\tau}\in\mathcal{T}^h\}.$$

We introduce the set $\{\eta_j\}_{j=0}^{J}$ as the standard basis functions for X^h, where X^h is a finite element space. These basis functions satisfy the property $\eta_j(p_i) = \delta_{ij}$ for $i, j = 0, \ldots, J,$. Here, $\mathcal{N}^h := \{p_j\}_{j=0}^{J}$ denotes the set of nodes of the partitioning $\mathcal{T}^h$. Moreover, we present that

$$\begin{aligned} X^h_{\geq 0} &:= \{\omega \in X^h : \omega(p_j) \geq 0,\ j = 0, \ldots, J\} \\ &\subset H^1_{\geq 0} := \{\Sigma \in H^1(\mathfrak{O}) : \Sigma \geq 0\ a.e. \in \mathfrak{O}\}. \end{aligned}$$

Define $\pi^h : C(\overline{\mathfrak{O}}) \to X^h$ as the Lagrange interpolation operator, also known as the piecewise linear interpolant, which has the form

$$\pi^h \Sigma(p_j) := \Sigma(p_j), \quad \text{for } j = 0, \ldots, J.$$

We can define a discrete L^2 inner (semi-inner) product on $X^h(C(\overline{\mathfrak{O}}))$ as follows:

$$(\vartheta_1, \vartheta_2)^h := \int_{\mathfrak{O}} \pi^h(\vartheta_1(x)\vartheta_2(x))d\mathbf{x} = \sum_{j=0}^{J} \widehat{M}_{jj}\ \vartheta_1(p_j)\,\vartheta_2(p_j), \tag{44}$$

where $\widehat{M}_{jj} = (\eta_j, \eta_j)^h = (1, \eta_j) > 0$. Taking (44) into consideration, it follows that

$$(\vartheta_1, \vartheta_2)^h = (\pi^h\vartheta_1, \vartheta_2)^h = (\pi^h\vartheta_1, \pi^h\vartheta_2)^h, \qquad \text{for all } \vartheta_1, \vartheta_2 \in C(\overline{\mathfrak{O}}). \tag{45}$$

Here are some well-known results regarding the finite element space X^h and its induced discrete semi-norm on $C(\overline{\mathfrak{O}})$ and norm on X^h. The discrete semi-norm, denoted as $|\cdot|_h$, is defined as $|\cdot|_h := [(\cdot,\cdot)^h]^{1/2}$. It is widely recognized that $|\cdot|_h$ is equivalent to the norm $|\cdot|_0$, defined as $|\cdot|_0 := [(\cdot,\cdot)]^{1/2}$, in the following manner:

$$\|\omega\|_0^2 \leq |\omega|_h^2 \leq (d+2)\|\omega\|_0^2, \qquad \text{for all } \omega \in X^h. \tag{46}$$

As a consequence of the quasi-uniform condition (see [19, Theorem 3.2.6]), we include extra inverse inequalities for future reference.

$$|\vartheta|_{1,\varsigma,\text{"}\tilde{\tau}\text{"}} \leq C h_{\text{"}\tilde{\tau}\text{"}}^{-1} |\vartheta|_{0,\varsigma,\text{"}\tilde{\tau}\text{"}}, \quad 1 \leq \varsigma \leq \infty, \tag{47}$$

$$|\vartheta|_{k,\varsigma_1,\text{"}\tilde{\tau}\text{"}} \leq C h_{\text{"}\tilde{\tau}\text{"}}^{-d(\frac{1}{\varsigma_2} - \frac{1}{\varsigma_1})} |\vartheta|_{k,\varsigma_2,\text{"}\tilde{\tau}\text{"}}, \quad 1 \leq \varsigma_2 \leq \varsigma_1 \leq \infty,\ k \in \{0, 1\}. \tag{48}$$

The interpolation results that we will need to use are as follows: For all $\vartheta \in W^{1,\gamma}(\mathfrak{O}),\ \gamma \in [2, \infty]$ if $d = 1$ and $\gamma \in (d, \infty]$ if $d = 2$ or 3:

$$|(I-\pi^h)\vartheta|_{k,\gamma} \le Ch^{1-k}|\vartheta|_{1,\gamma}, \quad \gamma \in \{0,1\}, \tag{49}$$

$$\lim_{h\to 0} |(I-\pi^h)\vartheta|_{1,\gamma} = 0. \tag{50}$$

For all $\Im_1, \Im_2 \in X^h$, we will also require the following result (e.g., [20]):

$$|(\Im_1,\Im_2)-(\Im_1,\Im_2)^h| \le Ch^{1+m}|\Im_1|_{m,n_1}\,|\Im_2|_{1,n_2}, \tag{51}$$

for $m \in \{0,1\}$ and $1 \le n_1,\ n_2 \le \infty$ with $\frac{1}{n_1}+\frac{1}{n_2}=1$.

The generalized version of (51) for future use is provided below. For any $\Im_1,, \Im_2,, \Im_3 \in X^h$, the following result holds:

$$|(\Im_1,\Im_2,\Im_3)-(\Im_1,\Im_2,\Im_3)^h| \le Ch^2|\Im_1|_{1,n_1}\,|\Im_2|_{1,n_2}\,|\Im_3|_{1,n_3}, \tag{52}$$

where $1 \le n_1,\ n_2,\ n_3 \le \infty$ with $\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}=1$.

We introduce, for any $\varrho \in (0,e^{-1}),\ \Xi_\varrho : X^h \to [L^\infty(\mathfrak{O})]^{d\times d}$ such that for all $\omega \in X^h$ and a.e in $\mathfrak{O}$, we have that

$$\Xi_\varrho(\omega) \text{ is symmetric and positive definite,} \tag{53}$$

$$\Xi_\varrho(\omega)\nabla\pi^h[\mathfrak{G}'_\varrho(\omega)] = \nabla\omega. \tag{54}$$

In the simple case of $d=1$, we construct Ξ_ϱ. Let $\omega \in X^h$ and $\tilde{\tau} \in \mathcal{T}^h$, with vertices p_j and p_k. We define the following:

$$\Xi_\varrho(\omega)|_{\tilde{\tau}} := \begin{cases} \frac{\omega(p_k)-\omega(p_j)}{\mathfrak{G}'_\varrho(\omega(p_k))-\mathfrak{G}'_\varrho(\omega(p_j))} = \frac{1}{\mathfrak{G}''_\varrho(\omega(\hat{\zeta}))}, \text{ for}\hat{\zeta} \in \tilde{\tau} \text{ if } \omega(p_k) \neq \omega(p_j), \\ \frac{1}{\mathfrak{G}''_\varrho(\omega(p_k))} \qquad \text{if } \omega(p_k) = \omega(p_j). \end{cases} \tag{55}$$

Since $\mathfrak{G}''_\varepsilon(s) > 0$ and $\sum_{j=0}^{J}\nabla\eta_j = 0$, it can be easily seen that the piecewise constant function Ξ_ε satisfies the conditions (53) and (54). Following [22], we extend the above construction to $d=2$ or 3. Let $\{e_i\}_{i=1}^d$ be the orthonormal vectors in $\mathbb{R}^d$, such that the j-th component of e_i is $\delta_{ij}, i,j = 1 \to d$. Given non-zero constants $\theta_i, i = 1 \to d$, let $\widehat{\tilde{\tau}}(\{\theta_i\}_{i=1}^d)$ be a reference simplex in $\mathbb{R}^d$ with vertices $\{\widehat{p}_i\}_{i=1}^d$, where $\widehat{p}_0$ is the origin and $\widehat{p}_i = \widehat{p}_{i-1} + \theta_i e_i, i = 1 \to d$. Given a $\tilde{\tau} \in \mathcal{T}^h$ with vertices $\{p_{j_i}\}_{i=0}^d$, such that p_{j_0} is not a right-angled vertex, then there exists a rotation/reflection matrix $R_{\tilde{\tau}}$ and non-zero constants $\{\widehat{p}_i\}_{i=1}^d$ such that the mapping $R_{\tilde{\tau}} : \widehat{x} \in \mathbb{R}^d \to p_{j_0} + R_{\tilde{\tau}}\widehat{x} \in \mathbb{R}^d$ maps the vertex $\widehat{p}_i$ to p_{j_i}, and hence $\widehat{\tilde{\tau}} \equiv \widehat{\tilde{\tau}}(\{\theta_i\}_{i=1}^d)$ to $\tilde{\tau}$. For all $\tilde{\tau} \in \mathcal{T}^h$ and $\omega \in V^h$, we set

$$\Xi_\varrho(\omega)|_{\tilde{\tau}} := R_{\tilde{\tau}}\widehat{\Xi_\varrho}(\widehat{\omega})|_{\widehat{\tilde{\tau}}}R_{\tilde{\tau}}^T, \tag{56}$$

where $\widehat{\Xi_\varrho}(\widehat{\omega})|_{\widehat{\tilde{\tau}}}$ is the $d \times d$ diagonal matrix with diagonal entries, $k = 1,\ldots,d$,

$$[\widehat{\Xi_\varrho}(\widehat{\omega})|_{\widehat{\tilde{\tau}}}]_{kk} := \begin{cases} \frac{\widehat{\omega}(\widehat{p}_k)-\widehat{\omega}(\widehat{p}_j)}{\mathfrak{G}'_\varrho(\widehat{\omega}(\widehat{p}_k))-\mathfrak{G}'_\varrho(\widehat{\omega}(\widehat{p}_j))} = & \frac{\omega(p_{j_k})-\omega(p_{j_0})}{\mathfrak{G}'_\varrho(\omega(p_{j_k}))-\mathfrak{G}'_\varrho(\omega(p_{j_0}))} \\ = \frac{1}{\mathfrak{G}''_\varrho(\omega(\hat{\zeta}))} & \text{for some } \hat{\zeta} \text{ between } p_{j_k} \text{ and } p_{j_0} \\ & \text{if } \omega(p_{j_k}) \neq \omega(p_{j_0}), \\ \frac{1}{\mathfrak{G}''_\varrho(\widehat{\omega}(\widehat{p}_0))} = \frac{1}{\mathfrak{G}''_\varrho(\omega(p_{j_0}))} & \text{if} \omega(p_{j_k}) = \omega(p_{j_0}). \end{cases} \tag{57}$$

For any given $\varrho \in (0, e^{-1})$, we consider the following fully discrete FEM of (Q_ϱ):
$(\mathrm{Q}_\varrho^{h,\delta t})$ For $n \geq 1$ find $\{\widehat{\Phi}_\varrho^n, \widehat{\Psi}_\varrho^n, \widehat{\Theta}_\varrho^n\} \in [X^h]^3$ such that for all $\omega \in X^h$,

$$\begin{aligned} &\Big(\frac{\widehat{\Phi}_\varrho^n - \widehat{\Phi}_\varrho^{n-1}}{\delta t}, \omega\Big)^h + (\theta_1 \nabla \widehat{\Phi}_\varrho^n + 2\theta_{11} \Xi_\varrho(\widehat{\Phi}_\varrho^n) \nabla \widehat{\Phi}_\varrho^n, \nabla \omega) \\ &= (a_1 \widehat{\Phi}_\varrho^n - b_1 \xi_\varrho(\widehat{\Phi}_\varrho^{n-1}) \xi_\varrho(\widehat{\Phi}_\varrho^n) - c_1 \xi_\varrho(\widehat{\Psi}_\varrho^{n-1}) \xi_\varrho(\widehat{\Phi}_\varrho^n) \\ &\quad - d_1 \xi_\varrho(\widehat{\Theta}_\varrho^{n-1}) \xi_\varrho(\widehat{\Phi}_\varrho^n), \omega)^h, \end{aligned} \tag{58}$$

$$\begin{aligned} &\Big(\frac{\widehat{\Psi}_\varrho^n - \widehat{\Psi}_\varrho^{n-1}}{\delta t}, \omega\Big)^h + (\theta_2 \nabla \widehat{\Psi}_\varrho^n + \theta_{21} \Xi_\varrho(\widehat{\Psi}_\varrho^n) \nabla \widehat{\Phi}_\varrho^n + \theta_{21} \Xi_\varrho(\widehat{\Phi}_\varrho^n) \nabla \widehat{\Psi}_\varrho^n \\ &+ 2\theta_{22} \Xi_\varrho(\widehat{\Psi}_\varrho^n) \nabla \widehat{\Psi}_\varrho^n + \theta_{23} \Xi_\varrho(\widehat{\Psi}_\varrho^n) \nabla \widehat{\Theta}_\varrho^n + \theta_{23} \Xi_\varrho(\widehat{\Theta}_\varrho^n) \nabla \widehat{\Psi}_\varrho^n, \nabla \omega) \\ &= (a_2 \widehat{\Psi}_\varrho^n - b_2 \xi_\varrho(\widehat{\Phi}_\varrho^{n-1}) \xi_\varrho(\widehat{\Psi}_\varrho^n) - c_2 \xi_\varrho(\widehat{\Psi}_\varrho^{n-1}) \xi_\varrho(\widehat{\Psi}_\varrho^n) \\ &\quad - d_2 \xi_\varrho(\widehat{\Theta}_\varrho^{n-1}) \xi_\varrho(\widehat{\Psi}_\varrho^n), \omega)^h, \end{aligned} \tag{59}$$

and

$$\begin{aligned} &\Big(\frac{\widehat{\Theta}_\varrho^n - \widehat{\Theta}_\varrho^{n-1}}{\delta t}, \omega\Big)^h + (\theta_3 \nabla \widehat{\Theta}_\varrho^n + 2\theta_{33} \Xi_\varrho(\widehat{\Theta}_\varrho^n) \nabla \widehat{\Theta}_\varrho^n, \nabla \omega) \\ &= (a_3 \widehat{\Theta}_\varrho^n - b_3 \xi_\varrho(\widehat{\Phi}_\varrho^{n-1}) \xi_\varrho(\widehat{\Theta}_\varrho^n) - c_3 \xi_\varrho(\widehat{\Psi}_\varrho^{n-1}) \xi_\varrho(\widehat{\Theta}_\varrho^n) - d_3 \xi_\varrho(\widehat{\Theta}_\varrho^{n-1}) \xi_\varrho(\widehat{\Theta}_\varrho^n), \omega)^h, \end{aligned} \tag{60}$$

where $\widehat{\Phi}_\varrho^0, \widehat{\Psi}_\varrho^0, \widehat{\Theta}_\varrho^0 \in X^h$ are given approximation of $u^0, v^0, w^0 \in C(\mathfrak{O})$, respectively.

Lemma 2. Assuming the validity of assumptions (**A**), the function $\Xi_\varrho : X^h \to [L^\infty(\mathfrak{O})]^{d\times d}$ satisfies the following result for *a.e.* in $\mathfrak{O}$ and for all $\varrho \in (0,\ e^{-1})$:

$$\varrho \upsilon^T \upsilon \leq \upsilon^T \Xi_\varrho(\omega) \upsilon \leq \frac{1}{\varrho} \upsilon^T \upsilon, \qquad \text{for all } \upsilon \in \mathbb{R}^d, \qquad \text{for all } \omega \in X^h. \tag{61}$$

Lemma 3. The function $\Xi_\varrho : X^h \to [L^\infty(\mathfrak{O})]^{d\times d}$ is continuous for any $\varrho \in (0, e^{-1})$. This continuity is defined in the sense that

$$\begin{aligned} &\|(\Xi_\varrho(\Im_1) - \Xi_\varrho(\Im_2))|_{\tilde{\tau}}\| \leq \frac{2}{\varrho} \|\Im_1 - \Im_2\|_{0,\infty}, \\ &\qquad\qquad \text{for all } \Im_1,\ \Im_2 \in X^h \quad \text{and} \quad \text{for all } \tilde{\tau} \in \mathcal{T}^h. \end{aligned} \tag{62}$$

Lemma 4. Assume the assumptions (A) are valid. Then, the function ξ_ϱ satisfies Let us suppose that the assumptions (A) hold true. Under this assumption, the function ξ_ϱ fulfills the following results

$$\|\nabla\pi^h[\xi_\varrho(\omega)]\|_0^2 \leq (\nabla\omega, \nabla\pi^h[\xi_\varrho(\omega)]), \qquad \text{for all } \omega \in X^h. \tag{63}$$

The objective of this document is to establish the existence of solutions to problem (Q) by taking the limit $\varrho \to 0$ in the discrete problem. In order to accomplish this, bounds on the approximate solution $(\mathrm{Q}_M^{\delta t})$ have been derived, which are independent of ϱ. It is important to note that the goal is to simultaneously approach the limits $\varrho \to 0$ and $\delta t \to 0$ in the discrete model, where the relationship between ϱ and δt is given by the condition $\delta t = o(\varrho)$ as $\varrho \to 0$.

4.2 Existence of the approximations

Ensuring the correctness and dependability of computational models requires investigating the existence of numerical solutions for PDEs using FEM. The existence of a numerical solution establishes a strong mathematical basis and validates that the FEM formulation is well-posed and able to approximate the system's actual physical behavior. Numerical results could be unstable or physically meaningless without this kind of verification. In order to create reliable simulations in the fields of engineering, physics, and applied sciences, it is important to analyze the existence and stability of FEM solutions.

To demonstrate the existence of $\widehat{\Phi}_\varrho^n$, $\widehat{\Psi}_\varrho^n$ and $\widehat{\Theta}_\varrho^n, n \geq 1$ to the system (58), (59) and (60), for given $\widehat{\Phi}_\varrho^{n-1}$, $\widehat{\Psi}_\varrho^{n-1}$ and $\widehat{\Theta}_\varrho^{n-1}$, it is useful to define the functions $M_u : X^h \times X^h \to X^h$, $M_v : X^h \times X^h \to X^h$ and $M_w : X^h \times X^h \to X^h$ so that for all $\omega \in X^h$, we have that

$$\begin{aligned}(M_u(\widehat{\Phi}, \widehat{\Psi}, \widehat{\Theta}),\ \omega)^h :=& (\widehat{\Phi} - \widehat{\Phi}_\varrho^{n-1},\ \omega)^h + \delta t(\theta_1\nabla\widehat{\Phi} + 2\theta_{11}\Xi_\varrho(\widehat{\Phi})\nabla\widehat{\Phi}, \nabla\omega)\\ &- \delta t(a_1\widehat{\Phi} - b_1\xi_\varrho(\widehat{\Phi}_\varrho^{n-1})\xi_\varrho(\widehat{\Phi}) - c_1\xi_\varrho(\widehat{\Psi}_\varrho^{n-1})\xi_\varrho(\widehat{\Phi})\\ &- d_1\xi_\varrho(\widehat{\Theta}_\varrho^{n-1})\xi_\varrho(\widehat{\Phi}), \omega)^h,\end{aligned} \tag{64}$$

$$\begin{aligned}(M_v(\widehat{\Phi}, \widehat{\Psi}, \widehat{\Theta}),\ \omega)^h :=& (\widehat{\Psi} - \widehat{\Psi}_\varrho^{n-1}, \omega)^h + \delta t(\theta_2\nabla\widehat{\Psi} + \theta_{21}\Xi_\varrho(\widehat{\Psi})\nabla\widehat{\Phi}\\ &+\theta_{21}\Xi_\varrho(\widehat{\Phi})\nabla\widehat{\Psi} + 2\theta_{22}\Xi_\varrho(\widehat{\Psi})\nabla\widehat{\Psi}\\ &+\theta_{23}\Xi_\varrho(\widehat{\Psi})\nabla\widehat{\Theta} + \theta_{23}\Xi_\varrho(\widehat{\Theta})\nabla\widehat{\Psi}, \nabla\omega)\\ &-\delta t(a_2\widehat{\Psi} - b_2\xi_\varrho(\widehat{\Phi}_\varrho^{n-1})\xi_\varrho(\widehat{\Psi}) - c_2\xi_\varrho(\widehat{\Psi}_\varrho^{n-1})\xi_\varrho(\widehat{\Psi})\\ &-d_2\xi_\varrho(\widehat{\Theta}_\varrho^{n-1})\xi_\varrho(\widehat{\Psi}), \omega)^h,\end{aligned} \tag{65}$$

and

$$\begin{aligned}(M_w(\widehat{\Phi},\widehat{\Psi},\widehat{\Theta}),\ \omega)^h :=&(\widehat{\Theta}-\widehat{\Theta}^{n-1}_{\varrho},\omega)^h+\delta t(\theta_3\nabla\widehat{\Theta}+2\theta_{33}\Xi_{\varrho}(\widehat{\Theta})\nabla\widehat{\Theta},\nabla\omega)\\ &-\delta t(a_3\widehat{\Theta}-b_3\xi_{\varrho}(\widehat{\Phi}^{n-1}_{\varrho})\xi_{\varrho}(\widehat{\Theta})\\ &-c_3\xi_{\varrho}(\widehat{\Psi}^{n-1}_{\varrho})\xi_{\varrho}(\widehat{\Theta})-d_3\xi_{\varrho}(\widehat{\Theta}^{n-1}_{\varrho})\xi_{\varrho}(\widehat{\Theta}),\omega)^h. \end{aligned} \tag{66}$$

Firstly, it is worth noting that the continuous piecewise linear functions M_u, M_v and M_w can be uniquely determined based on their values at the node points. This uniqueness can be observed by setting $\omega\equiv\eta_j$, for $j=0,\dots,J$ in (64), which subsequently leads to solvable square matrix systems

$$\widehat{M}M_u(\widehat{\Phi},\widehat{\Psi},\widehat{\Theta})=\mathcal{S}_1,$$

$$\widehat{M}M_v(\widehat{\Phi},\widehat{\Psi},\widehat{\Theta})=\mathcal{S}_2,$$

$$\widehat{M}M_w(\widehat{\Phi},\widehat{\Psi},\widehat{\Theta})=\mathcal{S}_3,$$

where $\widehat{M}$ is the lumped mass matrix and $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ are given vectors in terms of the nodal values of $\widehat{\Phi},\widehat{\Psi},\widehat{\Theta},\widehat{\Phi}^{n-1}_{\varrho},\widehat{\Psi}^{n-1}_{\varrho}$ and $\widehat{\Theta}^{n-1}_{\varrho}$ thus, the functions M_u, M_v and M_w are well defined.

The problem $(\mathrm{Q}^{h,\delta t}_{\varrho})$ can be restated based on (64), (65), and (66), in the following form: For given $\{\widehat{\Phi}^0_{\varrho},\widehat{\Psi}^0_{\varrho},\widehat{\Theta}^0_{\varrho}\}\in X^h\times X^h\times X^h$, find $\{\widehat{\Phi}^n_{\varrho},\widehat{\Psi}^n_{\varrho},\widehat{\Theta}^n_{\varrho}\}\in X^h\times X^h\times X^h,\ n\geq 1$, such that

$$M_u(\widehat{\Phi},\widehat{\Psi},\widehat{\Theta})=0,\qquad M_v(\widehat{\Phi},\widehat{\Psi},\widehat{\Theta})=0,\qquad M_w(\widehat{\Phi},\widehat{\Psi},\widehat{\Theta})=0. \tag{67}$$

Lemma 5. The functions $M_u,M_v,M_w:[X^h]^3_R\to X^h$, are continuous for any $R>0$, where

$$[X^h]^3_R=\bigg\{\{\Im_1,\Im_2,\Im_3\}\in X^h\times X^h\times X^h:|\Im_1|^3_h+|\Im_3|^2_h+|\Im_3|^3_h\leq R^2\bigg\}.$$

Proof. By using a method similar to that used in [27, 28, 32], the aforementioned theorems can be demonstrated. □

Theorem 1. Assume the assumptions (A) are valid. Suppose that $\{\widehat{\Phi}^{n-1}_{\varrho},\widehat{\Psi}^{n-1}_{\varrho},\widehat{\Theta}^{n-1}_{\varrho}\}\in X^h\times X^h\times X^h$ be given a solution to the $(n-1)$-th step of $(Q^{h,\delta t}_{\varrho})$, for some $n=1,2,\dots,N$, then for all $\varrho\in(0,e^{-1})$, for all $h>0$, and for all $\delta t\leq\frac{1}{2\theta_i(2+a_i)}$, for all $i=1,2$ there exists a solution $\{\widehat{\Phi}^n_{\varrho},\widehat{\Psi}^n_{\varrho},\widehat{\Theta}^n_{\varrho}\}\in[X^h]^3_R$ to the n-th step of $(Q^{h,\delta t}_{\varrho})$.

Proof. The proofs of the above theorems can be established using a methodology analogous to that employed in [30, 33, 31]. □

To process the initial data in the fully-discrete approximation, we introduce the discrete L^2-projection $P^h : L^2(\mathfrak{O}) \to X^h$, where Σ is given by

$$(P^h\Sigma, \omega)^h = (\Sigma, \omega), \qquad \text{for all } \omega \in X^h. \tag{68}$$

The above projection fulfills the following key outcomes [11]:

$$\|P^h\Sigma\|_{0,\infty} \leq \|\Sigma\|_{0,\infty}, \qquad \text{for all } \Sigma \in L^\infty(\mathfrak{O}), \tag{69}$$

$$\|(I - P^h)\Sigma\|_{m,s} \leq Ch^{1-m}\|\Sigma\|_{1,s}, \quad \text{for all } \Sigma \in W^{1,s}(\mathfrak{O}), \tag{70}$$

$$\text{for any } s \in [2, \infty] \qquad \text{and } m \in \{0, 1\}. \tag{71}$$

We define the "inverse Laplacian" operator $\mathfrak{G}_q$ for any $q \in (1, 2]$ as the mapping $\mathfrak{G}_q : (W^{1,q'}(\mathfrak{O}))' \to W^{1,q}(\mathfrak{O})$ in the form

$$(\nabla\mathfrak{G}_q\Pi, \nabla\Sigma) + (\mathfrak{G}_q\Pi, \Sigma) = \langle\Pi, \Sigma\rangle_{q'}, \qquad \text{for all } \Sigma \in W^{1,q'}(\mathfrak{O}), \tag{72}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$ and $\langle\cdot, \cdot\rangle_{q'}$ denotes the duality pairing between $(W^{1,q'}(\mathfrak{O}))'$ and $W^{1,q'}(\mathfrak{O})$ such that

$$\langle\Pi, \Sigma\rangle_{q'} = (\Pi, \Sigma), \qquad \text{for all } \Pi \in L^2(\mathfrak{O}),\ \Sigma \in W^{1,q'}(\mathfrak{O}). \tag{73}$$

The operator $\mathfrak{G}$ is well-posed, which can be shown by the generalized Lax-Milgram theorem, which guarantees the existence of a positive constant C satisfying the following inequality:

$$\|\mathfrak{G}_q\Pi\|_{1,q} \leq C\|\Pi\|_{(W^{1,q'}(\mathfrak{O}))'}, \qquad \text{for all } \Pi \in (W^{1,q'}(\mathfrak{O}))'. \tag{74}$$

We also introduce the operator $\mathfrak{G} : (H^1(\mathfrak{O}))' \to H^1(\mathfrak{O})$ in the form

$$(\nabla\mathfrak{G}\Pi, \nabla\Sigma) + (\mathfrak{G}\Pi, \Sigma) = \langle\Pi, \Sigma\rangle, \qquad \text{for all } \Sigma \in H^1(\mathfrak{O}), \tag{75}$$

where $\langle\cdot, \cdot\rangle$ represents the duality pairing between $(H^1(\mathfrak{O}))'$ and $H^1(\mathfrak{O})$, so that

$$\langle\Pi, \Sigma\rangle = (\Pi, \Sigma), \qquad \text{for all } \Pi \in L^2(\mathfrak{O}),\ \Sigma \in H^1(\mathfrak{O}). \tag{76}$$

We can also conclude, for $q \in (1, 2]$,, that

$$\|\omega\|_{0,q} \leq Ch^{-1}\|\mathfrak{G}_q\omega\|_{1,q}, \quad \text{for all } \quad \omega \in X^h. \tag{77}$$

5 Stability bounds

To guarantee the accuracy, stability, and convergence of the results, bounds for the numerical solution using FEM in various spaces must be established. These bounds ensure that the approximations meet the governing equations and boundary conditions across a range of spatial domains, help regulate the behavior of the solution, and stop non-physical oscillations. The numerical model may yield erratic or imprecise results in the absence of appropriate bounds, particularly in complex geometries or heterogeneous media.

We develop a discrete version of the a priori estimates referred to in Lemma 4 in the next section of this paper. The solutions $\widehat{\Phi}_\varrho^n, \widehat{\Psi}_\varrho^n, \widehat{\Theta}_\varrho^n$ are also shown to have a number of uniform bounds that are independent of the parameters ϱ, h, and δt. Because they guarantee the stability of different norms of the numerical solutions, these constraints are extremely significant. Our main objective is to show that the issue (Q) has weak solutions. The next stage in our proof is to determine the limits $\delta t \to 0^+$ and $h, \varrho \to 0$ in the system $(\mathrm{Q}_\varrho^{h,\delta t})$. To achieve this, we establish a connection between the cut-off parameter $\varrho < 1$ and the time step δt, ensuring that $\delta t = o(\varrho)$ as $\varrho \to 0$.

Lemma 6. Given $\widehat{\Phi}_\varrho^{n-1}, \widehat{\Psi}_\varrho^{n-1}, \widehat{\Theta}_\varrho^{n-1} \in X^h \times X^h \times X^h$ for $n = 1, \ldots, N$, there exists a solution $\widehat{\Phi}_\varrho^n, \widehat{\Psi}_\varrho^n, \widehat{\Theta}_\varrho^n \in X^h \times X^h \times X^h$ to the n-th step of $(\mathrm{Q}_\varrho^{h,\delta t})$ for all $\varrho \in (0, e^{-1})$, $h > 0$, and $\delta t > 0$, such that

$$
\begin{aligned}
&\frac{(\theta_{21})^2}{\theta_{22}\theta_{11}}\big[1 - 2\delta t a_1 - \delta t b_1 - \delta t c_1 - \delta t d_1\big](\mathfrak{G}_\varrho(\widehat{\Phi}_\varrho^n), 1)^h \\
&+ \big(1 - 2a_2\delta t - b_2\delta t - c_2\delta t - d_2\delta t\big)(\mathfrak{G}_\varrho(\widehat{\Psi}_\varrho^n), 1)^h \\
&+ \frac{(\theta_{23})^2}{\theta_{22}\theta_{33}}\big[1 - 2\delta t a_3 - \delta t b_3 - \delta t c_3 - \delta t d_3\big](\mathfrak{G}_\varrho(\widehat{\Theta}_\varrho^n), 1)^h \\
&+ \frac{\delta t\theta_1(\theta_{21})^2}{\theta_{22}\theta_{11}} \int_{\mathfrak{O}} \frac{|\nabla\widehat{\Phi}_\varrho^n|^2}{\xi_\varrho(\widehat{\Phi}_\varrho^n)} d\mathbf{x} \\
&+ \frac{3\delta t(\theta_{21})^2}{2\theta_{22}} |\widehat{\Phi}_\varrho^n|_1^2 + \delta t\theta_2 \int_{\mathfrak{O}} \frac{|\nabla\widehat{\Psi}_\varrho^n|^2}{\xi_\varrho(\widehat{\Psi}_\varrho^n)} d\mathbf{x} \\
&+ \delta t\theta_{21} \int_{\mathfrak{O}} |\nabla\widehat{\Psi}_\varrho^n|^2 \frac{\Xi_\varrho(\widehat{\Phi}_\varrho^n)}{\Xi_\varrho(\widehat{\Psi}_\varrho^n)} d\mathbf{x} + \delta t\theta_{22}|\widehat{\Psi}_\varrho^n|_1^2 \\
&+ \delta t\theta_{23} \int_{\mathfrak{O}} |\nabla\widehat{\Psi}_\varrho^n|^2 \frac{\Xi_\varrho(\widehat{\Theta}_\varrho^n)}{\Xi_\varrho(\widehat{\Psi}_\varrho^n)} d\mathbf{x} + \frac{\delta t\theta_3(\theta_{23})^2}{\theta_{22}\theta_{33}} \int_{\mathfrak{O}} \frac{|\nabla\widehat{\Theta}_\varrho^n|^2}{\xi_\varrho(\widehat{\Theta}_\varrho^n)} d\mathbf{x} + \frac{3\delta t(\theta_{23})^2}{2\theta_{22}} |\widehat{\Theta}_\varrho^n|_1^2 \\
&\quad \le \Big[\frac{(\theta_{21})^2}{\theta_{22}\theta_{11}} + \frac{3\delta t b_1(\theta_{21})^2}{\theta_{22}\theta_{11}} + 3b_2\delta t + \frac{3b_3\delta t(\theta_{23})^2}{\theta_{22}\theta_{33}}\Big](\mathfrak{G}_\varrho(\widehat{\Phi}_\varrho^{n-1}), 1)^h
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\frac{3c_1\delta t(\theta_{21})^2}{\theta_{22}\theta_{11}}+1+3c_2\delta t+\frac{3c_3\delta t(\theta_{23})^2}{\theta_{22}\theta_{33}}\right](\mathfrak{G}_{\varrho}(\widehat{\Psi}_{\varrho}^{n-1}),1)^h\\
&+\left[\frac{3d_1\delta t(\theta_{21})^2}{\theta_{22}\theta_{11}}+3d_2\delta t+\frac{(\theta_{23})^2}{\theta_{22}\theta_{33}}+\frac{3\delta t d_3(\theta_{23})^2}{\theta_{22}\theta_{33}}\right](\mathfrak{G}_{\varrho}(\widehat{\Theta}_{\varrho}^{n-1}),1)^h\\
&+\delta t C(|\mathfrak{O}|,a_1,a_2,a_3,\theta_{21},\theta_{22},\theta_{11},\theta_{23},\theta_{22},\theta_{33}). \qquad (78)
\end{aligned}
$$

Proof. Using $\omega\equiv\delta t\pi^h[\mathfrak{G}'_{\varrho}(\widehat{\Phi}_{\varrho}^n)]$ as a test function in (58), and considering (45) and (54), we obtain that

$$
\begin{aligned}
&(\widehat{\Phi}_{\varrho}^n-\widehat{\Phi}_{\varrho}^{n-1},\mathfrak{G}'_{\varrho}(\widehat{\Phi}_{\varrho}^n))^h+\delta t\theta_1\int_{\mathfrak{O}}\frac{|\nabla\widehat{\Phi}_{\varrho}^n|^2}{\xi_{\varrho}(\widehat{\Phi}_{\varrho}^n)}d\mathbf{x}+2\delta t\theta_{11}|\widehat{\Phi}_{\varrho}^n|_1^2\\
&=\delta t(a_1\widehat{\Phi}_{\varrho}^n-b_1\xi_{\varrho}(\widehat{\Phi}_{\varrho}^{n-1})\xi_{\varrho}(\widehat{\Phi}_{\varrho}^n)-c_1\xi_{\varrho}(\widehat{\Psi}_{\varrho}^{n-1})\xi_{\varrho}(\widehat{\Phi}_{\varrho}^n)\\
&\quad-d_1\xi_{\varrho}(\widehat{\Theta}_{\varrho}^{n-1})\xi_{\varrho}(\widehat{\Phi}_{\varrho}^n),\mathfrak{G}'_{\varrho}(\widehat{\Phi}_{\varrho}^n))^h. \qquad (79)
\end{aligned}
$$

By utilizing (27), (28), (14), Young's inequality, and the fact that $\mathfrak{G}_{\varrho}(\cdot)\geq 0$, we can deduce that

$$
\begin{aligned}
&\delta t(a_1\widehat{\Phi}_{\varrho}^n-b_1\xi_{\varrho}(\widehat{\Phi}_{\varrho}^n)\xi_{\varrho}(\widehat{\Phi}_{\varrho}^{n-1})-c_1\xi_{\varrho}(\widehat{\Phi}_{\varrho}^n)\xi_{\varrho}(\widehat{\Psi}_{\varrho}^{n-1})-d_1\xi_{\varrho}(\widehat{\Phi}_{\varrho}^n)\xi_{\varrho}(\widehat{\Theta}_{\varrho}^{n-1}),\mathfrak{G}'_{\varrho}(\widehat{\Phi}_{\varrho}^n))^h\\
&\leq 2a_1\delta t(\mathfrak{G}_{\varrho}(\widehat{\Phi}_{\varrho}^n),1)^h+b_1\delta t(\mathfrak{G}_{\varrho}(\widehat{\Phi}_{\varrho}^n),1)^h+c_1\delta t(\mathfrak{G}_{\varrho}(\widehat{\Phi}_{\varrho}^n),1)^h+d_1\delta t(\mathfrak{G}_{\varrho}(\widehat{\Phi}_{\varrho}^n),1)^h\\
&\quad+3b_1\delta t(\mathfrak{G}_{\varrho}(\widehat{\Phi}_{\varrho}^{n-1}),1)^h+3c_1\delta t(\mathfrak{G}_{\varrho}(\widehat{\Psi}_{\varrho}^{n-1}),1)^h\\
&\quad+3d_1\delta t(\mathfrak{G}_{\varrho}(\widehat{\Theta}_{\varrho}^{n-1}),1)^h+\delta tC(|\mathfrak{O}|,a_1). \qquad (80)
\end{aligned}
$$

By substituting (80) into (79), we obtain the following expression:

$$
\begin{aligned}
&(\widehat{\Phi}_{\varrho}^n-\widehat{\Phi}_{\varrho}^{n-1},\mathfrak{G}'_{\varrho}(\widehat{\Phi}_{\varrho}^n))^h+\delta t\theta_1\int_{\mathfrak{O}}\frac{|\nabla\widehat{\Phi}_{\varrho}^n|^2}{\xi_{\varrho}(\widehat{\Phi}_{\varrho}^n)}d\mathbf{x}+2\delta t\theta_{11}|\widehat{\Phi}_{\varrho}^n|_1^2\\
&\leq 2a_1\delta t(\mathfrak{G}_{\varrho}(\widehat{\Phi}_{\varrho}^n),1)^h+b_1\delta t(\mathfrak{G}_{\varrho}(\widehat{\Phi}_{\varrho}^n),1)^h+c_1\delta t(\mathfrak{G}_{\varrho}(\widehat{\Phi}_{\varrho}^n),1)^h+d_1\delta t(\mathfrak{G}_{\varrho}(\widehat{\Phi}_{\varrho}^n),1)^h\\
&\quad+3b_1\delta t(\mathfrak{G}_{\varrho}(\widehat{\Phi}_{\varrho}^{n-1}),1)^h+3c_1\delta t(\mathfrak{G}_{\varrho}(\widehat{\Psi}_{\varrho}^{n-1}),1)^h\\
&\quad+3d_1\delta t(\mathfrak{G}_{\varrho}(\widehat{\Theta}_{\varrho}^{n-1}),1)^h+\delta tC(|\mathfrak{O}|,a_1). \qquad (81)
\end{aligned}
$$

Therefore, it can be deduced from the second inequality in (81) that

$$
\begin{aligned}
&\big[1-2\delta ta_1-\delta tb_1-\delta tc_1-\delta td_1\big](\mathfrak{G}_{\varrho}(\widehat{\Phi}_{\varrho}^n),1)^h+\delta t\theta_1\int_{\mathfrak{O}}\frac{|\nabla\widehat{\Phi}_{\varrho}^n|^2}{\xi_{\varrho}(\widehat{\Phi}_{\varrho}^n)}d\mathbf{x}+2\delta t\theta_{11}|\widehat{\Phi}_{\varrho}^n|_1^2\\
&\leq\big[1+3\delta tb_1\big](\mathfrak{G}_{\varrho}(\widehat{\Phi}_{\varrho}^{n-1}),1)^h+3c_1\delta t(\mathfrak{G}_{\varrho}(\widehat{\Psi}_{\varrho}^{n-1}),1)^h+3d_1\delta t(\mathfrak{G}_{\varrho}(\widehat{\Theta}_{\varrho}^{n-1}),1)^h\\
&\qquad+\delta tC(|\mathfrak{O}|,a_1). \qquad (82)
\end{aligned}
$$

Using $\omega \equiv \delta t \pi^h[\mathfrak{G}'_\varrho(\widehat{\Psi}^n_\varrho)]$ and $\omega \equiv \delta t \pi^h[\mathfrak{G}'_\varrho(\widehat{\Theta}^n_\varrho)]$ as a test function in (59) and (60), respectively, and then by repeating the techniques utilized to construct (5), it follows that

$$\begin{aligned}
&\big(1 - 2a_2\delta t - b_2\delta t - c_2\delta t - d_2\delta t\big)(\mathfrak{G}_\varrho(\widehat{\Psi}^n_\varrho), 1)^h + \delta t\theta_2 \int_{\mathfrak{O}} \frac{|\nabla \widehat{\Psi}^n_\varrho|^2}{\xi_\varrho(\widehat{\Psi}^n_\varrho)} d\mathbf{x} \\
&- \frac{\delta t(\theta_{21})^2}{2\theta_{22}} |\widehat{\Phi}^n_\varrho|_1^2 + \delta t\theta_{21} \int_{\mathfrak{O}} |\nabla \widehat{\Psi}^n_\varrho|^2 \frac{\Xi_\varrho(\widehat{\Phi}^n_\varrho)}{\Xi_\varrho(\widehat{\Psi}^n_\varrho)} d\mathbf{x} + \delta t\theta_{22} |\widehat{\Psi}^n_\varrho|_1^2 \\
&\quad \le -\frac{\delta t(\theta_{23})^2}{2\theta_{22}} |\widehat{\Theta}^n_\varrho|_1^2 + \delta t\theta_{23} \int_{\mathfrak{O}} |\nabla \widehat{\Psi}^n_\varrho|^2 \frac{\Xi_\varrho(\widehat{\Theta}^n_\varrho)}{\Xi_\varrho(\widehat{\Psi}^n_\varrho)} d\mathbf{x} 3b_2\delta t(\mathfrak{G}_\varrho(\widehat{\Phi}^{n-1}_\varrho), 1)^h \\
&\qquad + (1+3c_2)\delta t(\mathfrak{G}_\varrho(\widehat{\Psi}^{n-1}_\varrho), 1)^h + 3d_2\delta t(\mathfrak{G}_\varrho(\widehat{\Theta}^{n-1}_\varrho), 1)^h + \delta t C(|\mathfrak{O}|, a_2),
\end{aligned} \tag{83}$$

and

$$\begin{aligned}
&\big[1 - 2\delta t a_3 - \delta t b_3 - \delta t c_3 - \delta t d_3\big](\mathfrak{G}_\varrho(\widehat{\Theta}^n_\varrho), 1)^h + \delta t\theta_3 \int_{\mathfrak{O}} \frac{|\nabla \widehat{\Theta}^n_\varrho|^2}{\xi_\varrho(\widehat{\Theta}^n_\varrho)} d\mathbf{x} + 2\delta t\theta_{33} |\widehat{\Theta}^n_\varrho|_1^2 \\
&\le 3b_3\delta t(\mathfrak{G}_\varrho(\widehat{\Phi}^{n-1}_\varrho), 1)^h + 3c_3\delta t(\mathfrak{G}_\varrho(\widehat{\Psi}^{n-1}_\varrho), 1)^h + \big[1 + 3\delta t d_3\big](\mathfrak{G}_\varrho(\widehat{\Theta}^{n-1}_\varrho), 1)^h \\
&\qquad + \delta t C(|\mathfrak{O}|, a_3).
\end{aligned} \tag{84}$$

By multiplying (5) by $\frac{(\theta_{21})^2}{\theta_{22}\theta_{11}}$ and (5) by $\frac{(\theta_{23})^2}{\theta_{22}\theta_{33}}$, and then combining the results with (83), we obtain the final result (78). □

Lemma 7. Under the assumptions of Lemma 6, suppose u^0, v^0, $w^0 \in L^\infty(\mathfrak{O})$, where u^0, v^0, $w^0 \ge 0$ for almost every $x \in \mathfrak{O}$. Let $\widehat{\Phi}^0_\varrho \equiv P^h u^0$, $\widehat{\Psi}^0_\varrho \equiv P^h v^0$, $\widehat{\Theta}^0_\varrho \equiv P^h w^0$ or $\widehat{\Phi}^0_\varrho \equiv \pi^h u^0$, $\widehat{\Psi}^0_\varrho \equiv \pi^h v^0$, $\widehat{\Theta}^0_\varrho \equiv \pi^h w^0$ if $u^0, v^0, w^0 \in C(\mathfrak{O})$. Then, for any $\varrho \in (0,, e^{-1})$, $h > 0$, and $\delta t \le \frac{1-\delta}{2\widehat{a}+\widehat{b}+\widehat{c}+\widehat{d}}$, where $\delta \in (0,1)$, $\widehat{a} = \max\{a_1, a_2, a_3\}$,, $\widehat{b} = \max\{b_1, b_2, b_3\}$, $\widehat{c} = \max\{c_1, c_2, c_3\}$, $\widehat{d} = \max\{d_1, d_2, d_3\}$, the problem $(\mathrm{Q}^{h,\delta t}_\varrho)$ has a solution $\widehat{\Phi}^n_\varrho, \widehat{\Psi}^n_\varrho, \widehat{\Theta}^n_\varrho$, $n = 1, \dots, N$, satisfying

$$\begin{aligned}
&\max_{n=1,\dots,N} \Bigg[\left(\frac{(\theta_{21})^2}{\theta_{22}\theta_{11}} \mathfrak{G}_\varrho(\widehat{\Phi}^n_\varrho), 1 \right)^h + (\mathfrak{G}_\varrho(\widehat{\Psi}^n_\varrho), 1)^h \\
&+ \left(\frac{(\theta_{23})^2}{\theta_{22}\theta_{33}} \mathfrak{G}_\varrho(\widehat{\Theta}^n_\varrho), 1 \right)^h + \varrho^{-1} \|\pi^h[\widehat{\Phi}^n_\varrho]_-\|_0^2 + \varrho^{-1} \|\pi^h[\widehat{\Psi}^n_\varrho]_-\|_0^2 \\
&+ \varrho^{-1} \|\pi^h[\widehat{\Theta}^n_\varrho]_-\|_0^2 + \|\widehat{\Phi}^n_\varrho\|_{0,1}^2 + \|\widehat{\Psi}^n_\varrho\|_{0,1} + \|\widehat{\Theta}^n_\varrho\|_{0,1} \Bigg] \\
&+ \sum_{n=1}^N \delta t \Big[\frac{3(\theta_{21})^2}{2\theta_{22}} |\widehat{\Phi}^n_\varrho|_1^2 + \theta_{22} |\widehat{\Psi}^n_\varrho|_1^2 + \frac{3(\theta_{23})^2}{2\theta_{22}} |\widehat{\Theta}^n_\varrho|_1^2 \Big] \le C.
\end{aligned} \tag{85}$$

Proof. Based on (78) and our assumptions regarding δt, we can conclude that for $n = 1, \dots, N$ and $i = 1, 2, 3$, the following holds:

$$\begin{aligned}
&\big[1 - 2\delta t\widehat{a} - \delta t\widehat{b} - \delta t\widehat{c} - \delta t\widehat{d}\big]\left(\frac{(\theta_{21})^2}{\theta_{22}\theta_{11}}\mathfrak{G}_\varrho(\widehat{\Phi}^n_\varrho) + \mathfrak{G}_\varrho(\widehat{\Psi}^n_\varrho) + \frac{(\theta_{23})^2}{\theta_{22}\theta_{33}}\mathfrak{G}_\varrho(\widehat{\Theta}^n_\varrho), 1\right)^h \\
&+ \frac{\delta t\theta_1(\theta_{21})^2}{\theta_{22}\theta_{11}}\int_{\mathfrak{D}}\frac{|\nabla\widehat{\Phi}^n_\varrho|^2}{\xi_\varrho(\widehat{\Phi}^n_\varrho)}d\mathbf{x} + \frac{3\delta t(\theta_{21})^2}{2\theta_{22}}|\widehat{\Phi}^n_\varrho|_1^2 + \delta t\theta_2\int_{\mathfrak{D}}\frac{|\nabla\widehat{\Psi}^n_\varrho|^2}{\xi_\varrho(\widehat{\Psi}^n_\varrho)}d\mathbf{x} \\
&+ \delta t\theta_{21}\int_{\mathfrak{D}}|\nabla\widehat{\Psi}^n_\varrho|^2\frac{\Xi_\varrho(\widehat{\Phi}^n_\varrho)}{\Xi_\varrho(\widehat{\Psi}^n_\varrho)}d\mathbf{x} + \delta t\theta_{22}|\widehat{\Psi}^n_\varrho|_1^2 + \delta t\theta_{23}\int_{\mathfrak{D}}|\nabla\widehat{\Psi}^n_\varrho|^2\frac{\Xi_\varrho(\widehat{\Theta}^n_\varrho)}{\Xi_\varrho(\widehat{\Psi}^n_\varrho)}d\mathbf{x} \\
&+ \frac{\delta t\theta_3(\theta_{23})^2}{\theta_{22}\theta_{33}}\int_{\mathfrak{D}}\frac{|\nabla\widehat{\Theta}^n_\varrho|^2}{\xi_\varrho(\widehat{\Theta}^n_\varrho)}d\mathbf{x} + \frac{3\delta t(\theta_{23})^2}{2\theta_{22}}|\widehat{\Theta}^n_\varrho|_1^2 \\
&\quad\le \Big[\frac{(\theta_{21})^2}{\theta_{22}\theta_{11}} + \frac{3\delta tb_1(\theta_{21})^2}{\theta_{22}\theta_{11}} + 3b_2\delta t + \frac{3b_3\delta t(\theta_{23})^2}{\theta_{22}\theta_{33}}\Big](\mathfrak{G}_\varrho(\widehat{\Phi}^{n-1}_\varrho), 1)^h \\
&\qquad + \left[\frac{3c_1\delta t(\theta_{21})^2}{\theta_{22}\theta_{11}} + 1 + 3c_2\delta t + \frac{3c_3\delta t(\theta_{23})^2}{\theta_{22}\theta_{33}}\right](\mathfrak{G}_\varrho(\widehat{\Psi}^{n-1}_\varrho), 1)^h \\
&\qquad + \left[\frac{3d_1\delta t(\theta_{21})^2}{\theta_{22}\theta_{11}} + 3d_2\delta t + \frac{(\theta_{23})^2}{\theta_{22}\theta_{33}} + \frac{3\delta td_3(\theta_{23})^2}{\theta_{22}\theta_{33}}\right](\mathfrak{G}_\varrho(\widehat{\Theta}^{n-1}_\varrho), 1)^h + C.
\end{aligned} \tag{86}$$

Based on our assumptions regarding the initial data u^0, v^0, w^0, and (21), we can conclude that

$$(\mathfrak{G}_\varrho(\widehat{\Phi}^0_\varrho),\ 1)^h \le C, \quad (\mathfrak{G}_\varrho(\widehat{\Psi}^0_\varrho),\ 1)^h \le C \quad \textit{and} \quad (\mathfrak{G}_\varrho(\widehat{\Theta}^0_\varrho),\ 1)^h \le C. \tag{87}$$

Therefore, it can be inferred from (86) and (87) that

$$\begin{aligned}
&\left(\frac{(\theta_{21})^2}{\theta_{22}\theta_{11}}\mathfrak{G}_\varrho(\widehat{\Phi}^n_\varrho) + \mathfrak{G}_\varrho(\widehat{\Psi}^n_\varrho) + \frac{(\theta_{23})^2}{\theta_{22}\theta_{33}}\mathfrak{G}_\varrho(\widehat{\Theta}^n_\varrho), 1\right)^h \\
&\le C\Big(\frac{1}{1 - (2\widehat{a} + \widehat{b} + \widehat{c} + \widehat{d})\delta t}\Big)\left[\frac{(\theta_{21})^2}{\theta_{22}\theta_{11}}(\mathfrak{G}_\varrho(\widehat{\Phi}^{n-1}_\varrho), 1)^h\right. \\
&\qquad \left. + (\mathfrak{G}_\varrho(\widehat{\Psi}^{n-1}_\varrho), 1)^h + \frac{(\theta_{23})^2}{\theta_{22}\theta_{33}}(\mathfrak{G}_\varrho(\widehat{\Theta}^{n-1}_\varrho), 1)^h + 1\right] \\
&\le C\Big(1 + \frac{(2\widehat{a} + \widehat{b} + \widehat{c} + \widehat{d})\delta t}{1 - (2\widehat{a} + \widehat{b} + \widehat{c} + \widehat{d})\delta t}\Big)\left[\frac{(\theta_{21})^2}{\theta_{22}\theta_{11}}(\mathfrak{G}_\varrho(\widehat{\Phi}^{n-1}_\varrho), 1)^h\right. \\
&\qquad \left. + (\mathfrak{G}_\varrho(\widehat{\Psi}^{n-1}_\varrho), 1)^h + \frac{(\theta_{23})^2}{\theta_{22}\theta_{33}}(\mathfrak{G}_\varrho(\widehat{\Theta}^{n-1}_\varrho), 1)^h + 1\right] \\
&\le C\Big(1 + \frac{(2\widehat{a} + \widehat{b} + \widehat{c} + \widehat{d})\delta t}{\delta}\Big)\left[\frac{(\theta_{21})^2}{\theta_{22}\theta_{11}}(\mathfrak{G}_\varrho(\widehat{\Phi}^{n-1}_\varrho), 1)^h\right. \\
&\qquad \left. + (\mathfrak{G}_\varrho(\widehat{\Psi}^{n-1}_\varrho), 1)^h + \frac{(\theta_{23})^2}{\theta_{22}\theta_{33}}(\mathfrak{G}_\varrho(\widehat{\Theta}^{n-1}_\varrho), 1)^h + 1\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq C e^{\frac{(2\widehat{a}+\widehat{b}+\widehat{c}+\widehat{d})\delta t}{\delta}} \left[\frac{(\theta_{21})^2}{\theta_{22}\theta_{11}} (\mathfrak{G}_\varrho(\widehat{\Phi}^{n-1}_\varrho), 1)^h \right. \\
&\qquad \left. + (\mathfrak{G}_\varrho(\widehat{\Psi}^{n-1}_\varrho), 1)^h + \frac{(\theta_{23})^2}{\theta_{22}\theta_{33}} (\mathfrak{G}_\varrho(\widehat{\Theta}^{n-1}_\varrho), 1)^h + 1 \right] \\
&\vdots \\
&\leq C e^{T\frac{(2\widehat{a}+\widehat{b}+\widehat{c}+\widehat{d})}{\delta}} \left[\frac{(\theta_{21})^2}{\theta_{22}\theta_{11}} (\mathfrak{G}_\varrho(\widehat{\Phi}^{0}_\varrho), 1)^h + (\mathfrak{G}_\varrho(\widehat{\Psi}^{0}_\varrho), 1)^h \right. \\
&\qquad \left. + \frac{(\theta_{23})^2}{\theta_{22}\theta_{33}} (\mathfrak{G}_\varrho(\widehat{\Theta}^{0}_\varrho), 1)^h + 1 \right] \leq C, \quad \text{for all } n \geq 1, \;\; T = n\,\delta t. \qquad (88)
\end{aligned}
$$

By setting $\omega \equiv 1$ in (58) and considering the positivity of $\xi_\varrho(\gamma)$ and the given assumptions on δt, we obtain, for $n = 1, \ldots, N$, the following results:

$$
(\widehat{\Phi}^n_\varrho, 1)^h \leq \frac{1}{1 - a_1\delta t} (\widehat{\Phi}^{n-1}_\varrho, 1)^h \leq (1 + \frac{a_1\delta t}{\delta_1})(\widehat{\Phi}^{n-1}_\varrho, 1)^h \leq e^{\frac{a_1\delta t}{\delta_1}} (\widehat{\Phi}^{n-1}_\varrho, 1)^h. \qquad (89)
$$

From (89), the definition of the interpolation π^h, (69), and the assumption on u^0, it can be inferred that

$$
\max_{n=1,\ldots,N} (\widehat{\Phi}^n_\varrho, 1)^h \leq e^{\frac{a_1 T}{\delta_1}} (\widehat{\Phi}^0_\varrho, 1)^h \leq |\mathfrak{O}| e^{\frac{a_1 T}{\delta_1}} \|u^0\|_{0,\infty} \leq C. \qquad (90)
$$

For $n = 1, \ldots, N$, using the given equation $|s| = s - 2[s]_-$, along with (90) and Young's inequality, we can derive the following:

$$
\begin{aligned}
\|\widehat{\Phi}^n_\varrho\|_{0,1} =& (|\widehat{\Phi}^n_\varrho|, 1) \leq (\pi^h|\widehat{\Phi}^n_\varrho|, 1) \leq (\widehat{\Phi}^n_\varrho - 2\pi^h[\widehat{\Phi}^n_\varrho]_-, 1) \\
&= (\widehat{\Phi}^n_\varrho, 1)^h - 2(\pi^h[\widehat{\Phi}^n_\varrho]_-, 1) \leq C(1 + \|\pi[\widehat{\Phi}^n_\varrho]_-\|_0^2). \qquad (91)
\end{aligned}
$$

From (46), (45), (25), and (88), it can be deduced, after recalling the expressions $\gamma = [\gamma]_+ + [\gamma]_-$ and $\mathfrak{G}_\varrho(\gamma) \geq 0$ for $n = 1, \ldots, N$, that

$$
\|\pi^h[\widehat{\Phi}^n_\varrho]_-\|_0^2 \leq |\pi^h[\widehat{\Phi}^n_\varrho]_-|_h^2 = ([\widehat{\Phi}^n_\varrho]_-^2, 1)^h \leq 2\,\varrho(\mathfrak{G}_\varrho(\widehat{\Phi}^n_\varrho), 1)^h \leq C\,\varrho. \qquad (92)
$$

The other bounds, namely $\|\pi^h[\widehat{\Psi}^n_\varrho]_-\|_0^2$, $\|\pi^h[\widehat{\Theta}^n_\varrho]_-\|_0^2$, $\|\widehat{\Psi}^n_\varrho\|_{0,1}$, and $\|\widehat{\Theta}^n_\varrho\|_{0,1}$, can be obtained using the same methodology that has used to establish the above bounds. The bounds (1-9) in (85) have now been derived, and the last three bounds in (85) are obtained by summing (78) over n, with the assistance of (87). $\square$

Theorem 2. Under the assumptions of Lemma 7, the solution $\widehat{\Phi}^n_\varrho, \widehat{\Psi}^n_\varrho, \widehat{\Theta}^n_\varrho$ to problem $(\mathrm{Q}^{h,\delta t}_\varrho)$ meets

$$
\begin{aligned}
&\sum_{n=1}^{N}\delta t\Big[\|\widehat{\Phi}_{\varrho}^{n}\|_{0,\sigma}^{\sigma}+\|\xi_{\varrho}(\widehat{\Phi}_{\varrho}^{n})\|_{0,\sigma}^{\sigma}+\|\pi^{h}\xi_{\varrho}(\widehat{\Phi}_{\varrho}^{n})\|_{0,\sigma}^{\sigma}+\|\Xi_{\varrho}(\widehat{\Phi}_{\varrho}^{n})\|_{0,\sigma}^{\sigma}+\|\widehat{\Psi}_{\varrho}^{n}\|_{0,\sigma}^{\sigma}\\
&+\|\xi_{\varrho}(\widehat{\Psi}_{\varrho}^{n})\|_{0,\sigma}^{\sigma}+\|\pi^{h}\xi_{\varrho}(\widehat{\Psi}_{\varrho}^{n})\|_{0,\sigma}^{\sigma}+\|\Xi_{\varrho}(\widehat{\Psi}_{\varrho}^{n})\|_{0,\sigma}^{\sigma}+\|\widehat{\Theta}_{\varrho}^{n}\|_{0,\sigma}^{\sigma}+\|\xi_{\varrho}(\widehat{\Theta}_{\varrho}^{n})\|_{0,\sigma}^{\sigma}\\
&+\|\pi^{h}\xi_{\varrho}(\widehat{\Theta}_{\varrho}^{n})\|_{0,\sigma}^{\sigma}+\|\Xi_{\varrho}(\widehat{\Theta}_{\varrho}^{n})\|_{0,\sigma}^{\sigma}\Big]+\sum_{n=1}^{N}\delta t\Big[\|\frac{\widehat{\Phi}_{\varrho}^{n}-\widehat{\Phi}_{\varrho}^{n-1}}{\delta t}\|_{(W^{1,q'}(\mathfrak{O}))'}^{2}\\
&+\|\frac{\widehat{\Psi}_{\varrho}^{n}-\widehat{\Psi}_{\varrho}^{n-1}}{\delta t}\|_{(W^{1,q'}(\mathfrak{O}))'}^{2}+\|\frac{\widehat{\Theta}_{\varrho}^{n}-\widehat{\Theta}_{\varrho}^{n-1}}{\delta t}\|_{(W^{1,q'}(\mathfrak{O}))'}^{2}\Big]\\
&+\sum_{n=1}^{N}\delta t\Big[\|\ \mathfrak{G}_{q}[\frac{\widehat{\Phi}_{\varrho}^{n}-\widehat{\Phi}_{\varrho}^{n-1}}{\delta t}]\|_{1,q}^{q}+\|\ \mathfrak{G}_{q}[\frac{\widehat{\Psi}_{\varrho}^{n}-\widehat{\Psi}_{\varrho}^{n-1}}{\delta t}]\|_{1,q}^{q}\\
&+\|\ \mathfrak{G}_{q}[\frac{\widehat{\Theta}_{\varrho}^{n}-\widehat{\Theta}_{\varrho}^{n-1}}{\delta t}]\|_{1,q}^{q}\Big]\leq C,
\end{aligned}
\tag{93}
$$

where $\sigma=\frac{2(d+1)}{d},\ q=\frac{2(d+1)}{2d+1},\ q'=2(d+1)$.

Proof. For $n=1,\dots,N$, the relation (9) and the seventh bound in (85) imply that

$$\|\widehat{\Phi}_{\varrho}^{n}\|_{0,\sigma}^{\sigma}\leq\|\widehat{\Phi}_{\varrho}^{n}\|_{0,1}^{\sigma-2}\|\widehat{\Phi}_{\varrho}^{n}\|_{1}^{2}\leq C\|\widehat{\Phi}_{\varrho}^{n}\|_{1}^{2},\tag{94}$$

where $\sigma(\frac{2d(\sigma-1)}{\sigma(d+2)})=2$; that is $\sigma=\frac{2(d+1)}{d}$.
Given the inequality $\xi_{\varrho}(s)\leq|s|+\varrho$, equation (94) implies, for $n=1,\dots,N$, the following equation:

$$\|\xi_{\varrho}(\widehat{\Phi}_{\varrho}^{n})\|_{0,\sigma}^{\sigma}\leq C+\|\widehat{\Phi}_{\varrho}^{n}\|_{0,\sigma}^{\sigma}\leq C(1+\|\widehat{\Phi}_{\varrho}^{n}\|_{1}^{2}).\tag{95}$$

Based on $\xi_{\varrho}(s)\leq[s]_{+}+\varrho=s-[s]_{-}+\varrho$, the Young's inequality, and the fourth and seventh bounds in (85), we can conclude, for $n=1,\dots,N$, that

$$
\begin{aligned}
\|\pi^{h}\xi_{\varrho}(\widehat{\Phi}_{\varrho}^{n})\|_{0,1}&\leq\int_{\mathfrak{O}}(\widehat{\Phi}_{\varrho}^{n}-\pi^{h}[\widehat{\Phi}_{\varrho}^{n}]_{-}+\varrho)d\mathbf{x}\\
&\leq C(1+\|\widehat{\Phi}_{\varrho}^{n}\|_{0,1}+\|\pi^{h}[\widehat{\Phi}_{\varrho}^{n}]_{-}\|_{0}^{2})\leq C,
\end{aligned}
\tag{96}
$$

and

$$\|\pi^{h}\xi_{\varrho}(\widehat{\Phi}_{\varrho}^{n})\|_{0}^{2}\leq C(1+\|\widehat{\Phi}_{\varrho}^{n}\|_{0}^{2}+\|\pi^{h}[\widehat{\Phi}_{\varrho}^{n}]_{-}\|_{0}^{2})\leq C(1+\|\widehat{\Phi}_{\varrho}^{n}\|_{0}^{2}).\tag{97}$$

For $n=1,\dots,N$, equations (9), (96), (97), and (63) yield the following result:

$$\|\pi^{h}\xi_{\varrho}(\widehat{\Phi}_{\varrho}^{n})\|_{0,\sigma}^{\sigma}\leq C\|\pi^{h}\xi_{\varrho}(\widehat{\Phi}_{\varrho}^{n})\|_{0,1}^{\sigma-2}\|\pi^{h}\xi_{\varrho}(\widehat{\Phi}_{\varrho}^{n})\|_{1}^{2}\leq C(1+\|\widehat{\Phi}_{\varrho}^{n}\|_{1}^{2}).\tag{98}$$

The bounds $1 \to 3$ in (93) can be derived from (85) and noting (94)–(98).

The fourth bound in (93) can be derived from the third bound in (93) by utilizing (56), (57), (24), and (48) in the form

$$\begin{aligned}\|\Xi_\varrho(\widehat{\Phi}^n_\varrho)\|^\sigma_{0,\sigma} &:= \int_{\mathfrak{O}} \|\Xi_\varrho(\widehat{\Phi}^n_\varrho)\|^\sigma d\mathbf{x} = \sum_{\tilde{\tau}\in\mathcal{T}^h} \int_{\tilde{\tau}} \|\Xi_\varrho(\widehat{\Phi}^n_\varrho)\|^\sigma d\mathbf{x} \\ &\le C \sum_{\tilde{\tau}\in\mathcal{T}^h} h^d_{\tilde{\tau}} \|\pi^h \xi_\varrho(\widehat{\Phi}^n_\varrho)\|^\sigma_{0,\infty,\tilde{\tau}} \le C \sum_{\tilde{\tau}\in\mathcal{T}^h} \|\pi^h \xi_\varrho(\widehat{\Phi}^n_\varrho)\|^\sigma_{0,\sigma,\tilde{\tau}} \\ &\le C\|\pi^h \xi_\varrho(\widehat{\Phi}^n_\varrho)\|^\sigma_{0,\sigma}.\end{aligned} \tag{99}$$

Based on the definition of π^h and (69), as well as the assumption of u^0, v^0, and w^0, it follows that

$$\|\widehat{\Phi}^0_\varrho\|_0 + \|\widehat{\Psi}^0_\varrho\|_0 + \|\widehat{\Theta}^0_\varrho\|_0 \le C(\|u^0\|_{0,\infty} + \|v^0\|_{0,\infty} + \|w^0\|_{0,\infty}) \le C. \tag{100}$$

From (70), we can conclude that

$$\begin{aligned}(\Xi_\varrho(\widehat{\Phi}^n_\varrho)\nabla\widehat{\Phi}^n_\varrho, \nabla P^h\Sigma) &\le \|\Xi_\varrho(\widehat{\Phi}^n_\varrho)\|_{0,\sigma}|\widehat{\Phi}^n_\varrho|_1|P^h\Sigma|_{1,q'} \\ &\le C\|\Xi_\varrho(\widehat{\Phi}^n_\varrho)\|_{0,\sigma}|\widehat{\Phi}^n_\varrho|_1\left[|(I-P^h)\Sigma|_{1,q'} + |\Sigma|_{1,q'}\right] \\ &\le C\|\Xi_\varrho(\widehat{\Phi}^n_\varrho)\|_{0,\sigma}|\widehat{\Phi}^n_\varrho|_1\|\Sigma\|_{1,q'},\end{aligned} \tag{101}$$

where $\frac{1}{2} + \frac{1}{\sigma} + \frac{1}{q'} = 1$, that is $\frac{1}{q} + \frac{1}{q'} = 1$ and $q = \frac{2(d+1)}{2d+1}$. From (45), (46), (97), and (70), we obtain the following results for any $\Sigma \in W^{1,q'}(\mathfrak{O})$ and $n = 1, \dots, N$:

$$\begin{aligned}&(\xi_\varrho(\widehat{\Phi}^n_\varrho)\xi_\varrho(\widehat{\Phi}^{n-1}_\varrho), P^h\Sigma)^h \\ &\le C\|\pi^h\xi_\varrho(\widehat{\Phi}^n_\varrho)\|_{0,\sigma}\|\pi^h\xi_\varrho(\widehat{\Phi}^{n-1}_\varrho)\|_0\|P^h\Sigma\|_{0,q'} \\ &\le C\|\pi^h\xi_\varrho(\widehat{\Phi}^n_\varrho)\|_{0,\sigma}(1+\|\widehat{\Phi}^{n-1}_\varrho\|_0)\left[\|(I-P^h)\Sigma\|_{0,q'} + \|\Sigma\|_{0,q'}\right] \\ &\le C\|\pi^h\xi_\varrho(\widehat{\Phi}^n_\varrho)\|_{0,\sigma}(1+\|\widehat{\Phi}^{n-1}_\varrho\|_0)\|\Sigma\|_{1,q'}.\end{aligned} \tag{102}$$

Similarly to (102), we can derive that

$$(\xi_\varrho(\widehat{\Phi}^n_\varrho)\xi_\varrho(\widehat{\Psi}^{n-1}_\varrho), P^h\Sigma)^h \le C\|\pi^h\xi_\varrho(\widehat{\Phi}^n_\varrho)\|_{0,\sigma}(1+\|\widehat{\Psi}^{n-1}_\varrho\|_0)\|\Sigma\|_{1,q'}, \tag{103}$$

and

$$(\xi_\varrho(\widehat{\Phi}^n_\varrho)\xi_\varrho(\widehat{\Theta}^{n-1}_\varrho), P^h\Sigma)^h \le C\|\pi^h\xi_\varrho(\widehat{\Phi}^n_\varrho)\|_{0,\sigma}(1+\|\widehat{\Theta}^{n-1}_\varrho\|_0)\|\Sigma\|_{1,q'}. \tag{104}$$

Using (76), (68), (58), (70), (101), (102), (103), and (104), we can determine that for any $\Sigma \in W^{1,q'}(\mathfrak{O})$ and $n = 1, \ldots, N$, the following holds:

$$\begin{aligned}
\langle \frac{\widehat{\Phi}_\varrho^n - \widehat{\Phi}_\varrho^{n-1}}{\delta t},\ \Sigma\rangle_{q'} &= (\frac{\widehat{\Phi}_\varrho^n - \widehat{\Phi}_\varrho^{n-1}}{\delta t}, \Sigma) = (\frac{\widehat{\Phi}_\varrho^n - \widehat{\Phi}_\varrho^{n-1}}{\delta t},\ P^h\Sigma)^h \\
&= (a_1\widehat{\Phi}_\varrho^n - b_1\xi_\varrho(\widehat{\Phi}_\varrho^n)\xi_\varrho(\widehat{\Phi}_\varrho^{n-1}) - c_1\xi_\varrho(\widehat{\Phi}_\varrho^n)\xi_\varrho(\widehat{\Psi}_\varrho^{n-1}) \\
&\quad -d_1\xi_\varrho(\widehat{\Phi}_\varrho^n)\xi_\varrho(\widehat{\Theta}_\varrho^{n-1}),\ P^h\Sigma)^h \\
&\quad -(\nabla\widehat{\Phi}_\varrho^n + \Xi_\varrho(\widehat{\Phi}_\varrho^n)\nabla\widehat{\Phi}_\varrho^n,\ \nabla\ P^h\Sigma) \\
&\leq C\big(\|\widehat{\Phi}_\varrho^n\|_1 + \|\widehat{\Phi}_\varrho^{n-1}\|_0 + \|\widehat{\Psi}_\varrho^{n-1} + \|\widehat{\Theta}_\varrho^{n-1}\|_0 + 1\big) \\
&\quad \times\big(\|\pi^h\xi_\varrho(\widehat{\Phi}_\varrho^n)\|_{0,\sigma} + \|\Xi_\varrho(\widehat{\Phi}_\varrho^n)\|_{0,\sigma} + 1\big)\|\Sigma\|_{1,q'},
\end{aligned} \tag{105}$$

where we have utilized the continuous embedding of $W^{1,q'}(\mathfrak{O}) \hookrightarrow L^2(\mathfrak{O})$ and $W^{1,q'}(\mathfrak{O}) \hookrightarrow H^1(\mathfrak{O})$. Hence, we can conclude that

$$\|\frac{\widehat{\Phi}_\varrho^n - \widehat{\Phi}_\varrho^{n-1}}{\delta t}\|_{(W^{1,q'}(\mathfrak{O}))'} \leq CA_nB_n, \tag{106}$$

where

$$A_n = \big(\|\widehat{\Phi}_\varrho^n\|_1 + \|\widehat{\Phi}_\varrho^{n-1}\|_0 + \|\widehat{\Psi}_\varrho^{n-1} + \|\widehat{\Theta}_\varrho^{n-1}\|_0 + 1\big),$$

$$B_n = \big(\|\pi^h\xi_\varrho(\widehat{\Phi}_\varrho^n)\|_{0,\sigma} + \|\Xi_\varrho(\widehat{\Phi}_\varrho^n)\|_{0,\sigma} + 1\big).$$

Based on (106), the Cauchy–Schwarz inequality, the third and fourth bounds in (93), and the assumption on the discretization and (100), we can conclude that

$$\begin{aligned}
\sum_{n=1}^N \delta t_n \bigg\|\frac{\widehat{\Phi}_\varrho^n - \widehat{\Phi}_\varrho^{n-1}}{\delta t_n}\bigg\|^q_{(W^{1,q'}(\mathfrak{O}))'} &\leq C\sum_{n=1}^N \delta t_n A_n^q B_n^q \\
&\leq C\big[\sum_{n=1}^N \delta t_n A_n^2\big]^{\frac{q}{2}}\big[\sum_{n=1}^N \delta t_n B_n^\sigma\big]^{\frac{q}{\sigma}} \leq C,
\end{aligned} \tag{107}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$ and $q = \frac{2(d+1)}{2d+1}$. The final three results in (93) can be obtained from the thirteenth, fourteenth, and fifteenth results in (93), respectively, by considering the implications of (74). □

6 Uniqueness

To guarantee the stability and dependability of computational models, the uniqueness of numerical solutions for PDEs utilizing FEM must be investigated. Establishing uniqueness ensures that the numerical approximation depicts a single, physically significant answer as opposed to several potential possibilities. In scientific and technical applications, where precise predictions of physical behavior rely on consistent and well-posed mathematical formulations, this aids in the validation of simulations.

Theorem 3. Assuming the conditions of Theorem 2 are met, and $\widehat{\Phi}_\varrho^n, \widehat{\Psi}_\varrho^n, \widehat{\Theta}_\varrho^n$, $n = 1, \ldots, N$, is a solution of $(\mathrm{Q}_\varrho^{h,\delta t})$. If $\delta t \in (0, \varpi)$ and

$$\varpi < \max\{\frac{1}{e_1 + e_4 + e_9}, \frac{1}{e_2 + e_5 + e_7 + e_8}, \frac{1}{e_3 + e_6 + e_{10}}\}, \tag{108}$$

where the values of the parameters $e_i, i = 1, \ldots, 6$, were stated in the proof, then the solution $\{\widehat{\Phi}_\varrho^n, \widehat{\Psi}_\varrho^n, \widehat{\Theta}_\varrho^n\}$, $n = 1, \ldots, N$, is unique.

Proof. The above theorems are proved by employing an approach similar to that used in [32, 30, 2]. □

7 Existence of a weak solution

Since many physical and engineering issues involve functions that might not have classical derivatives, it is crucial to investigate whether a weak solution exists for PDEs. These issues can be stated and examined within a more comprehensive functional framework thanks to the weak formulation, which guarantees that solutions are found even in cases when classical ones are not.

We commence by employing the following definitions:

$$\widehat{\Phi}_\varrho(t) := (\frac{t - t_{n-1}}{\delta t_n})\widehat{\Phi}_\varrho^n + (\frac{t_n - t}{\delta t_n})\widehat{\Phi}_\varrho^{n-1}, \quad t \in [t_{n-1}, t_n], \quad n \geq 1, \tag{109}$$

$$\widehat{\Psi}_\varrho(t) := (\frac{t - t_{n-1}}{\delta t_n})\widehat{\Psi}_\varrho^n + (\frac{t_n - t}{\delta t_n})\widehat{\Psi}_\varrho^{n-1}, \quad t \in [t_{n-1}, t_n], \quad n \geq 1, \tag{110}$$

$$\widehat{\Theta}_\varrho(t) := (\frac{t - t_{n-1}}{\delta t_n})\widehat{\Theta}_\varrho^n + (\frac{t_n - t}{\delta t_n})\widehat{\Theta}_\varrho^{n-1}, \quad t \in [t_{n-1}, t_n], \quad n \geq 1, \tag{111}$$

and

$$\widehat{\Phi}_\varrho^+(t) := \widehat{\Phi}_\varrho^n, \quad \widehat{\Phi}_\varrho^-(t) := \widehat{\Phi}_\varrho^{n-1}, \qquad t \in (t_{n-1}, t_n], \qquad n \geq 1, \tag{112}$$

$$\widehat{\Psi}_\varrho^+(t) := \widehat{\Psi}_\varrho^n, \quad \widehat{\Psi}_\varrho^-(t) := \widehat{\Psi}_\varrho^{n-1}, \qquad t \in (t_{n-1}, t_n], \qquad n \geq 1, \tag{113}$$

$$\widehat{\Theta}_\varrho^+(t) := \widehat{\Theta}_\varrho^n, \quad \widehat{\Theta}_\varrho^-(t) := \widehat{\Theta}_\varrho^{n-1}, \qquad t \in (t_{n-1}, t_n], \qquad n \geq 1. \tag{114}$$

By considering (109), (110), (112), and (113), we can conclude that

$$\frac{\partial \widehat{\Phi}_\varrho}{\partial t} = \frac{\widehat{\Phi}_\varrho^+ - \widehat{\Phi}_\varrho^-}{\delta t} = \frac{\widehat{\Phi}_\varrho^+ - \widehat{\Phi}_\varrho}{t_n - t} = \frac{\widehat{\Phi}_\varrho^- - \widehat{\Phi}_\varrho}{t - t_{n-1}}, \ t \in (t_{n-1}, t_n), \ n \geq 1, \tag{115}$$

$$\frac{\partial \widehat{\Psi}_\varrho}{\partial t} = \frac{\widehat{\Psi}_\varrho^+ - \widehat{\Psi}_\varrho^-}{\delta t} = \frac{\widehat{\Psi}_\varrho^+ - \widehat{\Psi}_\varrho}{t_n - t} = \frac{\widehat{\Psi}_\varrho^- - \widehat{\Psi}_\varrho}{t - t_{n-1}}, \ t \in (t_{n-1}, t_n), \ n \geq 1, \tag{116}$$

$$\frac{\partial \widehat{\Theta}_\varrho}{\partial t} = \frac{\widehat{\Theta}_\varrho^+ - \widehat{\Theta}_\varrho^-}{\delta t} = \frac{\widehat{\Theta}_\varrho^+ - \widehat{\Theta}_\varrho}{t_n - t} = \frac{\widehat{\Theta}_\varrho^- - \widehat{\Theta}_\varrho}{t - t_{n-1}}, \ t \in (t_{n-1}, t_n), \ \ n \geq 1. \tag{117}$$

Using the definitions provided above, we can rewrite the problem $(Q_\varrho^{h,\delta t})$ as follows: Find $\{\widehat{\Phi}_\varrho, \widehat{\Psi}_\varrho, \widehat{\Theta}_\varrho\} \in C([0,T]; X^h) \times C([0,T]; X^h \times C([0,T]; X^h)$ so that for all $\omega \in L^2(0,T; X^h)$,

$$\begin{aligned}
&\int_0^T \left[(\frac{\partial \widehat{\Phi}_\varrho}{\partial t}, \omega)^h + (\theta_1 \nabla \widehat{\Phi}_\varrho^+ + 2\theta_{11} \Xi_\varrho(\widehat{\Phi}_\varrho^+) \nabla \widehat{\Phi}_\varrho^+, \ \nabla \omega) \right] dt \\
&= \int_0^T \left[(a_1 \widehat{\Phi}_\varrho^+ - b_1 \xi_\varrho(\widehat{\Phi}_\varrho^+) \xi_\varrho(\widehat{\Phi}_\varrho^-) - c_1 \xi_\varrho(\widehat{\Phi}_\varrho^+) \xi_\varrho(\widehat{\Psi}_\varrho^-) - d_1 \xi_\varrho(\widehat{\Phi}_\varrho^+) \xi_\varrho(\widehat{\Theta}_\varrho^-), \omega)^h \right] dt,
\end{aligned} \tag{118}$$

$$\begin{aligned}
&\int_0^T \Big[(\frac{\partial \widehat{\Psi}_\varrho}{\partial t}, \omega)^h + (\theta_2 \nabla \widehat{\Psi}_\varrho^+ + \theta_{21} \Xi_\varrho(\widehat{\Psi}_\varrho^+) \nabla \widehat{\Phi}_\varrho^+ + \theta_{21} \Xi_\varrho(\widehat{\Phi}_\varrho^+) \nabla \widehat{\Psi}_\varrho^+ \\
&+ 2\theta_{22} \Xi_\varrho(\widehat{\Psi}_\varrho^+) \nabla \widehat{\Psi}_\varrho^+ + \theta_{23} \Xi_\varrho(\widehat{\Psi}_\varrho^+) \nabla \widehat{\Theta}_\varrho^+ + \theta_{23} \Xi_\varrho(\widehat{\Theta}_\varrho^+) \nabla \widehat{\Psi}_\varrho^+, \nabla \omega) \Big] dt \\
&= \int_0^T \left[(a_2 \widehat{\Psi}_\varrho^+ - b_2 \xi_\varrho(\widehat{\Psi}_\varrho^+) \xi_\varrho(\widehat{\Phi}_\varrho^-) - c_2 \xi_\varrho(\widehat{\Psi}_\varrho^+) \xi_\varrho(\widehat{\Psi}_\varrho^- - d_2 \xi_\varrho(\widehat{\Psi}_\varrho^+) \xi_\varrho(\widehat{\Theta}_\varrho^-), \omega)^h \right] dt,
\end{aligned} \tag{119}$$

and

$$\begin{aligned}
&\int_0^T \left[(\frac{\partial \widehat{\Theta}_\varrho}{\partial t}, \omega)^h + (\theta_3 \nabla \widehat{\Theta}_\varrho^+ + 2\theta_{33} \Xi_\varrho(\widehat{\Theta}_\varrho^+) \nabla \widehat{\Theta}_\varrho^+, \ \nabla \omega) \right] dt \\
&= \int_0^T \left[(a_3 \widehat{\Theta}_\varrho^+ - b_3 \xi_\varrho(\widehat{\Theta}_\varrho^+) \xi_\varrho(\widehat{\Phi}_\varrho^-) - c_3 \xi_\varrho(\widehat{\Theta}_\varrho^+) \xi_\varrho(\widehat{\Psi}_\varrho^-) - d_3 \xi_\varrho(\widehat{\Theta}_\varrho^+) \xi_\varrho(\widehat{\Theta}_\varrho^-), \omega)^h \right] dt.
\end{aligned} \tag{120}$$

Theorem 4. Let the assumption (A) be valid and given $u^0, v^0, w^0 \in H^1_{\geq 0}(\mathfrak{O}) \cap L^\infty(\mathfrak{O})$, we consider the values of $\delta t, \varrho$, and h that satisfy the following conditions:

(i) either $\widehat{\Phi}_\varrho^0 \equiv P^h u^0$, $\widehat{\Psi}_\varrho^0 \equiv P^h v^0$, $\widehat{\Theta}_\varrho^0 \equiv P^h w^0$ or $\widehat{\Phi}_\varrho^0 \equiv \pi^h u^0$, $\widehat{\Psi}_\varrho^0 \equiv \pi^h v^0$, $\widehat{\Theta}_\varrho^0 \equiv \pi^h w^0$, if either $d = 1$ or $u^0, v^0, w^0 \in W^{1,r}(\mathfrak{O})$ with $r > d$,

(ii) $\delta t \leq \frac{1-\delta}{2\widehat{a}+\widehat{b}+\widehat{c}+\widehat{d}}$,

(iii) $\delta t \to 0, \ \ \varrho \to 0$ as $h \to 0$.

There is a subsequence of $\widehat{\Phi}_\varrho, \widehat{\Psi}\varrho, \widehat{\Theta}_\varrho$ that satisfies (118), (119), and (120), along with corresponding functions

$$u \in L^2(0,T;H^1(\mathfrak{O})) \cap L^\sigma(\mathfrak{O}_T) \cap W^{1,q}(0,T;(W^{1,q'}(\mathfrak{O}))'), \tag{121}$$

$$v \in L^2(0,T;H^1(\mathfrak{O})) \cap L^\sigma(\mathfrak{O}_T) \cap W^{1,q}(0,T;(W^{1,q'}(\mathfrak{O}))'), \tag{122}$$

also

$$w \in L^2(0,T;H^1(\mathfrak{O})) \cap L^\sigma(\mathfrak{O}_T) \cap W^{1,q}(0,T;(W^{1,q'}(\mathfrak{O}))'), \tag{123}$$

with

$$u(x,t), v(x,t), w(x,t) \geq 0, \quad a.e. \ on \ \ \mathfrak{O},$$

and

$$u(.,0) = u^0, \qquad v(.,0) = v^0, \qquad w(.,0) = w^0, \qquad \text{in } (W^{1,q'}(\mathfrak{O}))'. \tag{124}$$

As δt, ϱ, and h tend to zero, we can deduce that

$$\widehat{\Phi}_\varrho, \widehat{\Phi}_\varrho^\pm \rightharpoonup u, \ \widehat{\Psi}_\varrho, \widehat{\Psi}_\varrho^\pm \rightharpoonup v, \ \widehat{\Theta}_\varrho, \widehat{\Theta}_\varrho^\pm \rightharpoonup w, \ \text{in } L^2(0,T;H^1(\mathfrak{O})) \cap L^\sigma(\mathfrak{O}_T), \tag{125}$$

$$\widehat{\Phi}_\varrho, \widehat{\Phi}_\varrho^\pm \rightharpoonup^* u, \ \widehat{\Psi}_\varrho, \widehat{\Psi}_\varrho^\pm \rightharpoonup^* v, \ \widehat{\Theta}_\varrho, \widehat{\Theta}_\varrho^\pm \rightharpoonup^* w, \ \text{in } L^\infty(0,T;L^2(\mathfrak{O})), \tag{126}$$

$$\frac{\partial \widehat{\Phi}_\varrho}{\partial t} \rightharpoonup \frac{\partial u}{\partial t}, \ \frac{\partial \widehat{\Psi}_\varrho}{\partial t} \rightharpoonup \frac{\partial v}{\partial t}, \ \frac{\partial \widehat{\Theta}_\varrho}{\partial t} \rightharpoonup \frac{\partial w}{\partial t}, \ \text{in } L^q(0,T;(W^{1,q'}(\mathfrak{O}))'), \tag{127}$$

$$\widehat{\Phi}_\varrho, \widehat{\Phi}_\varrho^\pm \to u, \ \widehat{\Psi}_\varrho, \widehat{\Psi}_\varrho^\pm \to v, \ \widehat{\Theta}_\varrho, \widehat{\Theta}_\varrho^\pm \to w, \ \text{in } L^2(0,T;L^{s_1}(\mathfrak{O})), \tag{128}$$

$$\xi_\varrho(\widehat{\Phi}_\varrho^\pm) \to u, \ \xi_\varrho(\widehat{\Psi}_\varrho^\pm) \to v, \ \ \xi_\varrho(\widehat{\Theta}_\varrho^\pm) \to w, \ \text{in } L^2(0,T;L^{s_1}(\mathfrak{O})), \tag{129}$$

$$\pi^h \xi_\varrho(\widehat{\Phi}_\varrho^\pm) \to u, \ \pi^h \xi_\varrho(\widehat{\Psi}_\varrho^\pm) \to v, \ \ \pi^h \xi_\varrho(\widehat{\Theta}_\varrho^\pm) \to w, \ \text{in } L^2(0,T;L^{s_1}(\mathfrak{O})), \tag{130}$$

$$\Xi_\varrho(\widehat{\Phi}_\varrho^\pm) \to u\mathcal{I} \ \Xi_\varrho(\widehat{\Psi}_\varrho^\pm) \to v\mathcal{I} \ \Xi_\varrho(\widehat{\Theta}_\varrho^\pm) \to w\mathcal{I} \text{ in } L^2(0,T;L^{s_1}(\mathfrak{O})), \tag{131}$$

for any

$$s_1 \in \begin{cases} [2,\infty] & \text{if } d=1, \\ [2,\infty) & \text{if } d=2, \\ [2,6) & \text{if } d=3. \end{cases}$$

Proof. Under the assumptions (i)-(ii) and utilizing (85), (93), (24), (56), (57), and (109)–(117), it can be proved that the following results, which do not depend on ϱ, h, or δt, are hold:

$$\begin{aligned}
&\|\widehat{\Phi}_\varrho^\pm\|_{L^2(0,T;H^1(\mathfrak{D}))} + \|\widehat{\Phi}_\varrho^\pm\|_{L^\sigma(\mathfrak{D}_T)} + \|\widehat{\Phi}_\varrho^\pm\|_{L^\infty(0,T;L^1(\mathfrak{D}))}\\
&+\varrho^{\frac{-1}{2}}\|\pi^h[\widehat{\Phi}_\varrho^\pm]_-\|_{L^\infty(0,T;L^2(\mathfrak{D}))} + \|\frac{\partial\widehat{\Phi}_\varrho}{\partial t}\|_{L^q(0,T;(W^{1,q'}(\mathfrak{D}))')}\\
&+\|\mathfrak{G}\frac{\partial\widehat{\Phi}_\varrho}{\partial t}\|_{L^q(0,T;W^{1,q}(\mathfrak{D}))} + \|\xi_\varrho(\widehat{\Phi}_\varrho^\pm)\|_{L^\sigma(\mathfrak{D}_T)}\\
&+\|\pi^h\xi_\varrho(\widehat{\Phi}_\varrho^\pm)\|_{L^\sigma(\mathfrak{D}_T)} + \|\Xi_\varrho(\widehat{\Phi}_\varrho^\pm)\|_{L^\sigma(\mathfrak{D}_T)} \le C,
\end{aligned}\tag{132}$$

$$\begin{aligned}
&\|\widehat{\Psi}_\varrho^\pm\|_{L^2(0,T;H^1(\mathfrak{D}))} + \|\widehat{\Psi}_\varrho^\pm\|_{L^\sigma(\mathfrak{D}_T)} + \|\widehat{\Psi}_\varrho^\pm\|_{L^\infty(0,T;L^1(\mathfrak{D}))}\\
&+\varrho^{\frac{-1}{2}}\|\pi^h[\widehat{\Psi}_\varrho^\pm]_-\|_{L^\infty(0,T;L^2(\mathfrak{D}))} + \|\frac{\partial\widehat{\Psi}_\varrho}{\partial t}\|_{L^q(0,T;(W^{1,q'}(\mathfrak{D}))')}\\
&+\|\mathfrak{G}\frac{\partial\widehat{\Psi}_\varrho}{\partial t}\|_{L^q(0,T;W^{1,q}(\mathfrak{D}))} + \|\xi_\varrho(\widehat{\Psi}_\varrho^\pm)\|_{L^\sigma(\mathfrak{D}_T)}\\
&+\|\pi^h\xi_\varrho(\widehat{\Psi}_\varrho^\pm)\|_{L^\sigma(\mathfrak{D}_T)} + \|\Xi_\varrho(\widehat{\Psi}_\varrho^\pm)\|_{L^\sigma(\mathfrak{D}_T)} \le C,
\end{aligned}\tag{133}$$

and

$$\begin{aligned}
&\|\widehat{\Theta}_\varrho^\pm\|_{L^2(0,T;H^1(\mathfrak{D}))} + \|\widehat{\Theta}_\varrho^\pm\|_{L^\sigma(\mathfrak{D}_T)} + \|\widehat{\Theta}_\varrho^\pm\|_{L^\infty(0,T;L^1(\mathfrak{D}))}\\
&+\varrho^{\frac{-1}{2}}\|\pi^h[\widehat{\Theta}_\varrho^\pm]_-\|_{L^\infty(0,T;L^2(\mathfrak{D}))} + \|\frac{\partial\widehat{\Theta}_\varrho}{\partial t}\|_{L^q(0,T;(W^{1,q'}(\mathfrak{D}))')}\\
&+\|\mathfrak{G}\frac{\partial\widehat{\Theta}_\varrho}{\partial t}\|_{L^q(0,T;W^{1,q}(\mathfrak{D}))} + \|\xi_\varrho(\widehat{\Theta}_\varrho^\pm)\|_{L^\sigma(\mathfrak{D}_T)}\\
&+\|\pi^h\xi_\varrho(\widehat{\Theta}_\varrho^\pm)\|_{L^\sigma(\mathfrak{D}_T)} + \|\Xi_\varrho(\widehat{\Theta}_\varrho^\pm)\|_{L^\sigma(\mathfrak{D}_T)} \le C,
\end{aligned}\tag{134}$$

where $(\pm)$ is an adopted abbreviation for "with" and "without" the superscripts "+" and "-". Based on (115) and the fifth bound in (132), we observe that

$$\begin{aligned}
&\|\widehat{\Phi}_\varrho^\pm - \widehat{\Phi}_\varrho\|^q_{L^q(0,T;(W^{1,q'}(\mathfrak{D}))')}\\
&= \int_0^T \|\widehat{\Phi}_\varrho^\pm - \widehat{\Phi}_\varrho\|^q_{(W^{1,q'}(\mathfrak{D}))'}\,dt\\
&= \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \|\widehat{\Phi}_\varrho^\pm - \widehat{\Phi}_\varrho\|^q_{(W^{1,q'}(\mathfrak{D}))'}\,dt = \sum_{n=1}^N \int_{t_{n-1}}^{t_n} |t-t_n^\pm|^q\|\frac{\partial\widehat{\Phi}_\varrho}{\partial t}\|^q_{(W^{1,q'}(\mathfrak{D}))'}\,dt\\
&\le \sum_{n=1}^N (\delta t)^q \int_{t_{n-1}}^{t_n} \|\frac{\partial\widehat{\Phi}_\varrho}{\partial t}\|^q_{(W^{1,q'}(\mathfrak{D}))'}\,dt \le (\delta t)^q \int_0^T \|\frac{\partial\widehat{\Phi}_\varrho}{\partial t}\|^q_{(W^{1,q'}(\mathfrak{D}))'}\,dt
\end{aligned}$$

$$\leq (\delta t)^q \|\frac{\partial \widehat{\Phi}_\varrho}{\partial t}\|^q_{L^q(0,T;(W^{1,q'}(\mathfrak{O}))')} \leq C(\delta t)^q. \tag{135}$$

Similarly to (135), we can deduce, from (135), (116), (117), (133), and (134), that

$$\|\widehat{\Psi}^{\pm}_\varrho - \widehat{\Psi}_\varrho\|^q_{L^q(0,T;(W^{1,q'}(\mathfrak{O}))')} \leq (\delta t)^q, \tag{136}$$

and

$$\|\widehat{\Theta}^{\pm}_\varrho - \widehat{\Theta}_\varrho\|^q_{L^q(0,T;(W^{1,q'}(\mathfrak{O}))')} \leq (\delta t)^q. \tag{137}$$

Before completing the proof, it is necessary to state the following facts. The space $L^\infty(0,T;L^2(\mathfrak{O}))$ is the dual space of $L^1(0,T;L^2(\mathfrak{O}))$, which is a separable Banach space but not reflexive. On the other hand, the spaces $L^2(0,T;H^1(\mathfrak{O}))$ and $L^\sigma(\mathfrak{O}_T)$ are reflexive Banach spaces. By employing compactness arguments and utilizing the bounds (132), we can extract subsequences denoted as $\{\widehat{\Phi}^{\pm}_\varrho\}_h$ and $\{\widehat{\Phi}_\varrho\}_h$. These subsequences satisfy the results (125) and (126) as $h \to 0$. Furthermore, since $\{\frac{\partial \widehat{\Phi}_\varrho}{\partial t}\}_h \in L^q(0,T;(W^{1,q'}(\mathfrak{O}))')$, which are reflexive Banach spaces, by the weak compactness theorem, there exists a subsequence $\{\frac{\partial \widehat{\Phi}_\varrho}{\partial t}\}_h \in L^q(0,T;(W^{1,q'}(\mathfrak{O}))')$ and a function $\widetilde{\widehat{\eta}} \in L^q(0,T;(W^{1,q'}(\mathfrak{O}))')$ such that

$$\frac{\partial \widehat{\Phi}_\varrho}{\partial t} \rightharpoonup \widetilde{\widehat{\eta}} \qquad \text{in } L^q(0,T;(W^{1,q'}(\mathfrak{O}))').$$

Using a well-known method outlined in [37], page 204, it can be readily demonstrated that $\widetilde{\widehat{\eta}} = \frac{\partial \widehat{\Phi}}{\partial t}$. Consequently, the result (127) is obtained.

From (125) and (126), it follows that $\widehat{\Phi}$ belongs to $L^2(0,T;H^1(\mathfrak{O}))$, $L^\sigma(\mathfrak{O}_T)$, and $L^\infty(0,T;L^2(\mathfrak{O}))$. To prove (121), we need to demonstrate that $\widehat{\Phi}$ also belongs to $W^{1,q}(0,T;(W^{1,q'}(\mathfrak{O}))')$. Using the embedding $L^2(0,T;H^1(\mathfrak{O})) \hookrightarrow L^q(0,T;(W^{1,q'}(\mathfrak{O}))')$, we conclude that $\widehat{\Phi} \in L^q(0,T;(W^{1,q'}(\mathfrak{O}))')$. Furthermore, from (127), we have $\frac{\partial \widehat{\Phi}}{\partial t} \in L^q(0,T;(W^{1,q'}(\mathfrak{O}))')$, therefore, it follows that

$$\|\widehat{\Phi}\|_{W^{1,q}(0,T;(W^{1,q'}(\mathfrak{O}))')} = \|\widehat{\Phi}\|_{L^q(0,T;(W^{1,q'}(\mathfrak{O}))')} + \|\frac{\partial \widehat{\Phi}}{\partial t}\|_{L^q(0,T;(W^{1,q'}(\mathfrak{O}))')} \leq C.$$

Therefore, (121) has been demonstrated. We can establish (122) by employing a similar approach as in (121).

Based on the given compact embedding result

$$\begin{aligned}\mathcal{H}_u = \Big\{\Sigma, \Sigma \in L^2(0,T;H^1(\mathfrak{O})), \frac{\partial \Sigma}{\partial t} \in L^q(0,T;(W^{1,q'}(\mathfrak{O}))')\Big\} \\ \overset{c}{\hookrightarrow} L^2(0,T;L^{s_1}(\mathfrak{O})),\end{aligned}$$

we have, as $\widehat{\Phi}_\varrho \in L^2(0,T;H^1(\mathfrak{O}))$ and $\frac{\partial \widehat{\Phi}_\varrho}{\partial t} \in L^q(0,T;(W^{1,q'}(\mathfrak{O}))')$, that $\widehat{\Phi}_\varrho \in \mathcal{H}_e$, then We may extract a subsequence, still assigned $\widehat{\Phi}_\varrho$, that supports the convergence outcome (128).

By utilizing the strong convergence of $\widehat{\Phi}_\varrho$ to u in $L^2(0,T;L^{s_1}(\mathfrak{O}))$, along with the fourth bound in (132), we can extract a subsequence, denoted as $\widehat{\Phi}_\varrho$ as well, such that as $h \to 0$, the following condition holds

$$\widehat{\Phi}_\varrho \to u \qquad \text{and} \qquad \pi^h[\widehat{\Phi}_\varrho]_- \to 0 \qquad a.e. \text{ in } \mathfrak{O} \times (0,T). \tag{138}$$

The definition of π^h implies that

$$\widehat{\Phi}_\varrho = \pi^h[\widehat{\Phi}_\varrho]_+ + \pi^h[\widehat{\Phi}_\varrho]_-. \tag{139}$$

From (138) and (139), we can conclude that $u \geq 0$ almost everywhere in $\mathfrak{O}$.

To display the outcomes (129)–(131), it is important to initially observe that

$$\begin{aligned} &\|\xi_\varrho(\widehat{\Phi}^\pm_\varrho) - u\|_{L^2(0,T;L^{s_1}(\mathfrak{O}))} \\ &\leq \|\xi_\varrho(\widehat{\Phi}^\pm_\varrho) - \xi_\varrho(u)\|_{L^2(0,T;L^{s_1}(\mathfrak{O}))} + \|\xi_\varrho(u) - u\|_{L^2(0,T;L^{s_1}(\mathfrak{O}))}. \end{aligned} \tag{140}$$

By employing the fact that $u \in L^2(0,T;H^1(\mathfrak{O})) \hookrightarrow L^2(0,T;L^{s_1}(\mathfrak{O}))$, along with (24), the non-negativity of $\widehat{\Phi}$, assumption (iii), and applying the dominated convergence theorem, we can reach the conclusion that

$$\|\xi_\varrho(u) - u\|_{L^2(0,T;L^{s_1}(\mathfrak{O}))} \to 0 \quad \text{as} \quad h \to 0. \tag{141}$$

The fact that ξ_ϱ exhibits Lipschitz continuity, along with the relationship stated in (128), leads to the inference that

$$\|\xi_\varrho(\widehat{\Phi}^\pm_\varrho) - \xi_\varrho(u)\|_{L^2(0,T;L^{s_1}(\mathfrak{O}))} \leq \|\widehat{\Phi}^\pm_\varrho - u\|_{L^2(0,T;L^{s_1}(\mathfrak{O}))} \to 0 \quad \text{as} \quad h \to 0. \tag{142}$$

By employing (49), (24), (48), and the first bound stated in (132), it is possible to deduce that

$$\begin{aligned} &\|(I-\pi^h)\xi_\varrho(\widehat{\Phi}^\pm_\varrho)\|_{L^2(0,T;L^{s_1}(\mathfrak{O}))} + \|\Xi_\varrho(\widehat{\Phi}^\pm_\varrho) - u\,\mathcal{I}\|_{L^2(0,T;L^{s_1}(\mathfrak{O}))} \\ &\qquad \leq Ch^{1-d(\frac{1}{2}-\frac{1}{s_1})}\|\xi_\varrho(\widehat{\Phi}^\pm_\varrho) - u\|_{L^2(0,T;H^1(\mathfrak{O}))}. \end{aligned} \tag{143}$$

Combining (140), (141), (142), and (7) yields the convergence results (129)–(131). The convergence results for $\widehat{\Psi}^\pm_\varrho$ and $\widehat{\Theta}^\pm_\varrho$ can be proven using the same approach as the one followed to prove the convergence results for $\widehat{\Phi}^\pm_\varrho$.

To complete this proof, we need to show that the solution u, v, w, based on the initial approximations, satisfies (124). This can be deduced from the error estimates (49) and (70), as well as the given assumptions on u^0, where we have that

$$\|u^0 - P^h u^0\|_0 \leq Ch|u^0|_1 \leq Ch,$$

and

$$\|u^0 - \pi^h u^0\|_0 \leq \begin{cases} Ch|u^0|_1 \leq Ch, \ \text{for } d = 1, \\ \\ Ch|u^0|_{1,r} \leq Ch, \ \text{for } d = 1 \ \textit{or} \ d = 2, \end{cases} \tag{144}$$

and similarly for v^0 and w^0. These results exhibit strong convergence as $h \to 0$, in the form

$$\widehat{\Phi}^0_\varrho \to u^0, \quad \widehat{\Psi}^0_\varrho \to v^0, \quad \widehat{\Theta}^0_\varrho \to w^0, \quad \text{in} \quad L^2(\mathfrak{O}). \tag{145}$$

By (129), we can conclude that

$$\widehat{\Phi}_\varrho(t) \to u(t), \ \widehat{\Psi}_\varrho(t) \to v(t), \ \widehat{\Theta}_\varrho(t) \to w(t), \ \text{in } L^2(\mathfrak{O}), \ \textit{as } \varrho \to 0. \tag{146}$$

We observe that (145), and (146) alone do not suffice to prove the result in (124). This is because if $t = 0$ belongs to the null-set of the "almost everywhere" condition for (146), it is possible that $u(0) \neq u^0$, $v(0) \neq v^0$, and $w(0) \neq w^0$ (as discussed in [37], Section 7.4.4). Additionally, we rely on other properties of the solutions $\widehat{\Phi}_\varrho, \widehat{\Psi}\varrho, \widehat{\Theta}_\varrho$ and the function u, v, w to establish the validity of (124). Firstly, we note that

$$\widehat{\Phi}_\varrho, \, u, \widehat{\Psi}_\varrho, \, v, \widehat{\Theta}_\varrho, \, w \in W^{1,q}(0, T; (W^{1,q'}(\mathfrak{O}))').$$

Consequently, we have that

$$\widehat{\Phi}_\varrho, \, u, \ \widehat{\Psi}_\varrho, \, v, \ \widehat{\Theta}_\varrho, \, w \in C([0, T]; (W^{1,q'}(\mathfrak{O}))'); \tag{147}$$

see [37, Proposition 7.1 and Theorem 7.2], respectively. By noting the (145)–(147), the desired result (124) follows directly. □

Lemma 8. Assuming the conditions of Theorem 4 are satisfied, it follows, as $h \to 0$, that

$$\pi^h \xi_\varrho(\widehat{\Phi}^+_\varrho) \pi^h \xi_\varrho(\widehat{\Phi}^-_\varrho) \to u^2 \qquad \text{in } L^q(\mathfrak{O}_T), \tag{148}$$

$$\pi^h \xi_\varrho(\widehat{\Psi}^+_\varrho) \pi^h \xi_\varrho(\widehat{\Psi}^-_\varrho) \to v^2 \qquad \text{in } L^q(\mathfrak{O}_T), \tag{149}$$

$$\pi^h\xi_\varrho(\widehat{\Theta}_\varrho^+)\pi^h\xi_\varrho(\widehat{\Theta}_\varrho^-)\to w^2 \qquad \text{in } L^q(\mathfrak{O}_T), \tag{150}$$

$$\pi^h\xi_\varrho(\widehat{\Phi}_\varrho^+)\pi^h\xi_\varrho(\widehat{\Psi}_\varrho^-)\to uv \qquad \text{in } L^q(\mathfrak{O}_T), \tag{151}$$

$$\pi^h\xi_\varrho(\widehat{\Theta}_\varrho^+)\pi^h\xi_\varrho(\widehat{\Phi}_\varrho^-)\to wu \qquad \text{in } L^q(\mathfrak{O}_T), \tag{152}$$

$$\pi^h\xi_\varrho(\widehat{\Theta}_\varrho^+)\pi^h\xi_\varrho(\widehat{\Psi}_\varrho^-)\to wv \qquad \text{in } L^q(\mathfrak{O}_T), \tag{153}$$

where $q=\frac{2(d+1)}{2d+1}$.

Proof. By utilizing Hölder's inequality, together with (132) and (121), we can deduce that

$$\begin{aligned}
&\|\pi^h\xi_\varrho(\widehat{\Phi}_\varrho^+)\pi^h\xi_\varrho(\widehat{\Phi}_\varrho^-)-u^2\|_{L^q(\mathfrak{O}_T)}\le\|\pi^h\xi_\varrho(\widehat{\Phi}_\varrho^-)-u\|_{L^2(\mathfrak{O}_T)}\|\pi^h\xi_\varrho(\widehat{\Phi}_\varrho^+)\|_{L^\sigma(\mathfrak{O}_T)}\\
&+\|\pi^h\xi_\varrho(\widehat{\Phi}_\varrho^+)-u\|_{L^2(\mathfrak{O}_T)}\|u\|_{L^\sigma(\mathfrak{O}_T)}\le C\big(\|\pi^h\xi_\varrho(\widehat{\Phi}_\varrho^-)-u\|_{L^2(\mathfrak{O}_T)}\\
&+\|\pi^h\xi_\varrho(\widehat{\Phi}_\varrho^+)-u\|_{L^2(\mathfrak{O}_T)}\big).
\end{aligned}\tag{154}$$

Hence, the outcome (148) can be deduced from (154), (128), and (130), while taking into consideration that $L^2(0,T;L^{s_1}(\mathfrak{O}))$ is embedded in $L^2(\mathfrak{O}_T)$. In a similar manner, the results (149)–(151) can be established by employing analogous methods as those utilized in establishing the proof for (148). □

Theorem 5. Let the conditions of Theorem 4 be fulfilled, there exists a subsequence of $\{\widehat{\Phi}_\varrho,\widehat{\Psi}\varrho,\widehat{\Theta}_\varrho\}$ that satisfies (118), (119), and (120). Consequently, the functions $\{u,v,w\}$ represent a global weak solution in the specified context:

$$\begin{aligned}
&\int_0^T\Big[\langle\frac{\partial u}{\partial t},\ \Sigma\rangle+(\theta_1\nabla u+2\theta_{11}u\nabla u,\ \nabla\Sigma)\Big]dt\\
&=\int_0^T\Big[(a_1u-b_1u^2-c_1uv-d_1uw,\Sigma)\Big]dt,\qquad \text{for all}\quad \Sigma\in L^{q'}(0,T;W^{1,q'}(\mathfrak{O})),
\end{aligned}\tag{155}$$

$$\begin{aligned}
&\int_0^T\Big[\langle\frac{\partial v}{\partial t},\ \Sigma\rangle+(\theta_2\nabla v+\theta_{21}v\nabla u+\theta_{21}u\nabla v+2\theta_{22}v\nabla v+\theta_{23}v\nabla w+\theta_{23}w\nabla v,\nabla\Sigma)\Big]dt\\
&=\int_0^T\Big[(a_2v-b_2vu-c_2v^2-d_2vw,\Sigma)^h\Big]dt,\quad \text{for all}\quad \Sigma\in L^{q'}(0,T;W^{1,q'}(\mathfrak{O})),
\end{aligned}\tag{156}$$

and

$$\begin{aligned}
&\int_0^T\Big[\langle\frac{\partial w}{\partial t},\ \Sigma\rangle+(\theta_3\nabla w+2\theta_{33}w\nabla w,\ \nabla\Sigma)\Big]dt\\
&=\int_0^T\Big[(a_3w-b_3wu-c_3wv-d_3w^2,\Sigma)\Big]dt,\ \text{for all}\quad \Sigma\in L^{q'}(0,T;W^{1,q'}(\mathfrak{O})),
\end{aligned}\tag{157}$$

where $q' = 2(d+1)$.

Proof. A proof of the aforementioned theorems can be obtained by applying a procedure similar to the one described in [32, 30, 2]. □

8 Numerical results

This section concentrates on numerical results concerning the three-species SKT model. An iterative approach has been developed to solve the nonlinear system of equations arising from problem $(\mathrm{Q}_{\varrho}^{h,\delta t})$. Moreover, numerical error analyses have been presented and examined for the SKT model with Dirichlet non-homogeneous boundary conditions in one, two, and three dimensions. Additionally, numerical simulations have been performed for the one- and two-dimensional cases. The numerical simulations were performed using MATLAB.

8.1 Numerical algorithm

At each time step, we suggest the following practical algorithm for solving the nonlinear algebraic system that emerges from problem $(\mathrm{Q}_{\varrho}^{h,\delta t})$: $(\mathrm{Q}_{\varrho,k}^{h\,\delta t})$: Given $\{\widehat{\Phi}_{\varrho}^{n,0}, \widehat{\Psi}_{\varrho}^{n,0}\} \in X^h \times X^h$ for $k \geq 1$ find $\{\widehat{\Phi}_{\varrho}^{n,k}, \widehat{\Psi}_{\varrho}^{n,k}\} \in X^h \times X^h$ such that for all $\omega \in X^h$

$$\begin{aligned}
&\Big(\frac{\widehat{\Phi}_{\varrho}^{n,k} - \widehat{\Phi}_{\varrho}^{n-1}}{\delta t}, \omega\Big)^h + (\theta_1 \nabla \widehat{\Phi}_{\varrho}^{n,k} + 2\theta_{11} \Xi_{\varrho}(\widehat{\Phi}_{\varrho}^{n,k-1}) \nabla \widehat{\Phi}_{\varrho}^{n,k}, \nabla \omega) \\
&= (a_1 \widehat{\Phi}_{\varrho}^{n,k} - b_1 \xi_{\varrho}(\widehat{\Phi}_{\varrho}^{n-1}) \xi_{\varrho}(\widehat{\Phi}_{\varrho}^{n,k-1}) - c_1 \xi_{\varrho}(\widehat{\Psi}_{\varrho}^{n-1}) \xi_{\varrho}(\widehat{\Phi}_{\varrho}^{n,k-1}) \\
&\quad - d_1 \xi_{\varrho}(\widehat{\Theta}_{\varrho}^{n-1}) \xi_{\varrho}(\widehat{\Phi}_{\varrho}^{n,k-1}), \omega)^h,
\end{aligned} \tag{158}$$

$$\begin{aligned}
&\Big(\frac{\widehat{\Psi}_{\varrho}^{n,k} - \widehat{\Psi}_{\varrho}^{n-1}}{\delta t}, \omega\Big)^h + (\theta_2 \nabla \widehat{\Psi}_{\varrho}^{n,k} + \theta_{21} \Xi_{\varrho}(\widehat{\Psi}_{\varrho}^{n,k-1}) \nabla \widehat{\Phi}_{\varrho}^{n,k} \\
&+ \theta_{21} \Xi_{\varrho}(\widehat{\Phi}_{\varrho}^{n,k-1}) \nabla \widehat{\Psi}_{\varrho}^{n,k} + 2\theta_{22} \Xi_{\varrho}(\widehat{\Psi}_{\varrho}^{n,k-1}) \nabla \widehat{\Psi}_{\varrho}^{n,k} \\
&+ \theta_{23} \Xi_{\varrho}(\widehat{\Psi}_{\varrho}^{n,k-1}) \nabla \widehat{\Theta}_{\varrho}^{n,k} + \theta_{23} \Xi_{\varrho}(\widehat{\Theta}_{\varrho}^{n,k-1}) \nabla \widehat{\Psi}_{\varrho}^{n,k}, \nabla \omega) \\
&\quad = (a_2 \widehat{\Psi}_{\varrho}^{n,k} - b_2 \xi_{\varrho}(\widehat{\Phi}_{\varrho}^{n-1}) \xi_{\varrho}(\widehat{\Psi}_{\varrho}^{n,k-1}) - c_2 \xi_{\varrho}(\widehat{\Psi}_{\varrho}^{n-1}) \xi_{\varrho}(\widehat{\Psi}_{\varrho}^{n,k-1}) \\
&\quad\quad - d_2 \xi_{\varrho}(\widehat{\Theta}_{\varrho}^{n-1}) \xi_{\varrho}(\widehat{\Psi}_{\varrho}^{n,k-1}), \omega)^h,
\end{aligned} \tag{159}$$

$$\Big(\frac{\widehat{\Theta}_{\varrho}^{n,k}-\widehat{\Theta}_{\varrho}^{n-1}}{\delta t},\omega\Big)^h+(\theta_3\nabla\widehat{\Theta}_{\varrho}^{n,k}+2\theta_{33}\Xi_{\varrho}(\widehat{\Theta}_{\varrho}^{n,k-1})\nabla\widehat{\Theta}_{\varrho}^{n,k},\nabla\omega)$$
$$=(a_3\widehat{\Theta}_{\varrho}^{n,k}-b_3\xi_{\varrho}(\widehat{\Phi}_{\varrho}^{n-1})\xi_{\varrho}(\widehat{\Theta}_{\varrho}^{n,k-1})-c_3\xi_{\varrho}(\widehat{\Psi}_{\varrho}^{n-1})\xi_{\varrho}(\widehat{\Theta}_{\varrho}^{n,k-1})$$
$$-d_3\xi_{\varrho}(\widehat{\Theta}_{\varrho}^{n-1})\xi_{\varrho}(\widehat{\Theta}_{\varrho}^{n,k-1}),\omega)^h. \tag{160}$$

Let us start with $\widehat{\Phi}_{\varrho}^{0}=\pi^h u^0$, $\widehat{\Psi}_{\varrho}^{0}=\pi^h v^0$ and $\widehat{\Theta}_{\varrho}^{0}=\pi^h W^0$. For $n\geq 1$, we define $\widehat{\Phi}_{\varrho}^{n,0}=\widehat{\Phi}_{\varrho}^{n-1}$, $\widehat{\Psi}_{\varrho}^{n,0}=\widehat{\Psi}_{\varrho}^{n-1}$ and $\widehat{\Theta}_{\varrho}^{n,0}=\widehat{\Theta}_{\varrho}^{n-1}$. The standard method for solving (158) - (160) involves testing the equations with η_j, $j=0,\ldots,J$, to obtain a $(2J+2)^3$ linear equations with $(2J+2)^3$ unknowns. $\widehat{\Phi}^{n,k}\varrho,\widehat{\Psi}^{n,k}\varrho$ and $\widehat{\Theta}^{n,k}\varrho$. This resulting system can be solved using the Gauss-Seidel iterative method. In our numerical results, we set $TOL=10^{-8}$ and used the stopping criteria:

$$\max\Big\{|\widehat{\Phi}_{\varrho}^{n,k}-\widehat{\Phi}_{\varrho}^{n,k-1}|_{0,\infty},|\widehat{\Psi}_{\varrho}^{n,k}-\widehat{\Psi}_{\varrho}^{n,k-1}|_{0,\infty},|\widehat{\Theta}_{\varrho}^{n,k}-\widehat{\Theta}_{\varrho}^{n,k-1}|_{0,\infty}\Big\}<TOL. \tag{161}$$

For a given k that satisfies (161), we define $\widehat{\Phi}_{\varrho}^{n}=\widehat{\Phi}_{\varrho}^{n,k}$, $\widehat{\Psi}_{\varrho}^{n}=\widehat{\Psi}_{\varrho}^{n,k}$ and $\widehat{\Theta}_{\varrho}^{n}=\widehat{\Theta}_{\varrho}^{n,k}$.

Although we could not confirm the convergence of $\{\widehat{\Phi}_{\varrho}^{n,k},\widehat{\Psi}_{\varrho}^{n,k},\widehat{\Theta}_{\varrho}^{n,k}\}_{k=1}^{\infty}$ to $\{\widehat{\Phi}_{\varrho}^{n},\widehat{\Psi}_{\varrho}^{n},\widehat{\Theta}_{\varrho}^{n}\}$ for a fixed value of n, strong convergence characteristics were observed in practice. The iterative method consistently achieved good convergence, often requiring only a few steps at each stage to meet the termination criteria.

8.2 Error computations

To quantify the error, we can introduce source terms $f_1(\mathbf{x},t)$, $f_2(\mathbf{x},t)$ and $f_3(\mathbf{x},t)$ to the problem (Q). This modification transforms (1)–(3) into the following equations:
(Q) Find $\{u,v,w\}$ such that

$$\partial_t u=\Delta(\theta_1 u+\theta_{11}u^2)+(a_1-b_1u-c_1v-d_1w)u+f_1(\mathbf{x},t),\quad \text{in }\mathfrak{O},\quad t>0, \tag{162}$$

$$\partial_t v=\Delta(\theta_2 v+\theta_{21}uv+\theta_{22}v^2+\theta_{23}vw)+(a_2-b_2u-c_2v-d_2w)v+f_2(\mathbf{x},t)\ \text{ in }\mathfrak{O},\quad t>0,$$

$$\partial_t w=\Delta(\theta_3 w+\theta_{33}w^2)+(a_3-b_3u-c_3v-d_3w)w+f_3(\mathbf{x},t),\quad \text{in }\mathfrak{O},\quad t>0, \tag{163}$$

We consider the following discrete FEM of $(\mathrm{Q}_{\varrho,k}^{h,\delta t})$:
Given $\widehat{\Phi}_{\varrho}^{n,0},\widehat{\Psi}_{\varrho}^{n,0}\in X^h\times X^h$, we seek $\widehat{\Phi}_{\varrho}^{n,k},\widehat{\Psi}_{\varrho}^{n,k}\in X^h\times X^h$ for $k\geq 1$ such that for all $\omega\in X^h$, we have that

$$\Big(\frac{\widehat{\Phi}_{\varrho}^{n,k}-\widehat{\Phi}_{\varrho}^{n-1}}{\delta t},\omega\Big)^h+(\theta_1\nabla\widehat{\Phi}_{\varrho}^{n,k}+2\theta_{11}\Xi_{\varrho}(\widehat{\Phi}_{\varrho}^{n,k-1})\nabla\widehat{\Phi}_{\varrho}^{n,k},\nabla\omega)$$

$$-(a_1\widehat{\Phi}_\varrho^{n,k} - b_1\xi_\varrho(\widehat{\Phi}_\varrho^{n-1})\xi_\varrho(\widehat{\Phi}_\varrho^{n,k-1}) - c_1\xi_\varrho(\widehat{\Psi}_\varrho^{n-1})\xi_\varrho(\widehat{\Phi}_\varrho^{n,k-1})$$
$$-d_1\xi_\varrho(\widehat{\Theta}_\varrho^{n-1})\xi_\varrho(\widehat{\Phi}_\varrho^{n,k-1}),\omega)^h = (f_1(\mathbf{x},t_n),\ \omega)^h, \tag{164}$$

$$\begin{aligned}&\big(\frac{\widehat{\Psi}_\varrho^{n,k} - \widehat{\Psi}_\varrho^{n-1}}{\delta t},\omega\big)^h + (\theta_2\nabla\widehat{\Psi}_\varrho^{n,k} + \theta_{21}\Xi_\varrho(\widehat{\Psi}_\varrho^{n,k-1})\nabla\widehat{\Phi}_\varrho^{n,k}\\
&+\theta_{21}\Xi_\varrho(\widehat{\Phi}_\varrho^{n,k-1})\nabla\widehat{\Psi}_\varrho^{n,k} + 2\theta_{22}\Xi_\varrho(\widehat{\Psi}_\varrho^{n,k-1})\nabla\widehat{\Psi}_\varrho^{n,k} + \theta_{23}\Xi_\varrho(\widehat{\Psi}_\varrho^{n,k-1})\nabla\widehat{\Theta}_\varrho^{n,k}\\
&+\theta_{23}\Xi_\varrho(\widehat{\Theta}_\varrho^{n,k-1})\nabla\widehat{\Psi}_\varrho^{n,k},\nabla\omega) - (a_2\widehat{\Psi}_\varrho^{n,k} - b_2\xi_\varrho(\widehat{\Phi}_\varrho^{n-1})\xi_\varrho(\widehat{\Psi}_\varrho^{n,k-1})\\
&-c_2\xi_\varrho(\widehat{\Psi}_\varrho^{n-1})\xi_\varrho(\widehat{\Psi}_\varrho^{n,k-1}) - d_2\xi_\varrho(\widehat{\Theta}_\varrho^{n-1})\xi_\varrho(\widehat{\Psi}_\varrho^{n,k-1}),\omega)^h = (f_2(\mathbf{x},t_n),\ \omega)^h,\end{aligned} \tag{165}$$

$$\big(\frac{\widehat{\Theta}_\varrho^{n,k} - \widehat{\Theta}_\varrho^{n-1}}{\delta t},\omega\big)^h + (\theta_3\nabla\widehat{\Theta}_\varrho^{n,k} + 2\theta_{33}\Xi_\varrho(\widehat{\Theta}_\varrho^{n,k-1})\nabla\widehat{\Theta}_\varrho^{n,k},\nabla\omega)$$
$$-(a_3\widehat{\Theta}_\varrho^{n,k} - b_3\xi_\varrho(\widehat{\Phi}_\varrho^{n-1})\xi_\varrho(\widehat{\Theta}_\varrho^{n,k-1}) - c_3\xi_\varrho(\widehat{\Psi}_\varrho^{n-1})\xi_\varrho(\widehat{\Theta}_\varrho^{n,k-1})$$
$$-d_3\xi_\varrho(\widehat{\Theta}_\varrho^{n-1})\xi_\varrho(\widehat{\Theta}_\varrho^{n,k-1}),\omega)^h = (f_3(\mathbf{x},t_n),\ \omega)^h. \tag{166}$$

8.2.1 One-dimensional accuracy

To assess the accuracy of our approximation, we solve (162)–(163) with non-homogeneous Dirichlet boundary conditions. Let's illustrate this with a numerical example. We evaluate the error of the approximate solution when all parameters of the problem are equal to 1 and $\mathfrak{O} = [0,1]$. The initial conditions, boundary conditions, and source terms $f(x,t)$ and $g(x,t)$ can be obtained from the corresponding exact solution. We divide the interval $\mathfrak{O} = [0,1]$ into J equal sub-intervals, with $h = 1/J$ as the element mesh size, and $\delta t = 10^{-6}$ as the time step. We consider the following exact solution:

$$u = \exp(-t+x), \qquad v = \exp(-t+x^2), \qquad w = \exp(-t+x^3).$$

The error norms in L^2, L^∞ for a set of simulations were shown in Table 1 and the order of convergence of L^2, L^∞-error are introduced in Table 2.

8.2.2 Two-dimensional accuracy

Here is an example illustrating the error of approximate solutions for the two-dimensional case. We focus on investigating the Dirichlet boundary conditions for (162)–(163). In this numerical

Table 1: Discrete L^2, L^∞-error for one-dimensional case at $T = 1$ with $\theta_i = \theta_{ii} = \theta_{21} = \theta_{23} = a_i = b_i = c_i = d_i = 1$, $i = 1, 2, 3$.

	$\|u - \widehat{\Phi}_\varrho^n\|$		$\|v - \widehat{\Psi}_\varrho^n\|$		$\|w - \widehat{\Theta}_\varrho^n\|$	
J	L^2	L^∞	L^2	L^∞	L^2	L^∞
10	6.6E-05	2.7E-04	1.4E-03	5.3E-03	3.2E-03	1.3E-02
20	1.2E-05	7.0E-05	2.3E-04	1.3E-03	5.5E-04	3.3E-03
25	6.6E-06	4.5E-05	1.3E-04	8.6E-04	3.1E-04	2.1E-03
40	2.0E-06	1.7E-05	4.0E-05	3.4E-04	9.6E-05	8.4E-04
50	1.1E-06	1.1E-05	2.3E-05	2.2E-04	5.5E-05	5.4E-04
100	1.8E-07	2.6E-06	4.0E-06	5.4E-05	9.6E-06	1.3E-04

Table 2: The order of convergence of L^2, L^∞-error for one-dimensional case at $T = 1$ with $\theta_i = \theta_{ii} = \theta_{21} = \theta_{23} = a_i = b_i = c_i = d_i = 1$, $i = 1, 2, 3$.

	$\|u - \widehat{\Phi}_\varrho^n\|$		$\|v - \widehat{\Psi}_\varrho^n\|$		$\|w - \widehat{\Theta}_\varrho^n\|$	
J	L^2	L^∞	L^2	L^∞	L^2	L^∞
20	2.459	1.948	2.606	2.027	2.541	1.978
25	2.679	1.980	2.557	1.852	2.569	2.026
40	2.540	2.071	2.508	1.974	2.494	1.950
50	2.679	1.951	2.480	1.951	2.496	1.980
100	2.611	2.081	2.524	2.026	2.518	2.054

example, we consider a domain $\mathfrak{O} = [0,1]\times[0,1]$ and a time interval $[0,1]$. The problem parameter values were fixed to be 1. We utilize exact solutions to determine the initial and boundary conditions, as well as the source terms $f(x, y, t)$ and $g(x, y, t)$. For the numerical computation, we choose a time step of $\delta t = 10^{-6}$. The exact solutions are provided below:

$$u = \exp(-t + x + y), \qquad v = \exp(-t + x^2 + y^2), \qquad w = \exp(-t + x^3 + y^3).$$

Table 3 presents the errors in the L^2 and L^∞ norms. The order of convergence of L^2 and L^∞ norms are shown in Table 4.

8.2.3 Three-dimensional accuracy

This section introduces an example that demonstrates the error of approximate solutions for a three-dimensional case. The specific focus of investigation is on the Dirichlet boundary conditions applied to (162)–(163). The domain is selected as $\mathfrak{O} = [0,1]\times[0,1]\times[0,1]$, and the time interval is $[0,1]$. The problem's parameter values are set as follows: $\theta_i = 0.1, \theta_{ii} = 0.001, \theta_{21} = \theta_{23} = 0.01, a_i = b_i = c_i = d_i = 1$, where $i = 1, 2, 3$. The exact solutions are utilized to determine the

Table 3: Discrete L^2, L^∞-error for two-dimensional case at $T = 1$ with $\theta_i = \theta_{ii} = \theta_{21} = \theta_{23} = a_i = b_i = c_i = d_i = 1,\ i = 1, 2, 3$.

J	$\Vert u - \widehat{\Phi}_\varrho^n \Vert$		$\Vert v - \widehat{\Psi}_\varrho^n \Vert$		$\Vert w - \widehat{\Theta}_\varrho^n \Vert$	
	L^2	L^∞	L^2	L^∞	L^2	L^∞
10	7.1E-06	9.4E-05	6.3E-04	9.4E-03	1.4E-03	2.3E-02
20	7.3E-07	2.3E-05	7.2E-05	2.4E-03	1.6E-04	5.9E-03
25	3.6E-07	1.5E-05	3.6E-05	1.5E-03	8.3E-05	3.8E-03
40	9.7E-08	6.1E-06	8.6E-06	6.1E-04	2.0E-05	1.5E-03
50	5.3E-08	3.9E-06	4.4E-06	3.9E-04	1.0E-05	9.7E-04

Table 4: The order of convergence of L^2, L^∞-error for two-dimensional case at $T = 1$ with $\theta_i = \theta_{ii} = \theta_{21} = \theta_{23} = a_i = b_i = c_i = d_i = 1,\ i = 1, 2, 3$.

J	$\Vert u - \widehat{\Phi}_\varrho^n \Vert$		$\Vert v - \widehat{\Psi}_\varrho^n \Vert$		$\Vert w - \widehat{\Theta}_\varrho^n \Vert$	
	L^2	L^∞	L^2	L^∞	L^2	L^∞
20	3.282	2.031	3.129	1.970	3.129	1.963
25	3.168	1.916	3.106	2.106	2.941	1.972
40	2.790	1.914	3.046	1.914	3.028	1.978
50	2.970	2.005	3.003	2.005	3.018	1.954

initial and boundary conditions, as well as the source terms $f(x, y, z, t)$ and $g(x, y, z, t)$. The value of the time step is chosen such that $\delta t = 10^{-4}$. The exact solutions are

$$u = \exp(-t + x + y + z), \quad v = \exp(-t + x^2 + y^2 + z^2), \quad w = \exp(-t + x^3 + y^3 + z^3).$$

Table 5 displays the errors calculated using the L^2, and L^∞ norms and the order of convergence of L^2, and L^∞ norms are presented in Table 6.

Here we will make notes regarding the results of Tables 1-6. We note that the highest error values were obtained by the solution v, and the reason is that the second equation, through which

Table 5: Discrete L^2, L^∞-error for three-dimensional case at $T = 1$ with $\theta_i = 0.1, \theta_{ii} = 0.001, \theta_{21} = \theta_{23} = 0.01, a_i = b_i = c_i = d_i = 1,\ i = 1, 2, 3$.

J	$\Vert u - \widehat{\Phi}_\varrho^n \Vert$		$\Vert v - \widehat{\Psi}_\varrho^n \Vert$		$\Vert w - \widehat{\Theta}_\varrho^n \Vert$	
	L^2	L^∞	L^2	L^∞	L^2	L^∞
10	5.5E-05	3.4E-03	1.1E-04	7.5E-03	1.2E-04	7.7E-03
20	4.2E-06	9.1E-04	8.4E-06	2.1E-03	9.6E-06	2.2E-03
25	1.9E-06	5.8E-04	3.8E-06	1.4E-03	4.3E-06	1.5E-03
40	3.2E-07	2.2E-04	7.1E-07	5.3E-04	8.1E-07	5.9E-04

Table 6: The order of convergence of L^2, L^∞-error for three-dimensional case at $T = 1$ with $\theta_i = 0.1, \theta_{ii} = 0.001, \theta_{21} = \theta_{23} = 0.01, a_i = b_i = c_i = d_i = 1,\ i = 1, 2, 3$.

	$\Vert u - \widehat{\Phi}_\varrho^n \Vert$		$\Vert v - \widehat{\Psi}_\varrho^n \Vert$		$\Vert w - \widehat{\Theta}_\varrho^n \Vert$	
J	L^2	L^∞	L^2	L^∞	L^2	L^∞
20	3.711	1.902	3.711	1.837	3.644	1.807
25	3.555	2.019	3.555	1.817	3.599	1.716
40	3.790	2.063	3.569	2.067	3.552	1.985

the solution v is calculated, includes two additional cross-diffusion terms. This generates a higher approximation error than in the first and third equations. The lowest error value was recorded for the solution u, although the first equation, through which the solution u is calculated, is similar to the third equation, through which the solution w is calculated. The reason for this is that the form of the analytical solution associated with the solution w has a higher non-linearity than what is the case in the solution u. For the results that represent the error value of the three-dimensional problem, which were presented in Table 5, we were not able to choose the value of the time step $\delta t = 10^{-6}$ because this requires large calculations. Therefore, we chose the value of the diffusion parameters to be in the form $\theta_i = 0.1, \theta_{ii} = 0.001, \theta_{21} = \theta_{23} = 0.01$, where $i = 1, 2, 3$. This selection of diffusion parameter values makes the numerical method stable when choosing the time step $\delta t = 10^{-4}$.

8.3 One-dimensional simulations

In the subsequent numerical results, we performed simulations in one space dimension, utilizing a uniform partition of the domain $\mathfrak{O} = [0, 5]$ consisting of 256 subintervals ($J = 256$). The time step size employed was $\delta t = 10^{-5}$. The initial conditions for the simulations were set as follows: $u(x, 0) = 0.5 + 0.5\cos(\frac{4\pi x}{5})$, $v(x, 0) = 0.5 - 0.5\cos(\frac{4\pi x}{5})$, and $w(x, 0) = 0.5 + 0.3\cos(\frac{4\pi x}{5})$. The parameters used in these experiments are fixed as $\theta_{21} = \theta_{23} = a_i = 1$ for $i = 1, 2, 3$, and $b_2 = b_3 = c_1 = c_3 = d_1 = d_2 = 1$, while $b_1 = c_2 = d_3 = 0$. However, we varied the values of θ_{ii} and θ_i for each experiment. Throughout all the experiments, solutions were graphically represented at various carefully chosen time points to showcase the evolution of the interacting cells as time (t) increases. The selected time points were $T = 0, 0.01, 0.1, 0.5$, and 1, enabling a clear visualization of the system's behavior at different stages of time progression.

In the first experiment, we set $\theta_i = \theta_{ii} = 1$ for $i = 1, 2, 3$, and the numerical solutions of $(\mathrm{Q}_{\varrho,k}^{h\,\delta t})$ are displayed in Figures 1 (a), (b), and (c). A stable state was noticed in the solution. In the subsequent experiment, we performed a similar investigation, but this time, we altered the values of θ_i. Precisely, we assigned $\theta_i = 10$ and $\theta_{ii} = 1$ for $i = 1, 2, 3$. The outcomes are depicted in Figures 1 (d), (e), and (f). The general pattern remained in line with our earlier observations. An essential discovery in these figures is that the equilibrium solutions were attained more rapidly compared to the first experiment. Next, we conducted a similar experiment, but this time with $\theta_i = 0.1$ and $\theta_{ii} = 1$ for $i = 1, 2, 3$. The results are depicted in Figures 1 (g), (h), and (i). The general behavior of the system remained consistent with the observations made in previous cases. However, we noticed that the stationary solutions were reached at a later time compared to the previous experiments. In the fourth experiment, we selected $\theta_i = 1$ and $\theta_{ii} = 0$ for $i = 1, 2, 3$. The numerical results for various time points are shown in Figures 2 (a), (b), and (c). As in the third experiment, it was observed that reaching the stationary solution took longer compared to the first and second experiments.

8.4 Two-dimensional simulations

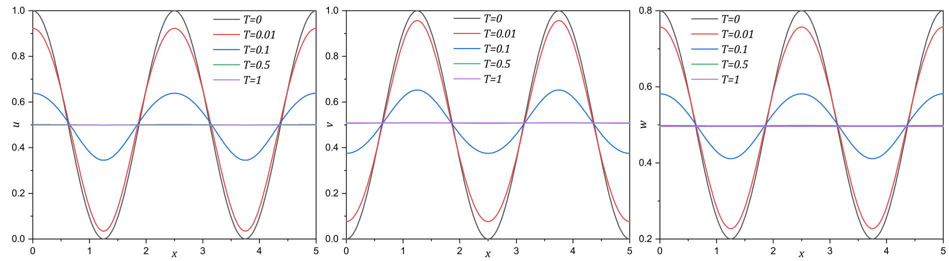
To showcase the development of spatial arrangements in the three-species SKT model, we examine a two-dimensional region denoted by $\mathfrak{O} = (0, 5)^2$, which is divided into a mesh containing 80000 triangles. The model incorporates the subsequent parameter values: $\theta_{21} = 0.1, \theta_{23} = 0.2, a_1 = b_1 = 1, a_2 = 0.2, a_3 = 0.15, b_2 = 0.0037, b_3 = 0.0033, c_1 = 0.001, c_2 = 0.25, c_3 = d_2 = 0.0025, d_1 = 0.002$ and $d_3 = 0.1667$. In the experiments, the values of θ_i and θ_{ii}, $i = 1, 2, 3$, are varied to observe their impact on the solutions. The system's stable equilibrium for the chosen parameter values is given by $(u^*, v^*, w^*) = (0.9749, 0.7769, 0.8689)$. The initial conditions are as follows:

$$u(x, 0) = 0.9749 + 0.5 \sin(0.8\pi x) \cos(0.8\pi y),$$

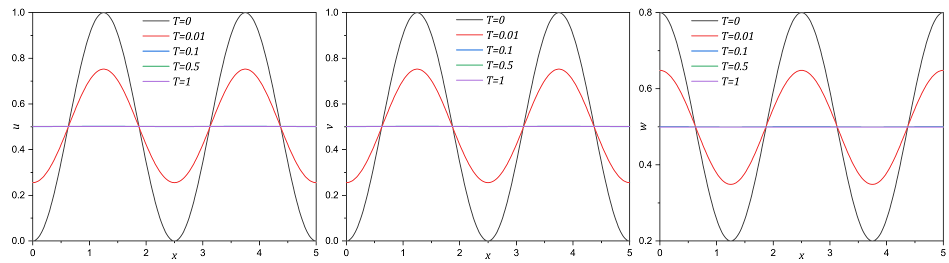
$$v(x, 0) = 0.7769 + 0.2 \cos(0.8\pi x) \sin(0.8\pi y),$$

$$w(x, 0) = 0.8689 + 0.3 \sin(0.8\pi x) \sin(0.8\pi y).$$

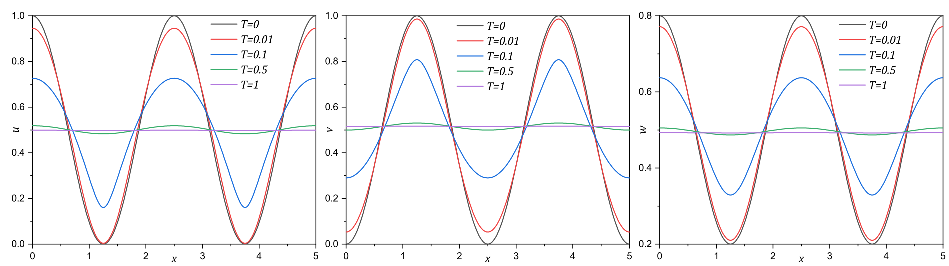
The results of all experiments were presented at the following times: $T = 0.5, 1, 2, 3, 6, 9, 12$ and 15. The values of the self-diffusion coefficients were chosen in Figures 3–6 such that $\theta_{ii} = 0.001$, $i = 1, 2, 3$. The values of diffusion coefficients were selected in Figures 3, 4, 5, and 6 such that $\theta_i = 0.005, 0.001, 0.0005$, and 0.0001, $i = 1, 2, 3$, respectively. In Figure 3, $\theta_i = 0.005$, it was observed that the solution tended to reach the equilibrium point, as the solution was very close to reaching it at the last time. Then, with the decrease in the values of θ_i, the solution was chosen to the equilibrium point as well, but it became slower. This indicates the significant stabilizing



(a) $u\&\theta_i = 1$ (b) $v\&\theta_i = 1$ (c) $w\&\theta_i = 1$

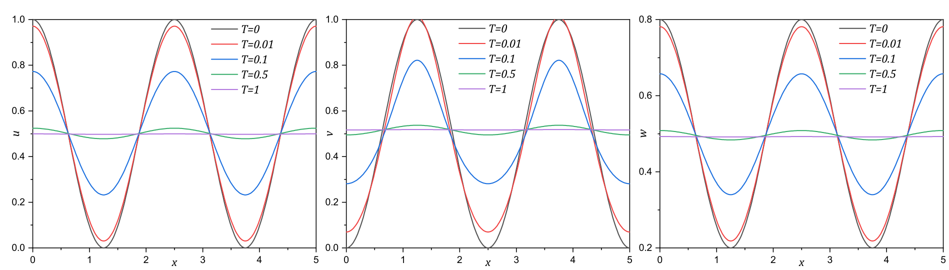


(d) $u\&\theta_i = 10$ (e) $v\&\theta_i = 10$ (f) $w\&\theta_i = 10$



(g) $u\&\theta_i = 0.1$ (h) $v\&\theta_i = 0.1$ (i) $w\&\theta_i = 0.1$

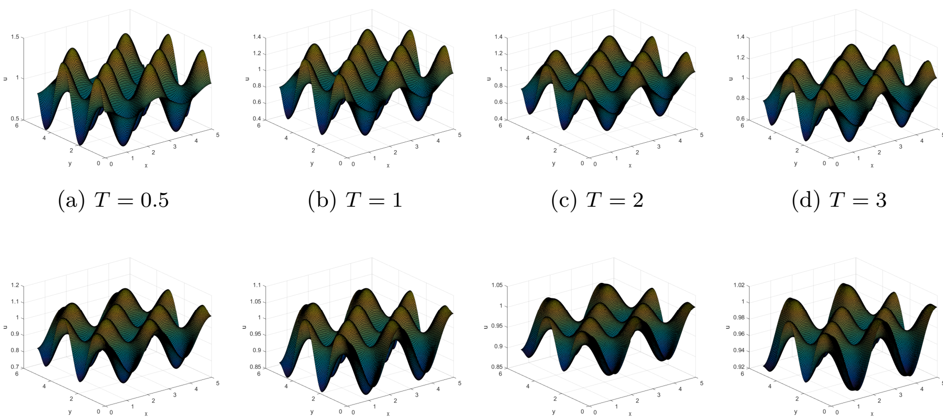
Figure 1: Numerical solutions for the one-dimensional case at different time points for $\theta_{ii} = \theta_{21} = \theta_{23} = a_i = 1$, $i = 1, 2, 3$, $b_2 = b_3 = c_1 = c_3 = d_1 = d_2 = 1$ and $b_1 = c_2 = d_3 = 0$.



(a) u (b) v (c) w

Figure 2: Numerical solutions for the one-dimensional case at different time points for $\theta_{ii} = 0$, $\theta_i = \theta_{21} = \theta_{23} = a_i = 1$, $i = 1, 2, 3$, $b_2 = b_3 = c_1 = c_3 = d_1 = d_2 = 1$ and $b_1 = c_2 = d_3 = 0$.

effect of the diffusion terms. In the last experiment, Figure 7, we fixed the values of diffusion coefficients at $\theta_i = 0.0001$, $i = 1, 2, 3$, and increased the values of self-diffusion coefficients to $\theta_{ii} = 0.01$, $i = 1, 2, 3$ to confirm the effect of the self-diffusion terms on the stability of the system. In this figure, it is observed that the solution moves faster than when the values of the self-diffusion coefficients are $\theta_{ii} = 0.001$, $i = 1, 2, 3$. This confirms the significant stabilizing effect of the self-diffusion terms.



(a) $T = 0.5$ (b) $T = 1$ (c) $T = 2$ (d) $T = 3$

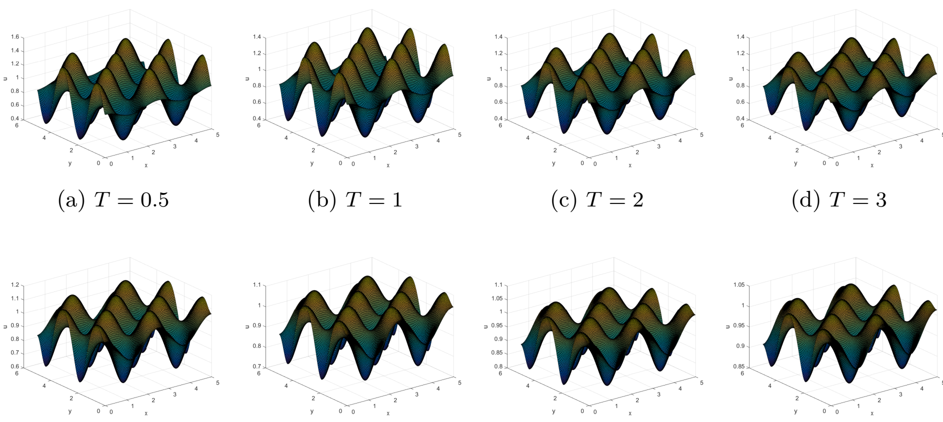
(e) $T = 6$ (f) $T = 9$ (g) $T = 12$ (h) $T = 15$

Figure 3: Numerical solution u for the two-dimensional case at different time points for $\theta_i = 0.005$ and $\theta_{ii} = 0.001$, $i = 1, 2, 3$.

9 Conclusions

This article looks into how we can use FEMs to understand the three-species SKT model better. It starts by outlining the main challenges of the model and explains a method to regularize it, ensuring some mathematical proofs to support this. Then, it describes an FEM to get approximate solutions for this adjusted model. The study shows how these approximations link back to the original model, proving that solutions for the SKT model do exist. Finally, it introduces a practical algorithm for solving the approximated problem and presents numerical results. This whole process helps us gain insight into the behavior of the SKT model in real-world scenarios.

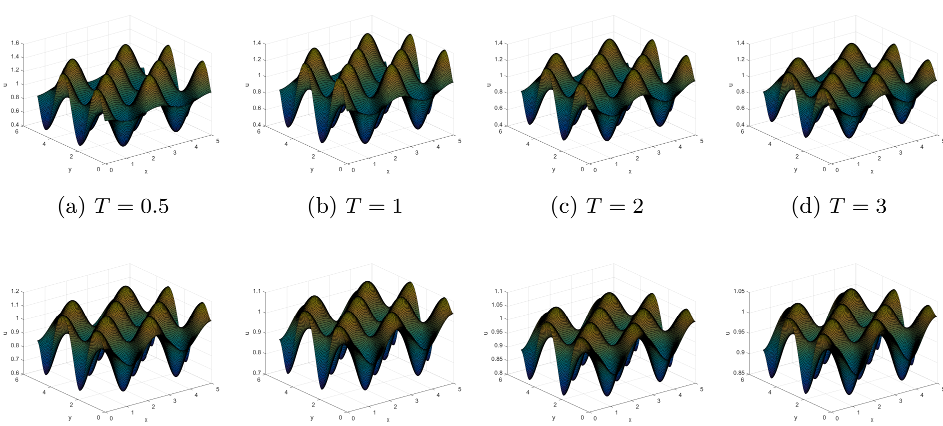
It has been found that using FEM is a reliable way to manage complicated boundary conditions and varied spaces. The numerical results showed interesting patterns formed by cross-diffusion, which lined up with what theory suggested about how species can segregate. Overall,



(a) $T = 0.5$ (b) $T = 1$ (c) $T = 2$ (d) $T = 3$

(e) $T = 6$ (f) $T = 9$ (g) $T = 12$ (h) $T = 15$

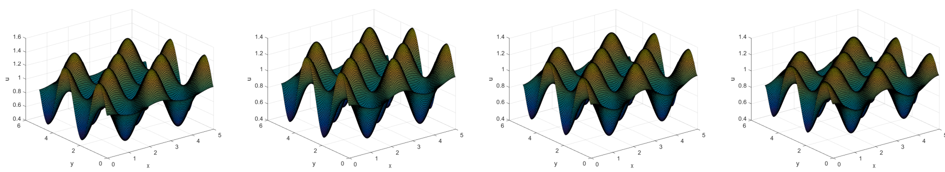
Figure 4: Numerical solution u for the two-dimensional case at different time points for $\theta_i = 0.001$ and $\theta_{ii} = 0.001$, $i = 1, 2, 3$.



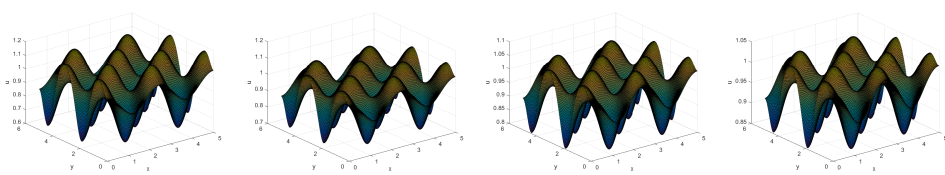
(a) $T = 0.5$ (b) $T = 1$ (c) $T = 2$ (d) $T = 3$

(e) $T = 6$ (f) $T = 9$ (g) $T = 12$ (h) $T = 15$

Figure 5: Numerical solution u for the two-dimensional case at different time points for $\theta_i = 0.0005$ and $\theta_{ii} = 0.001$, $i = 1, 2, 3$.

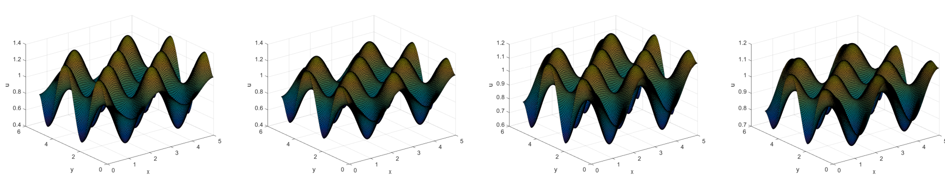


(a) $T = 0.5$ (b) $T = 1$ (c) $T = 2$ (d) $T = 3$

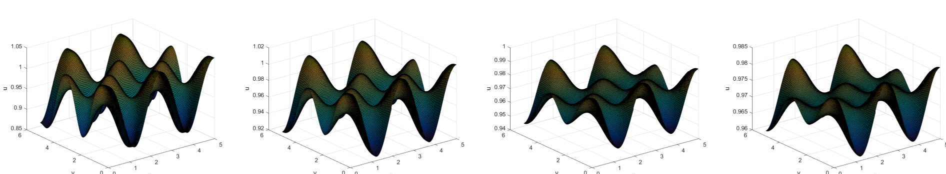


(e) $T = 6$ (f) $T = 9$ (g) $T = 12$ (h) $T = 15$

Figure 6: Numerical solution u for the two-dimensional case at different time points for $\theta_i = 0.0001$ and $\theta_{ii} = 0.001$, $i = 1, 2, 3$.



(a) $T = 0.5$ (b) $T = 1$ (c) $T = 2$ (d) $T = 3$



(e) $T = 6$ (f) $T = 9$ (g) $T = 12$ (h) $T = 15$

Figure 7: Numerical solution u for the two-dimensional case at different time points for $\theta_i = 0.0001$ and $\theta_{ii} = 0.01$, $i = 1, 2, 3$.

FEM is a great tool for studying ecological and biological systems based on cross-diffusion models, paving the way for more advanced future research.

The SKT cross-diffusion model is a prototype nonlinear parabolic system that depicts intricate diffusion-driven interactions across many species, making its analysis mathematically significant. Any significant numerical or analytical investigation requires that the model be mathematically well-posed, which is ensured by proving the existence, uniqueness, and regularity of weak and strong solutions. Numerical simulations have a strong theoretical basis thanks to the creation and validation of finite element approximations and rigorous proofs of convergence from the discrete weak formulation to the continuous weak solution. These findings advance the mathematical understanding of nonlinear PDEs in general, and cross-diffusion and coupled reaction terms in particular, which are known to pose analytical difficulties because of their nonlinearity and coupling structure.

From a computational perspective, numerical simulations of multicomponent diffusion systems gain trust when the convergence, stability, and correctness of the FEM are demonstrated. The effectiveness and dependability of the numerical method for real-world calculations are validated by the examination of error estimates in one, two, and three spatial dimensions. These results serve as standards for evaluating alternative numerical approaches used in complicated cross-diffusion systems, in addition to supporting the use of numerical approximations for theoretical research. Through simulations, numerical results and patterns can be shown, bridging the gap between mathematical theory and real-world phenomena and offering further insight into the qualitative behavior of the system.

The SKT model is essential to comprehending the spatial segregation and coexistence of competing biological species from a biological perspective. The cross-diffusion terms, which indicate avoidance behavior or territorial competition, explain why individuals of one species tend to migrate away from areas where substantial numbers of another species are present. These processes play a key role in the explanation of how spatial patterns like coexistence zones, segregation, and clustering develop in ecosystems. This kind of research allows for accurate population dynamics simulation by mathematically and numerically testing the model, offering predictive insight into how competition and diffusion mechanisms create ecological structures in genuine ecosystems.

This research is very interdisciplinary since it integrates biological modeling, numerical analysis, and mathematical rigor. It encourages the creation of quantitative tools for environmental scientists, ecologists, and biophysicists who want to simulate spatial population interactions in a range of environmental settings. Furthermore, the approach and findings can be applied to other systems, such as chemical reaction networks, tumor growth modeling, or bacterial colony formation, that are controlled by comparable reaction diffusion cross-diffusion structures. By strengthening the link between abstract analysis and applied sciences, the study advances both theoretical mathematics and applied biology modeling.

Acknowledgments

We would like to thank two anonymous reviewers for their informative remarks and helpful criticism, which have significantly improved this work.

Funding

This research received no external funding.

Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] Adams, R.A. and Fournier, J.J., *Sobolev spaces*, Elsevier, 2003.

[2] Al-Juaifri, G.A. and Harfash, A.J., *Finite element analysis of nonlinear reaction–diffusion system of Fitzhugh–Nagumo type with Robin boundary conditions*, Math. Comput. Simul., 203, (2023), 486–517.

[3] Al-Juaifri, G.A. and Harfash, A.J., *Numerical analysis of the brusselator model with Robin boundary conditions,* SeMA J., 82, (2025), 351–397.

[4] Al-Musawi, G.A. and Harfash, A.J., *Finite element analysis of extended Fisher-Kolmogorov equation with Neumann boundary conditions,* Appl. Numer. Math., 201, (2024), 41–71.

[5] Al-Musawi, G.A. and Harfash, A.J., *Finite element approximation of coupled Cahn-Hilliard equations with a logarithmic potential and nondegenerate mobility,* J. Math. Model., 13(1), (2025), 49–68.

[6] Al-Musawi, G.A. and Harfash, A.J., *Numerical approximation of the two-species Shigesada–Kawasaki–Teramoto model,* J. Math. Sci., (2025), 1–38.

[7] Baharlouei, S., Mokhtari, R., and Chegini, N., *Solving two-dimensional coupled Burgers equations via a stable hybridized discontinuous Galerkin method,* Iran. J. Numer. Anal. Optim., 13(3), (2023), 397–425.

[8] Barrett, J.W. and Blowey, J.F., *Finite element approximation of a nonlinear cross-diffusion population model,* Numer. Math., 98(2), (2004), 195–221.

[9] Barrett, J.W., Garcke, H., and Nürnberg, R., *Finite element approximation of surfactant spreading on a thin film*, SIAM J. Numer. Anal., 41(4), (2003), 1427–1464.

[10] Barrett, J.W. and Nürnberg, R., *Finite-element approximation of a nonlinear degenerate parabolic system describing bacterial pattern formation,* Interfaces and Free Boundaries, 4(3), (2002), 277–307.

[11] Barrett, J.W. and Nürnberg, R., *Convergence of a finite-element approximation of surfactant spreading on a thin film in the presence of van der waals forces,* IMA J. Numer. Anal., 24(2), (2004), 323–363.

[12] Barrett, J.W., Nürnberg, R., and Warner, M.R., *Finite element approximation of soluble surfactant spreading on a thin film,* SIAM J. Numer. Anal., 44(3), (2006), 1218–1247.

[13] Barrett, J.W., Schwab, C., and Süli, E., *Existence of global weak solutions for some polymeric flow models,* Math. Models Methods Appl. Sci., 15(06), (2005), 939–983.

[14] Cazenave, T., *Semilinear Schrodinger Equations*, vol. 10, American Mathematical Soc, 2003.

[15] Chand, A. and Mohapatra, J., *An analytical and numerical approach for the (1+ 1)-dimensional nonlinear Kolmogorov–Petrovskii–Piskunov equation,* Iran. J. Numer. Anal. Optim., 15(1), (2025), 79–98.

[16] Chen, L. and Jüngel, A., *Analysis of a multidimensional parabolic population model with strong cross-diffusion,* SIAM J. Math. Anal., 36(1), (2004), 301–322.

[17] Chen, L. and Jüngel, A., *Analysis of a parabolic cross-diffusion population model without self-diffusion,* J. Diff. Equ., 224(1), (2006), 39–59.

[18] Chen, X., Daus, E.S., and Jüngel, A., *Global existence analysis of cross-diffusion population systems for multiple species,* Arch. Rational Mech. Anal., 227, (2018), 715–747.

[19] Ciarlet, P.G., *The finite element method for elliptic problems*, SIAM, 2002.

[20] Ciavaldini, J., *Analyse numerique d un problème de stefan à deux phases par une methode d éléments finis,* SIAM J. Numer. Anal., 12(3), (1975), 464–487.

[21] Galiano, G. and Velasco, J., *Well-posedness of a cross-diffusion population model with nonlocal diffusion,* SIAM J. Math. Anal., 51(4), (2019), 2884–2902.

[22] Grün, G. and Rumpf, M., *Nonnegativity preserving convergent schemes for the thin film equation,* Numer. Math., 87(1), (2000), 113–152.

[23] Hashim, M.H. and Harfash, A.J., *Finite element analysis of a Keller–Segel model with additional cross-diffusion and logistic source. part I: Space convergence,* Comput. Math. Appl., 89(1), (2021), 44–56.

[24] Hashim, M.H. and Harfash, A.J., *Finite element analysis of a Keller-Segel model with additional cross-diffusion and logistic source. part II: Time convergence and numerical simulation,* Comput. Math. Appl., 109(1), (2022), 216–234.

[25] Hashim, M.H. and Harfash, A.J., *Finite element analysis of attraction-repulsion chemotaxis system. Part I: Space convergence,* Commun. Appl. Math. Comput., 4(3), (2022), 1011–1056.

[26] Hashim, M.H. and Harfash, A.J., *Finite element analysis of attraction-repulsion chemotaxis system. Part II: Time convergence, error analysis and numerical results,* Commun. Appl. Math. Comput., 4(3), (2022), 1057–1104.

[27] Hashim, M.H. and Harfash, A.J., *Finite element analysis for microscale heat equation with Neumann boundary conditions,* Iran. J. Numer. Anal. Optim., 14(3), (2024), 796–832.

[28] Hashim, M.H. and Harfash, A.J., *The convergence of the finite element approximation to the weak solution of attraction-repulsion chemotaxis system with additional self-diffusion term,* J. Math. Sci., 289(2), (2025), 273–322.

[29] Hashim, M.H., Hassan, S.M., Hameed, A.A., and Harfash, A.J., *Numerical approximations for damped wave equation with Neumann boundary conditions,* Math. Comput. Sci., 6(1), (2025), 1–18.

[30] Hassan, S.M. and Harfash, A.J., *Finite element analysis of a two-species chemotaxis system with two chemicals,* Appl. Numer. Math., 182, (2022), 148–175.

[31] Hassan, S.M. and Harfash, A.J., *Finite element analysis of the two-competing-species Keller–Segel chemotaxis model,* Comput. Math. Model., 33(4), (2022), 443–471.

[32] Hassan, S.M. and Harfash, A.J., *Finite element approximation of a Keller–Segel model with additional self-and cross-diffusion terms and a logistic source,* Commun. Nonlinear Sci. Numer. Simul., 104, (2022), 106063.

[33] Hassan, S.M. and Harfash, A.J., *Finite element analysis of chemotaxis-growth model with indirect attractant production and logistic source,* Int. J. Comput. Math., 100(4), (2023), 745–774.

[34] Jüngel, A. and Zamponi, N., *Qualitative behavior of solutions to cross-diffusion systems from population dynamics,* J. Math. Anal. Appl., 440(2), (2016), 794–809.

[35] Lions, J.L., *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Paris: Dunod, 1969.

[36] Ranjan, K.R. and Gowrisankar, S., *Discontinuous Galerkin approach for two-parametric convection-diffusion equation with discontinuous source term,* Iran. J. Numer. Anal. Optim., 14(2), (2024), 347–366.

[37] Robinson, J.C., *Infinite-dimensional dynamical systems: an introduction to dissipative parabolic PDEs and the theory of global attractors*, vol. 28, Cambridge University Press, 2001.

[38] Samy, H., Adel, W., Hanafy, I., and Ramadan, M., *A Petrov–Galerkin approach for the numerical analysis of soliton and multi-soliton solutions of the Kudryashov–Sinelshchikov equation,* Iran. J. Numer. Anal. Optim., 14(4), (2024), 1309–1335.

[39] Shigesada, N., Kawasaki, K., and Teramoto, E., *Spatial segregation of interacting species*, J. Theor. Biol., 79(1), (1979), 83–99.

[40] Youssri, Y. and Atta, A., *Modal spectral Tchebyshev Petrov–Galerkin stratagem for the time-fractional nonlinear Burgers' equation,* Iran. J. Numer. Anal. Optim., 14(1) (2024), 172–199.