A linearization technique for optimal design of the damping set with internal dissipation

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Abstract

Considering a damped wave system defined on a two-dimensional domain, with a dissipative term localized in an unknown subset with an unknown damping parameter, we address the ill-posed shape design problem which consists of optimizing the shape of the unknown subset in order to minimize the energy of the system at a given time. By using a new approach based on the embedding process, first the system is formulated in variational form. Then, by transferring the problem into polar coordinates and defining two positive Radon measures, we represent the problem in a space of measures. Hence, the shape design problem is changed into an infinite linear one whose solution is guaranteed. In this stage, by applying two subsequent approximation steps, the optimal solution (optimal control, optimal region, optimal damping parameter and optimal energy) is identified by a three-phase optimization search technique. Numerical simulations are also given in order to compare this new method with level set algorithm.

1 Introduction and Problem statement

In many technological situations, a given structure whose optimal position is at rest (for instance), starts to vibrate due to uncontrolled disturbances

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which we would like to stop. One possibility, although under ideal conditions, is described in [16] through damping mechanisms. In the literature, the problem of optimal stabilization for the 2-D wave equation has been extensively studied from different perspectives (see for instance, [4], [9] and [13]). The analysis performed by Hebrard et al. highlights the effect of the over-damping phenomenon characteristic of this damped wave equation [2]. Freitas [9] and Lopez [13] solved the mentioned problem in which the dissipation vanishes for large values of the constant damping coefficient. In 2006, Munch et al. used Young measures to solve a similar problem and presented a solution method. In that study, the damping coefficient was fixed and the best unknown internal region was determined by the use of the gradient descend method [15]. In sequence, the best damping coefficient and damping set were determined at different times using the level set method [16].

In this paper, we solve the problem of finding an optimal observation domain $\omega \subset \Omega \subset \mathbb{R}^2$ for general damping wave equations in a new way. We optimize not only the placement but also the shape of ω, over all possible measurable subsets of Ω having a certain prescribed measure. Such questions are frequently encountered in engineering applications but have rarely been treated from the mathematical point of view. In this regard, for the first time, we consider a shape optimization problem to find the optimal shape and place of a sensor, modeled by a two-dimensional wave equation. The objective is to find the shape of the damping set that minimizes the energy at some given end time (see [18] and [19]).

2 Optimal wave damping problem

Let $\Omega \subset \mathbb{R}^2$ be a domain with piecewise smooth boundary and consider the two-dimensional damping wave equation with Dirichlet boundary conditions. Consider additionally that ω is a subset of Ω with positive Lebesgue measure which is independent of time $t \in (0, T)$. The resulting equation for the displacement of the sensor is then ([2] and [16])

$$
\begin{aligned}
&\ddot{y}_{\omega,a} - \Delta y_{\omega,a} + a(x)\dot{y}_{\omega,a} = 0, &\quad (x,t) \in \Omega \times (0,T), \\
&y_{\omega,a} = 0, &\quad (x,t) \in \partial\Omega \times (0,T), \\
&y_{\omega,a}(x,0) = y_0(x), \quad \dot{y}_{\omega,a}(x,0) = y_1(x), &\quad x \in \Omega;
\end{aligned} \tag{1}
$$

here, $a(x) = a\chi_\omega(x) \in L^\infty(\Omega, \mathbb{R}^+)$ is a damping function where $a \in \mathbb{R}^+$ is unknown, $\varnothing \neq \omega$ is an unknown region in Ω, $\partial\omega$ is a smooth and simple closed curve boundary which must be identified, χ_ω is a characteristic function of ω and y_0 and y_1 also indicate the initial position and velocity, respectively. In addition, regarding the initial conditions, we assume that:

$$(y_0(x), y_1(x)) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega),$$

where $H_0^1(\Omega)$ (the Sobolev space of order 1 on Ω whose functions are zero on the boundary of Ω [22]). System (1) is well-posed [12], whose unique solution satisfies:

$$y_{\omega,a} \in C((0,T); H^2(\Omega) \cap H_0^1(\Omega) \cap C^1((0,T); H_0^1(\Omega) \cap C^2((0,T); L^2(\Omega)).$$

For every $t > 0$, the energy of system (1) is obtained through ([16])

$$E(\omega, a, t) = \frac{1}{2} \int_\Omega (|\dot{y}_{\omega,a}(x,t)|^2 + |\nabla y_{\omega,a}(x,t)|^2 dx, \qquad (2)$$

which satisfies the following dissipation law (see [23]):

$$\dot{E}(\omega, a, t) = -\int_\Omega a(x)|\dot{y}_{\omega,a}(x,t)|^2 dx \le 0.$$

Here, $y_{\omega,a}$ denotes the transversal displacement at point x in time t.

We attempt to find the unknown region ω and damping function $a(x)$, simultaneously, through a three-phase optimization procedure which is based on an embedding technique. To apply this method, first, we present the problem in variational form; next, it is transferred into a new theoretical measure problem in which two unknown positive Radon measures in a product space of measures are sought. Then, the solution procedure is explained and finally, by a three-phase optimization technique, a nearly optimal shape together with the optimal damping function as well as the minimized value of system energy are constructed.

The paper is organized as follows: the next section is devoted to the basic deformation in variational form. The aim of Section 4 is to state the problem in a polar system. Section 5 deals with the embedding process and approximation schemes. In Section 6, based on the previous discussions, we present the solution algorithm. Then, two numerical simulations are presented in Section 7. Finally, concluding remarks are offered in Section 8.

3 Basic Deformation

In general, it is difficult to identify a calssical solution for problem (1); thus attempts have usually been made to find a weak (or generalized) solution of the problem, which is more applicable in our work. The main idea in this replacement is to convert the problem into the variational form. To this end, by multiplying the first equation of system (1) with a function $\varphi \in H_0^1(\Omega \times (0,T))$ and using Green's theorem, to the initial conditions, for each t, one obtains:

$$\int_\Omega y \Delta\varphi dx - \int_\Omega \varphi \Delta y dx = \int_{\partial\Omega} (y \frac{\partial\varphi}{\partial n} - \varphi \frac{\partial y}{\partial n} ds) = 0,$$

therefore

$$\int_\Omega y\Delta\varphi dx = \int_\Omega \varphi\Delta y dx;$$

then, we have:

$$\int_\Omega \ddot{y}\varphi dx - \int_\Omega y\Delta\varphi dx + \int_\Omega a(x)\dot{y}\varphi dx = 0. \qquad (3)$$

Integrating both sides of (3) with respect to t over $[0, T]$ gives:

$$\int_0^T \int_\Omega \ddot{y}\varphi dxdt - \int_0^T \int_\Omega y\Delta\varphi dxdt + \int_0^T \int_\Omega a(x)\dot{y}\varphi dxdt = 0, \qquad (4)$$

Double integrating by parts with respect to t from the first left expression and integrating the third expression on the left-hand side of (4), we conclude:

$$
\begin{aligned}
\int_0^T \int_\Omega \ddot{y}\varphi dxdt &= \int_\Omega [\dot{y}(T)\varphi(T) - \dot{y}(0)\varphi(0) - y(T)\dot{\varphi}(T) + y(0)\dot{\varphi}(0)]dx \\
&\quad + \int_0^T \int_\Omega y\ddot{\varphi} dxdt; \\
\int_0^T \int_\Omega a(x)\dot{y}\varphi dxdt &= \int_\Omega a(x)[y(T)\varphi(T) - y(0)\varphi(0) - \int_0^T y\dot{\varphi}]dx \\
&= \int_\Omega a(x)[y(T)\varphi(T) - y(0)\varphi(0)]dx - \int_0^T \int_\Omega a(x)y\dot{\varphi} dxdt.
\end{aligned}
\qquad (5)
$$

Now, by substituting the initial conditions of system (1) in (5), we have:

$$
\begin{aligned}
\int_0^T \int_\Omega \ddot{y}\varphi dxdt &= \int_\Omega [\dot{y}(T)\varphi(T) - y_1(x)\varphi(0) - y(T)\dot{\varphi}(T) + y_0(x)\dot{\varphi}(0)]dx \\
&\quad + \int_0^T \int_\Omega y\ddot{\varphi} dxdt; \\
\int_0^T \int_\Omega a(x)\dot{y}\varphi dxdt &= \int_\Omega a(x)[y(T)\varphi(T) - y_0(x)\varphi(0)]dx - \int_0^T \int_\Omega a(x)y\dot{\varphi} dxdt.
\end{aligned}
\qquad (6)
$$

By applying (6), equation (4) is changed to:

$$
\begin{aligned}
\int_\Omega \dot{y}(T)\varphi(T)dx &- \int_\Omega y(T)\dot{\varphi}(T)dx - \int_0^T \int_\Omega y\Delta\varphi dxdt + \int_\Omega a(x)y(T)\varphi(T)dx \\
&- \int_\Omega a(x)y_0(x)\varphi(0)dx - \int_0^T \int_\Omega a(x)y\dot{\varphi} dxdt + \int_0^T \int_\Omega y\ddot{\varphi} dxdt \\
&= \int_\Omega [y_1(x)\varphi(0) - y_0(x)\dot{\varphi}(0)]dx.
\end{aligned}
\qquad (7)
$$

Moreover, for all $(x, t) \in \partial\Omega \times [0, T]$ by the initial condition, we have $y(x, t) = 0$; to apply this condition and using Green's theorem, we have:

$$\int_{\partial\Omega} y(x, t)\varphi(x, t).nd\sigma = \int_\Omega div(y(x, t)\varphi(x, t))dx = 0. \qquad (8)$$

Since the unknown region ω must lie in Ω and the measure of this unknown region must be non-zero, the set of admissible shapes for problem (1) can be shown as:

$$V_L = \{\omega \subset \Omega : |\omega| = L|\Omega| \ \}, \qquad 0 < L < 1 \qquad (9)$$

in which $|\omega|$ indicates the measure of ω and L is a fixed number. This constraint can be shown by the following integral relation:

$$\int_{\omega} dx = L \int_{\Omega} dx. \qquad (10)$$

4 Expressing the problem in polar system

The mentioned optimal shape design (OSD) problem is defined based on the unknown geometrical pair $(\omega, \partial\omega)$. This pair consists of a measurable set that can be regarded as a nonempty region, and a simple closed curve which is its boundary. Based on the simplicity property of the curve, our OSD problem depends on the geometry which is used. We prefer to solve the appropriate problems in polar coordinates since where $0 \le \theta \le 2\pi$ and $r \geqslant 0$, the curve $r = r(\theta)$ is simple. This simple fact is an essential part in our calculations and also in numerical simulations. Hence, let $x_1 = r\cos(\theta)$ and $x_2 = r\sin(\theta)$; then, we have:

$$|\nabla y|_T = [(\frac{\partial y}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial y}{\partial \theta})^2], \quad \Delta y|_T = \frac{\partial^2 y}{\partial r^2} + \frac{1}{r}\frac{\partial y}{\partial r} + \frac{1}{r^2}\frac{\partial^2 y}{\partial \theta^2},$$

and therefore:

$$E(\omega, a, T) = \frac{1}{2}\int_{\Omega}[|\dot{y}(r, \theta, T)|^2 + (\frac{\partial y}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial y}{\partial \theta})^2]rdrd\theta.$$

Since the nature of Ω has not changed, but rather its representation has changed, we use the same symbol and, in the end, the optimal shape is shown in polar coordinates.

Additionally, for every $\varphi \in H_0^1(\Omega \times (0, T))$, the mentioned constraint in (7) can be represented as:

$$\int_{\Omega} \dot{y}(T)\varphi(T)rdrd\theta - \int_{\Omega} y(T)\dot{\varphi}(T)rdrd\theta - \int_0^T \int_{\Omega} y\Delta\varphi rdrd\theta dt$$
$$+ \int_{\Omega} a(r, \theta)y(T)\varphi(T)rdrd\theta - \int_{\Omega} a(r, \theta)y_0(r, \theta)\varphi(0)rdrd\theta$$
$$- \int_0^T \int_{\Omega} a(r, \theta)y\dot{\varphi}rdrd\theta dt + \int_0^T \int_{\Omega} y\ddot{\varphi}rdrd\theta dt = \Phi,$$

$$(11)$$

in which

$$\Phi = \int_{\Omega}[y_1(r, \theta)\varphi(0) - y_0(r, \theta)\dot{\varphi}(0)]rdrd\theta.$$

Moreover, equations (8) and (9) can be represented in polar coordinates as:

$$\int_{\Omega} div(y(r,\theta,t)\varphi(r,\theta,t))rdrd\theta = 0,$$
$$\int_{\omega} rdrd\theta = L \int_{\Omega} rdrd\theta. \tag{12}$$

Therefore, the problem of obtaining the optimal shape and damping coefficient for minimizing energy in polar coordinates has the following presentation:

$$Min : E(\omega, a, T) = \tfrac{1}{2} \int_{\Omega} [|\dot{y}(r,\theta,T)|^2 + (\tfrac{\partial y}{\partial r})^2 + \tfrac{1}{r^2}(\tfrac{\partial y}{\partial \theta})^2] rdrd\theta,$$

$$S.\ to : \int_{\Omega} \dot{y}(T)\varphi(T)rdrd\theta - \int_{\Omega} y(T)\dot{\varphi}(T)rdrd\theta - \int_0^T \int_{\Omega} y\Delta\varphi rdrd\theta dt$$

$$+ \int_{\Omega} a(r,\theta)y(T)\varphi(T)rdrd\theta - \int_{\Omega} a(r,\theta)y_0(r,\theta)\varphi(0)rdrd\theta$$

$$- \int_0^T \int_{\Omega} a(r,\theta)y\dot{\varphi}rdrd\theta dt + \int_0^T \int_{\Omega} y\ddot{\varphi}rdrd\theta dt = \Phi; \tag{13}$$

$$\int_{\Omega} div(y(r,\theta,t)\varphi(r,\theta,t))rdrd\theta = 0,\ \forall \varphi \in H_0^1(\Omega \times (0,T));$$

$$\int_{\omega} rdrd\theta = L \int_{\Omega} rdrd\theta.$$

To solve (12), we change the problem and consider a new one with a different formulation. By applying this method, we show how one can obtain the optimal region ω, optimal damping function $a(x)$ and the amount of minimized energy simultaneously.

5 Embedding the solution space: metamorphosis

The solution method which is based on an embedding process involves several stages to set up a linear programming problem whose solution converges to the solution of the original problem (see [20]). This is one of the outstanding advantages of this method even for strongly nonlinear problems. Hence, we present a new version of shape measure method to solve the optimal shape design (12). First, by defining a new variational formulation, an optimal control problem equivalent to the original problem is obtained. Then, a measure theoretical approach and a two-stage approximation are used to convert the optimal control problem to a finite dimensional LP. The solution of this LP is used to construct an approximate solution to the original control problem. The proposed approach is practical and accurate enough and its accuracy can be improved as far as desired (see [8]).

5.1 Step 1: Displaying the problem in variational form

In order to transform the optimal shape design into variational form, we need to define some fundamental concepts. The conditions imposed on the functions and sets will serve two important purposes. First, they are reasonable

conditions which are usually met when considering classical problems. Second, they will allow the modification of these classical problems into ones which appear to have some advantages over the classical formulation.

Suppose that $J = [0, 2\pi]$ and $A_1 = [0, r_\Omega]$ are the domains of variables θ and r_1, respectively. For $\Omega = J \times A_1$, let $V \subseteq \mathbb{R}$ be a given bounded and closed set, and $A_2 = [0, r_\omega]$.

Definition 1. Consider the variable $r : J^0 \to A_2$ as an absolutely continuous trajectory function, where $J^0 = (0, 2\pi)$; then, we denote the boundary curve of the unknown region ω by $\partial\omega$ which is introduced by:

$$\partial\omega : r = r(\theta), \ \theta \in J^o. \tag{14}$$

Definition 2. By supposing that r is a simple and closed curve and U is a bounded closed subset in $\mathbb{R}$, we introduce an artificial control function u as follows:

$$u : J^o \to U,$$
$$u(\theta) = \dot{r}(\theta) \equiv g(\theta, r, u),$$

where $r(\theta) \in [0, r_\omega]$ and the boundary $\partial\omega$ is determined by this variable.

Definition 3. Let S' be a bounded closed subset of $\mathbb{R}$ and function $Y : \Omega \times [0, T] \to S'$ be defined in the following way:

$$Y = \frac{\partial y}{\partial \theta}. \tag{15}$$

Since $y = y(r_1, \theta)$ and $r_1 = r_1(\theta) \in \Omega$, we can write $y = y(\theta)$ and $\theta = \theta(r_1)$. In this case, we have $\partial y/\partial r_1 = (\partial y/\partial \theta)(d\theta/dr_1)$ and by introducing $v : J^o \to V$ where $v = dr_1/d\theta$, we can write:

$$\frac{\partial y}{\partial r_1} = \frac{\partial y}{\partial \theta}\frac{1}{v},$$

where $r_1(\theta) \in [0, r_\Omega]$ and $\partial\Omega$ is determined by this variable. From now on, for simiplicity we denote r_1 also with r by regarding that $r_1 = r \in \Omega$; hence the derivative of r in ω and Ω were shown above by u and v respectively.

To identify the relationship between r and v as variables, suppose $E' = J \times A_1 \times V$ and consider h in $C^1(E')$, then:

$$\frac{\partial h}{\partial \theta}(\theta, r, v) = \frac{\partial h}{\partial r}(\theta, r, v)v. \tag{16}$$

In the same way, to display the relationship between functions y and Y as two variables, we define $E = J \times S \times S'$ (where S is the range of y) and consider $f \in C^1(E)$; on the basis of $y = y(r, \theta)$, it is possible to express θ according to y implicitly. Therefore, function $f(\theta, y, Y)$ can basically be displayed as $f(\theta, Y)$. That is , y is not an independent variable of f; hence,

$\partial f / \partial y(\theta, Y) = (\partial f / \partial \theta(\theta, Y)).(\partial \theta / \partial y)$. Now, by regarding (14), we have $\partial f / \partial y(\theta, Y) = (\partial f / \partial \theta(\theta, Y)).(1/Y)$ [3]. Since this relationship is satisfied for every $f \in C^1(E)$, it can be concluded that:

$$\int_{\Omega} \frac{\partial f}{\partial y}(\theta, Y) r dr d\theta = \int_{\Omega} \frac{\partial f}{\partial \theta}(\theta, Y) \frac{1}{Y} r dr d\theta, \quad \forall f \in C^1(E). \quad (17)$$

We add this set of constraints to (12) in order to specify the relationship between y and Y when they are considered as variables in the problem.

Since our aim is to identify ω and its unknown boundary, we prefer to display the constraints of (12) as integrals on the boundary of ω as much as we can. Thus, the term on the right-hand side of (6), we have:

$$\int_{\Omega} a(x) y(T) \varphi(T) dx - \int_{\Omega} a(x) y_0(x) \varphi(0) dx = \int_{\omega} a(y(T) \varphi(T) - y_0(x) \varphi(0)) dx;$$

Regarding the Green's theorem :

$$\oint_{\partial \omega} M dx_1 + N dx_2 = \int_{\omega} (\frac{\partial N}{\partial x_1} - \frac{\partial M}{\partial x_2}) dA,$$

suppose $M = 0$ and $N = \int_0^{x_1} a(y(T) \varphi(T) - y_0(x) \varphi(0)) dx_1$. Considering the fact that $a(x) = a \chi_\omega(x)$, we have:

$$\int_{\partial \omega} a \int_0^{r cos\theta} [y(T) \varphi(T) - y_0(r, \tau) \varphi(0)] (\dot{r} cos\tau - r sin\tau)(\dot{r} sin\tau + r cos\tau) d\tau d\tau. \quad (18)$$

Also, the area constraint (9) can be presented as:

$$\frac{1}{2} \int_{\partial \omega} r^2 dr d\theta = L \int_{\Omega} r dr d\theta = L \frac{1}{T} \int_0^T \int_{\Omega} r dr d\theta.$$

The independency of the objective function from t results in:

$$E(\omega, a, T) = \frac{1}{2T} \int_0^T \int_{\Omega} [|\dot{y}(r, \theta, T)|^2 + (\frac{1}{v^2} + \frac{1}{r^2}) Y^2] r dr d\theta dt. \quad (19)$$

Now, by substituting (17) into (10) for every $\varphi \in H_0^1(\Omega \times (0, T))$, we have:

$\frac{1}{T} \int_0^T \int_{\Omega} \dot{y}(T) \varphi(T) r dr d\theta dt - \frac{1}{T} \int_0^T \int_{\Omega} y(T) \dot{\varphi}(T) r dr d\theta dt - \int_0^T \int_{\Omega} y \Delta \varphi r dr d\theta dt$

$+ \int_{\partial \omega} a \int_0^{r cos\theta} [y(T) \varphi(T) - y_0(r, \tau) \varphi(0)] (\dot{r} cos\tau - r sin\tau)(\dot{r} sin\tau + r cos\tau) d\tau d\tau$

$- \int_0^T \int_{\Omega} a(r, \theta) y \dot{\varphi} r dr d\theta dt + \int_0^T \int_{\Omega} y \ddot{\varphi} r dr d\theta dt = \Phi.$

Integrating (11), (15) and (16) over $[0, T]$ implies that:

$\int_0^T \int_\Omega div(y(r, \theta, t)\varphi(r, \theta, t))rdrd\theta dt = 0, \ \forall \varphi \in H_0^1(\Omega \times (0, T));$

$\frac{1}{T} \int_0^T \int_\Omega \frac{\partial f}{\partial y} rdrd\theta dt = \frac{1}{T} \int_0^T \int_\Omega \frac{\partial f}{\partial \theta} \frac{1}{Y} rdrd\theta dt, \ \forall f \in C^1(E);$

$\frac{1}{T} \int_0^T \int_\Omega \frac{\partial h}{\partial \theta}(\theta, r, v) rdrd\theta dt = \frac{1}{T} \int_0^T \int_\Omega \frac{\partial h}{\partial r}(\theta, r, v) vrdrd\theta dt, \ \forall h \in C^1(E').$

Therefore, problem (12) can be displayed in a new variational form as follows:

$Min : E(\omega, a, T) = \frac{1}{2T} \int_0^T \int_\Omega [|\dot{y}(r, \theta, T)|^2 + (\frac{1}{v^2} + \frac{1}{r^2})Y^2]rdrd\theta dt$

$S. \ to : \frac{1}{2} \int_{\partial \omega} r^2 drd\theta = L\frac{1}{T} \int_0^T \int_\Omega rdrd\theta dt,$

$\qquad \int_0^T \int_\Omega div(y(r, \theta, t)\varphi(r, \theta, t))rdrd\theta dt = 0, \ \forall \varphi \in H_0^1(\Omega \times (0, T));$

$\qquad \frac{1}{T} \int_0^T \int_\Omega \frac{\partial f}{\partial y}(\theta, Y) rdrd\theta dt = \frac{1}{T} \int_0^T \int_\Omega \frac{\partial f}{\partial \theta} \frac{1}{Y}(\theta, Y) rdrd\theta dt, \ \forall f \in C^1(E);$

$\qquad \frac{1}{T} \int_0^T \int_\Omega \frac{\partial h}{\partial \theta}(\theta, r, v) rdrd\theta dt = \frac{1}{T} \int_0^T \int_\Omega \frac{\partial h}{\partial r}(\theta, r, v) vrdrd\theta dt, \ \forall h \in C^1(E');$

$\qquad \frac{1}{T} \int_0^T \int_\Omega \dot{y}(T)\varphi(T)rdrd\theta dt - \frac{1}{T} \int_0^T \int_\Omega y(T)\dot{\varphi}(T)rdrd\theta dt$

$\qquad - \int_0^T \int_\Omega y\Delta\varphi rdrd\theta dt - \int_0^T \int_\Omega a(r, \theta)y\dot{\varphi}rdrd\theta dt;$

$\qquad + \int_{\partial \omega} a \int_0^{rcos\theta} [y(T)\varphi(T) - y_0\varphi(0)](\dot{r}cos\tau - rsin\tau)(\dot{r}sin\tau + rcos\tau)d\tau d\tau$

$\qquad + \int_0^T \int_\Omega y\ddot{\varphi}rdrd\theta dt = \Phi, \ \forall \varphi \in H_0^1(\Omega \times (0, T)).$

$$(20)$$

Now, by determining a suitable control function, we rewrite the problem in the form of an optimal control problem.

5.2 Step 2: Embedding into measure space

By considering the pair of functions (r, y) as the trajectory and the triple (u, v, Y) as the control vector, problem (25) can be considered as an optimal control problem. In this manner, we need to present the following definition:

Definition 4. Quintuplet $p = (r, u, v, y, Y)$ is called admissible when it satisfies the following conditions:

1. The control functions u, ν and Y are bounded and continuous and take their values on compact sets U, V and S';

2. r is a differentiable function and $r(0) = r(2\pi)$;

3. y is the bounded solution of the linear damped wave system (1);

4. The relations (15) and (16) are satisfied.

The set of all admissible quintuplets is denoted by P. We also suppose that P is nonempty; in other words, we suppose that the system is controllable (This can be seen in [20], for instance).

Let $D = [0,T] \times J \times A_1 \times S \times S' \times V$ and $D' = J \times A_2 \times U$; for any admissible quintuplets in P, we define the linear, positive and bounded functionals Λ_P and Γ_P on $C(D)$ and $C(D')$ in the following way:

$$\begin{aligned}
\Gamma(F) &= \int_0^T \int_\Omega F(t,\theta,r,v,y,Y)drd\theta dt, && \forall F \in C(D); \\
\Lambda(G) &= \int_{\partial\omega} G(\theta,r,u)d\theta, && \forall G \in C(D');
\end{aligned}$$
(21)

Since $\mathbb{R}^6$ is a locally compact space, by the Heine-Borel theorem ([21]), $D \subseteq \mathbb{R}^6$ is a compact Hausdorff space. Also, for the same reason, D' is a Hausdorff compact space. Therefore, for every given p, Riesz representation theorem ([22]) indicates uniquely two positive Radon measures , μ_P and λ_P, so that:

$$\begin{aligned}
\Gamma_P(F) &= \int_D F d\mu_P \equiv \mu_P(F), && \forall F \in C(D); \\
\Lambda_P(G) &= \int_{D'} G d\lambda_P \equiv \lambda_P(G), && \forall G \in C(D');
\end{aligned}$$
(22)

Consequently, any admissible quintuplets can be displayed as (27) by a unique pair of measures, say (μ_P, λ_P), in a subset F of $\mathcal{M}^+(D) \times \mathcal{M}^+(D')$, where $\mathcal{M}^+(X)$ is the set of all positive Radon measures on X. Therefore, one can transfer problem (25) into a measure space by:

$$(r,u,v,y,Y) \in P \longmapsto (\mu_P,\lambda_P) \in \mathcal{M}(D) \times \mathcal{M}(D').$$

It was proved by Rubio (1986) that such a transformation is an injection. To achieve something new, we enlarge the underlying space and consider the problem of finding a minimizer pair of measures, say (μ^*, λ^*), on the space of all positive related Radon measures which are just satisfied to the conditions of (25) (Not just those that are induced from Riesz Representation theorem). Therefore, our method is somehow global.

We now characterize some properties of admissible pairs. Suppose B is an open disc in $\mathbb{R}^2$ that includes $J \times A_2$; consider $C'(B)$ as the space of real-valued continuously differentiable functions on B. Then, for every ϕ in $C'(B)$, we define:

$$\phi^g(\theta,r,u) = \phi_r(\theta,r)u + \phi_\theta(\theta,r), \ \forall(\theta,r,u) \in D'.$$

Then, since the boundary $\partial\omega$ is a closed and simple curve, we have:

$$\int_J \phi^g(\theta,r,u)d\theta = \int_J \dot{\phi}(\theta,r)d\theta = d_\phi, \quad \forall\phi \in C'(B),$$
(23)

where $d_\phi = \phi(2\pi,r_d) - \phi(0,r_d)$, is still unknown since r_d in $(0,r_d) = (2\pi,r_d)$, which is the initial and final point of the closed curve ω, is unknown. We will explain later that it would be characteristic (see Section (6)).

Let $D(J^0)$ be the space of infinitely differentiable real-valued functions with compact support in J^0. Define

$$\psi^g(\theta, r, u) = r(\theta)\dot{\psi}(\theta) + u(\theta)\psi(\theta), \ \forall (\theta, r, u) \in D';$$

then,

$$\int_J \psi^g(\theta, r, u)d\theta = r(2\pi)\psi(2\pi) - r(0)\psi(0) = 0, \quad \forall \psi \in D(J^o). \qquad (24)$$

The same situation arises for another special choice of functions in $C'(B)$ which are only dependent on variable θ, denoted by $C_1(D')$. Thus, for $\phi(\theta, r, u) \equiv \nu(\theta)$, we can have:

$$\int_J \nu(\theta)d\theta = a_\nu, \ \forall \nu \in C_1(D'), \qquad (25)$$

where a_ν is the Lebesgue integral of ν over J.

Regarding the famous properties of admissible quintuplets in P which are looked at in (28), (29) and (25), and the definitions of the pair of measures (μ, λ) in (21), problem (25) can now be displayed as follows in which the measures λ and μ are its unknown variables:

$Min: \quad E(\mu, \lambda) = \mu(\frac{r}{2T}[|\dot{y}(r, \theta, T)|^2 + (\frac{1}{v^2} + \frac{1}{r^2})Y^2]$
$S. \ to: \ \lambda(\phi^g(\theta, r, u)) = d_\phi, \ \forall \phi \in C'(B);$

$\lambda(\psi^g(\theta, r, u)) = 0, \ \forall \psi \in D(J^o);$

$\lambda(\nu(\theta)) = a_\nu, \ \forall \nu \in C_1(D');$

$\lambda(\frac{1}{2}r^2) = L\mu(\frac{1}{T}r),$

$\mu(\frac{1}{T}\frac{\partial f}{\partial y}r) = \mu(\frac{1}{T}\frac{\partial f}{\partial \theta}\frac{1}{Y}r), \ \forall f \in C^1(E);$

$\mu(\frac{1}{T}\frac{\partial h}{\partial \theta}(\theta, r, v)r) = \mu(\frac{1}{T}\frac{\partial h}{\partial r}(\theta, r, v)vr), \ \forall h \in C^1(E');$

$\mu(div(y(r, \theta, t)\varphi(r, \theta, t))r) = 0, \ \forall \varphi \in H_0^1(\Omega \times (0, T));$

$\mu(\frac{1}{T}\dot{y}(T)\varphi(T)r) - \mu(\frac{1}{T}y(T)\dot{\varphi}(T)r) - \mu(y\Delta\varphi r) - \mu(a(r, \theta)y\dot{\varphi}r) + \mu(y\ddot{\varphi}r)$

$+\lambda(a\int_0^{rcos\theta}[y(T)\varphi(T) - y_0\varphi(0)](\dot{r}cos\tau - rsin\tau)(\dot{r}sin\tau + rcos\tau)d\tau) = \Phi;$
$$(26)$$

We remind that the theoretical measure problem (26) is linear even though the initial problem is highly nonlinear.

The space $M^+(D) \times M^+(D')$ is a linear space which will become a locally convex topological vector space when it gives the weak*topology. This can be defined by the family of semi-norms $(\mu, \lambda) \mapsto |\mu(F)| + |\lambda(G)|$ for $F \in C(D)$, $G \in C(D')$ and $\epsilon > 0$, which can be on the basis of a family of neighborhoods of zero for $M^+(D) \times M^+(D')$. This family is defined by:

$$U_\epsilon = \{(\mu, \lambda) \in M^+(D) \times M^+(D') : |\mu(F_j)| + |\lambda(G_j)| < \epsilon; j = 1, 2, ..., r\}$$

and makes a basis for a weak*topology on the space $M^+(D) \times M^+(D')$ (Many properties of this topology can be found in the literature such as [24]). In this way, $M^+(D) \times M^+(D')$ under this topology is a Hausdorff space ([21]).

The proof of the following theorems can be found in [6], [7] and [20].

Theorem 1.
a) $Q \subseteq M^+(D) \times M^+(D')$ is compact under the weak* topology on $M^+(D) \times M^+(D')$
b) The objective function $E(\mu, \lambda)$ in problem (26) is continuous.
c) There exists a pair of measures (μ^*, λ^*) which are optimal for (26) in the set $Q \subset M^+(D) \times M^+(D')$; that is, for every $(\mu, \lambda) \in Q$, we have:

$$E(\mu^*, \lambda^*) \leq E(\mu, \lambda).$$

Even though (26) has an optimal solution in Q, it is still very difficult to obtain the exact solution since the underlying spaces are not finite-dimensional, the number of equations is not finite and the unknowns are measures. Therefore, it is completely acceptable to seek for a suboptimal solution. Thus, first, by choosing suitable dense subsets in the appropriate spaces and then, by choosing a finite number of them, the problem is approximated by a semifinite linear programming.

5.3 Identifying a nearly optimal solution

It is possible to approximate the solution of problem (26) by the solution of a finite-dimensional linear one of sufficiently large dimensions. Besides, by increasing the dimension of the problem, the accuracy of the approximation can be increased. First, we consider the minimization of (26) not only over set Q, but also over its subset called $Q(M_1, M_2, ..., M_7)$ and defined by only a finite number of constraints to be satisfied. This will be achieved by choosing countable sets of functions whose linear combinations are dense in appropriate spaces and then by selecting a finite number of constraints. Let $\{\phi_i : i \in N\}$, $\{\psi_i : i \in N\}, \{\nu_i : i \in N\}, \{\varphi_i : i \in N\}, \{f_i : i \in N\}$ and $\{h_i : i \in N\}$ be countable dense (in the topological convergence sense) sets in spaces $C'(B)$, $D(J^o)$, $C_1(D')$, $H_0^1(\Omega \times (0, T))$, $C^1(E)$ and $C^1(E')$, respectively. By choosing a finite number of functions in each set, the solution of (29) can be approximated by the solution of the following one:

$$Min: \ E(\mu, \lambda) = \mu(\tfrac{r}{2T}[|\dot{y}(r, \theta, T)|^2 + (\tfrac{1}{v^2} + \tfrac{1}{r^2})Y^2])$$

$$S. \ to: \lambda(\phi_k^g(\theta, r, u)) = d_{\phi_k}, \ k = 1, 2, ..., M_1;$$

$$\lambda(\psi_l^g(\theta, r, u)) = 0, \ l = 1, 2, ..., M_2;$$

$$\lambda(\nu_s(\theta)) = a_s, \ s = 1, 2, ..., M_3;$$

$$\lambda(\tfrac{1}{2}r^2) - L\mu(\tfrac{1}{T}r) = 0,$$

$$\mu(F_i) = 0, \ i = 1, 2, ..., M_4;$$

$$\mu(G_j) - \mu(H_j) = 0, \ j = 1, 2, ..., M_5;$$

$$\mu(I_j) - \mu(K_j) = 0, \ j = 1, 2, ..., M_6;$$

$$\mu(L_i) - \mu(P_i) - \mu(Q_i) - \mu(R_i) + \lambda(T_i) + \mu(N_i) = \Phi_i,$$

$$i = 1, 2, ..., M_7,$$

(27)

where

$$
\begin{cases}
F_i = div(y(r, \theta, t)\varphi_i(r, \theta, t))r, \quad G_j = \tfrac{1}{T}\tfrac{\partial f_j}{\partial y}r, \\[6pt]
H_j = \tfrac{1}{T}\tfrac{\partial f_j}{\partial \theta}\tfrac{1}{Y}r, \quad I_j = \tfrac{1}{T}\tfrac{\partial h_j}{\partial \theta}(\theta, r, v)r, \\[6pt]
K_j = \tfrac{1}{T}\tfrac{\partial h_j}{\partial r}(\theta, r, v)vr, \quad L_i = \tfrac{1}{T}\dot{y}(T)\varphi_i(T)r, \\[6pt]
P_i = \tfrac{1}{T}y(T)\dot{\varphi}_i(T)r, \quad Q_i = y\Delta\varphi_i r, \\[6pt]
R_i = a(r, \theta)y\dot{\varphi}_i r, \quad N_i = y\ddot{\varphi}r, \\[6pt]
T_i = a\int_0^{rcos\theta}[y(T)\varphi_i(T) - y_0(r, \tau)\varphi_i(0)](\dot{r}cos\tau - rsin\tau)(\dot{r}sin\tau + rcos\tau)d\tau.
\end{cases}
$$

The density property of the selected sets in (27) causes its solution to tend to the solution of (26) when $M_1, M_2, ..., M_7 \to \infty$; thus, if numbers $M_1, ..., M_7$ are selected large enough, (27) is a good approximation of our main problem. Now, the number of constraints of the problem is finite, but the problem is still infinite since the underlying space is a subspace of measures. It would be more convenient if we could approximate the solution just by a solution of a simple finite LP. This is precisely our main attention.

Fakharzadeh et al. (1999) presented that the pair of the optimal measures of (25) are in the form of $\lambda^* = \sum_{m=1}^{M} \beta_m^* \delta(z_m^*)$ and $\mu^* = \sum_{n=1}^{N} \alpha_n^* \delta(Z_n^*)$ in which Z_n^* and z_m^* belong to dense subsets of D and D', respectively; moreover, $\delta(t)$ is a unitary atomic measure with support at the singleton set t. Substituting these forms in (27), it might seem that the problem has been made even more difficult, since, it is transformed into a non-linear one. But, if function $E(\mu, \lambda)$ can be minimized only with respect to the coefficients α_n^* and β_m^*, it will be turned to a linear programming problem. In other words, the solution can be obtained approximately by solving just the simple finite linear programming like below. If one chooses the points Z_n^* and z_m^* from the dense subsets of D and D', this fact could be achieved in the second step of our approximation. (see [6] for more details):

$$Min: \ E(\alpha, \beta, a, r_{d_a}) = \sum_{n=1}^{n=N} \alpha_n \Theta(Z_n)$$

$$S. \ to: \sum_{m=1}^{m=M} \beta_m \phi_k^g(z_m) = d_{\phi_k}, \qquad\qquad k = 1, 2, ..., M_1;$$

$$\sum_{m=1}^{m=M} \beta_m \psi_l^g(z_m) = 0, \qquad\qquad l = 1, 2, ..., M_2;$$

$$\sum_{m=1}^{m=M} \beta_m \nu_s(z_m) = a_s, \qquad\qquad s = 1, 2, ..., M_3;$$

$$\sum_{m=1}^{m=M} \beta_m \frac{1}{2} r_m^2 - \left(\frac{1}{T}\right) L \sum_{n=1}^{n=N} \alpha_n r_n = 0,$$

$$\sum_{n=1}^{n=N} \alpha_n F_i(Z_n) = 0, \qquad\qquad i = 1, 2, ..., M_4;$$

$$\sum_{n=1}^{n=N} \alpha_n G_j(Z_n) - \sum_{n=1}^{n=N} \alpha_n H_j(Z_n) = 0, \qquad j = 1, 2, ..., M_5;$$

$$\sum_{n=1}^{n=N} \alpha_n I_j(Z_n) - \sum_{n=1}^{n=N} \alpha_n K_j(Z_n) = 0, \qquad j = 1, 2, ..., M_6;$$

$$\sum_{n=1}^{n=N} \alpha_n [L_i(Z_n) - P_i(Z_n) - Q_i(Z_n) - R_i(Z_n) + N_i(Z_n)]$$

$$+ \sum_{m=1}^{m=M} \beta_m T_i(z_m) = \Phi_i, \qquad\qquad i = 1, 2, ..., M_7;$$

$$\alpha_n \geq 0, \qquad n = 1, 2, ..., N;$$

$$\beta_m \geq 0, \qquad m = 1, 2, ..., M.$$

$$(28)$$

Here, we defined $\Theta(Z) = (r/2T)[|\dot{y}(r, \theta, T)|^2 + (1/v^2 + 1/r^2)Y^2]$. Problem (28) is still non-linear because R_i and T_i are functions of the damping coefficient and d_{φ_k} is unknown since the constant point $(0, r_{d_a})$ of ω is unknown. Now, by using simultaneous three-phase search techniques for (28), the optimal damping coefficient, $r(0) = r(2\pi) = r_{d_a}$, and the optimal coefficients $\alpha_1^*, ..., \alpha_N^*, \beta_1^*, ..., \beta_M^*$ would be found as explained in next section. Thus, one is able to construct the pair of optimal shape and control function in the manner which has been explained in ([8], [7]).

6 Algorithm

To apply the mentioned method for solving problem (1) practically, here we present an algorithmic path for the solution procedure. Regarding the previous statements, we are able to identify the optimal control and optimal region by using the following 4 steps algorithm:

Step 1: The given sets $[0, T], J, A_1, S, S'$ and V which form Ω are divided into n_1, n_2, n_3, n_4, n_5 and n_6 equal parts, and the sets J, A_2 and U which form ω are divided into m_1, m_2 and m_3 equal parts, respectively; so that, the $N = n_1.n_2.n_3.n_4.n_5.n_6$, the number of 6-dimensional cells, and the $M = m_1.m_2.m_3$, the number of 3-dimensional cells in the related spaces are obtained. Then, in each of these 6-dimensional and 3-dimensional cells arbitrary points $Z_i = (t_i, \theta_i, r_i, y_i, Y_i, \nu_i)$ and $z_j = (\theta_j, r_j, u_j)$ are selected respectively.

Step 2: For fixed numbers M_1, M_2, M_3, we select M_1 number of $\phi_k^g(z)$, M_2 of $\psi_l^g(z)$, M_3 of $\nu_s(z)$, and for fixed numbers M_4, M_5, M_6, M_7, we define M_4 number of $F_i(Z)$, M_5 of $G_j(Z)$ and $H_j(Z)$, M_6 of I_j and K_j and M_7 of L_i, P_i, Q_i, R_i, N_i and T_i functions, respectively. Now, one is able to set up the finite linear programming (28) with $N + M$ variables and $M_1 + M_2 + ... + M_7$ constraints, which is dependent on the variables a and r_d.

Step 3: To solve problem (1), we use an iterative method with two inner loops and apply a three-phase optimization approach. In the first loop by giving a fixing the amount of a, the function $J_{1a} : [0, 1] \longrightarrow \mathbb{R}$ defined by $J_{1a}(r_d) = E^*(\alpha, \beta, r_d)$ is set up. Then, in the second loop, this function is minimized by the use of a standard minimization technique (like a line search method) as one of the optimization approaches. We remind that in each function, in calculating the standard minimization technique, its related LP (28) should be solved (one of the optimization phases).

If the minimizer of J_{1a} is called $r_{d_a}^*$ with the optimal value $J_{1a}^*(r_{d_a}) \equiv E^*(\alpha, \beta, a, r_{d_a}^*)$, one is able to set up the function $J_2 : [0, 1] \to \mathbb{R}$ by $J_2(a) = J_{1a}^*(r_{d_a}^*)$ in the first loop, use a search technique as the last phase of the optimization approach, determine the optimal damping coefficient, say a^*, with the optimal value of energy $J_2^*(a^*) \equiv E^*(\alpha, \beta, a^*, r_{d_a}^*)$. In this manner, the damping coefficient, the value of the energy of the system and the coefficients $\alpha^, s$ and $\beta^, s$ are simultaneously and optimally determined.

Remark 1. In each stage where alternative optimal cases happen, it suffices to select one arbitrarily.

Step 4: Regarding [20], [8] and [7], for the optimal values $\alpha_1^*, \alpha_2^*, ..., \alpha_N^*$, $\beta_1^*, \beta_2^*, ..., \beta_n^*$ obtained from Step 3, the optimal control and the optimal region are determined through the following instructions:

i) Let $\theta_0 = 0$ and $\theta_i = \theta_{i-1} + \beta_i^*$ for $i = 1, 2, ..., M$.
ii) For $\theta \in [\theta_{i-1}, \theta_i)$, set $u^*(\theta) = u_i$, where, u_i is the related component associated with point z_i. In this manner, according to [20], the nearly optimal control can be constructed as a piecewise constant function.
iii) Let $r_0 = r_{2\pi} = r_d$, using the differential equation $u(\theta) = \dot{r}(\theta)$, we take the following difference equation:

$$r_i = r_{i-1} + (\theta_i - \theta_{i-1}).u_i, \qquad i = 1, 2, .., M.$$

Therefore, M number points (θ_i, r_i), $i = 1, 2, ..., M$ of the nearly optimal region are determined. Using curve fitting or connecting them by line segments, we demonstrate the approximated optimal region.

Theorem 2. *If the used minimization techniques used in Step 3 of the*

above algorithm are convergent, then the algorithm converges to the optimal solution of (1) when $M, N, M_1, M_2, ..., M_7$ tend to infinity.

Proof. The proof of this theorem is given in Appendix A.

7 Examples

Now, to show the efficiency of our method and to explain how it works, we solve two numerical examples. It is worth mentioning that these examples are taken from [17] and [15] as well as from other studies cited by them in order for the readers to be able to compare the two methods.

Example 1: By defining $\Omega = (0,1) \times (0,1)$ and selecting $a = 10$ (constant), problem (1) was solved by Munch (2009) by the level set method. To apply this method, the author used the gradient descend method and also supplied some necessity relations by applying the finite difference method. In this manner, he used an initial shape to determine the optimal solution. We must mention that this approach is very time-consuming and the resulting optimal shape is also dependent on the number of iterations as shown on Page 25 of [17]. As it is also mentioned in Sub-section 5.1.2 Page 24, the results of the problem for variable a, depend on the initial shape. Moreover, in this case, the local minima have been obtained which are also completely dependent on the initial shape (Page 34 of [17]).

For the chosen initial conditions:

$$\begin{cases} y_0(\theta, r) = 100 sin(\pi r cos\theta) sin(\pi r sin\theta), \\ y_1(\theta, r) = 0, \quad (\theta, r) \in \Omega = J \times A_1, \end{cases}$$

The optimal value obtained by Munch (2009) for $a = 10$ and $T = 2$ was mentioned as $E(\omega, a, T) = 88.17$, and for $a = 10$ and $T = 1$, as $E(\omega, a, T) = 249.10$.

We considered the same condition as above, and additional conditions that are needed for our method as follows:

$$y(T = 1) = \dot{y}(T = 1) = 1.$$

We supposed $L = 0.11164$, the area of the unknown region ω was equal to 0.7 and $(0, r_{d_a})$ was a boundary point of ω which was determined optimally in domain Ω as mentioned in Step 3. Then, by selecting the following functions and setting them in (28) for $M_1 = 2, M_2 = 10, M_3 = 10, M_4 = M_5 = M_6 = M_7 = 2$, we set up the corresponding LP with:

$$\phi_1^g(\theta, r, u) = 2r\theta u + r^2; \qquad \phi_2^g(\theta, r, u) = 2ru;$$

$$\psi_l(\theta) = sin(l\theta), \ l = 1, 2, ..., 5; \ \psi_{l'}(\theta) = (1 - \cos(l'\theta)), \ l' = 1, 2, ..., 5;$$

$$J_s = [\frac{2\pi(s-1)}{10}, \frac{2\pi s}{10}]; \ a_s = \int_{J_s} d\theta = \frac{2\pi}{10}, \ s = 1, 2, ..., 10.$$

The functions introduced in (27), which expressed the relationship between y and Y and also between r and ν, were determined as below:

$$F_1(\theta, y, Y) = \theta y, \quad F_2(\theta, y, Y) = \theta^2 y Y, \quad G_1(\theta, y, Y) = r\theta,$$
$$G_2(\theta, y, Y) = \theta^2 r Y, \quad H_1(\theta, y, Y) = \frac{ry}{Y}, \quad H_2(\theta, y, Y) = 2r\theta y,$$
$$h_1(\theta, r, v) = \theta r, \quad h_2(\theta, r, v) = \theta r v, \quad I_1(\theta, r, v) = r^2,$$
$$I_2(\theta, r, v) = r^2 v, \quad K_1(\theta, r, v) = \theta r v, \quad K_2(\theta, r, v) = r\theta v^2.$$

Also, for $i = 1, 2$ we selected $\varphi_i = r^i sin(i\theta)t$; therefore:

$$\begin{cases} \varphi_i(T) = r^i sin(i\theta), \dot\varphi_i = r^i sin(i\theta), \\ \ddot\varphi_i = 0, \\ \Delta\varphi_i = \frac{\partial^2 \varphi_i}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi_i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi_i}{\partial \theta^2} \\ \quad = (i(i-1) + i - i^2)r^{i-2}sin(i\theta)t = 0. \end{cases}$$

Thus the functions in (27) were illustrated as:

$$\begin{cases} L_i = r^{i+1} sin(i\theta), \\ P_i = r^{i+1} sin(i\theta), \\ Q_i = 0, \\ R_i = ayr^{i+1} sin(i\theta) \\ T_i = a[\frac{(-1)(r^{i+1}u)}{2}(\frac{2i}{i^2-4} - (\frac{\cos((i+2)r\cos\theta)}{i+2} + \frac{\cos((i-2)r\cos\theta)}{i-2})) + \\ \quad \frac{(r^{i+2}+u^2 r^i)}{4}(\frac{sin(i-2)r\cos\theta}{i-2} - \frac{sin(i+2)r\cos\theta}{i+2})] \\ N_i = 0; \end{cases}$$

since:

$$T_i = a\int_0^{r\cos\theta}[y(T)\varphi_i(T) - y_0(r,\tau)\varphi_i(0)](\dot r sin\tau + r\cos\tau)(\dot r\cos\tau - r sin\tau)d\tau$$

$$= a\int_0^{r\cos\theta}[r^i sin(i\tau)](u\sin\tau + r\cos\tau)(u\cos\tau - r\sin\tau)d\tau$$

$$= a\int_0^{r\cos\theta}[r^i sin(i\tau)](ru(\cos 2\tau) + (u^2 - r^2)(\cos\tau)(\sin\tau))d\tau$$

$$= a\int_0^{r\cos\theta}[(r^{i+1}u)sin(i\tau)\cos 2\tau + (u^2 r^i - r^{i+2})(\sin(i\tau)\cos\tau\sin\tau)]d\tau$$

$$= a([(\frac{-1}{2})r^{i+1}u(\frac{\cos((i+2)\tau)}{i+2} + \frac{\cos((i-2)\tau)}{i-2})] + \frac{((ur)^i - r^{i+2})}{4}[\frac{sin(i-2)\tau}{i-2} - \frac{sin(i+2)\tau}{i+2}])|_0^{r\cos\theta}.$$

By dividing each of intervals $[0, 1]$ and $[0, 2\pi]$ into ten, $A_1 = [0, \sqrt{2}]$ and $S = [-2, 2]$ into five, $S' = [-2, 2]$ into four, $V = [-1, 1]$ and $A_2 = [0, 1]$ into ten and $U = [-0.6, 0.6]$ into eleven equal parts, we selected $N = 10^5$ points

of Z_i and $M = 1100$ points of z_i; thus, LP (28) was set up with $M + N$ variables and 30 constraints:

$$Min : E(\alpha, \beta, a, r_{d_a}) = \sum_{n=1}^{n=N} \alpha_n \frac{r_n}{2}[1 + (\frac{1}{v_n} + \frac{1}{r_n})Y_n^2]$$

$$S.\ to : \sum_{m=1}^{m=M} \beta_m(2r_m\theta_m u_m + r_m^2) = 2\pi(r_{d_a}^2);$$

$$\sum_{m=1}^{m=M} \beta_m 2r_m u_m = r_{d_a}^2;$$

$$\sum_{m=1}^{m=M} \beta_m(lr_m cos(l\theta_m) + u_m sin(l\theta_m)) = 0,\ l = 1, 2, ..., 5;$$

$$\sum_{m=1}^{m=M} \beta_m(l'r_m sin(l'\theta_m) + u_m(1 - cos(l'\theta_m))) = 0,\ l' = 1, 2, ..., 5;$$

$$\sum_{m=1}^{m=M} \frac{1}{2}r_m^2 \beta_m = 0.7;$$

$$\sum_{n=1}^{n=N} \alpha_n(r_n^2 - \theta_n r_n v_n) = 0;$$

$$\sum_{n=1}^{n=N} \alpha_n(r_n^2 v_n - \theta_n r_n v_n^2) = 0;$$

$$\sum_{n=1}^{n=N} \alpha_n(r_n\theta_n - \frac{r_n y_n}{Y_n}) = 0;$$

$$\sum_{n=1}^{n=N} \alpha_n(r_n\theta_n^2 Y_n - 2_n r_n y_n) = 0;$$

$$\sum_{n=1}^{n=N} \alpha_n(-ay_n r_n^4 sin(3\theta_n)) + \sum_{m=1}^{m=M} \beta_m(\frac{-1}{2}ar_m^4 u_m)[\frac{6}{5} - (\frac{\cos(5r_m cos\theta_m)}{5}$$
$$+ \cos(r_m cos\theta_m))]\frac{a(r_m^5 + u_m^2 r_m^3)}{4}[(sin(r_m cos\theta_m) - \frac{(sin(6r_m cos\theta_m)}{6}]$$
$$= \int_0^{2\pi} \int_0^{\sqrt{2}}(-r^2 sin(2\theta) sin(\pi r cos\theta) sin(\pi r sin\theta)drd\theta;$$

$$\sum_{n=1}^{n=N} \alpha_n(-ay_n r_n^5 sin(4\theta_n)) + \sum_{m=1}^{m=M} \beta_m(\frac{-1}{2}ar_m^5 u_m)[\frac{2}{3} - \frac{(\cos(6r_m cos\theta_m)}{6}$$
$$+ \frac{\cos(2r_m cos\theta_m)}{2})] + \frac{a(r_m^6 + u_m^2 r_m^4)}{4}(\frac{1}{2}[sin(2r_m cos\theta_m) - \frac{(sin(5r_m cos\theta_m)}{5}]$$
$$= \int_0^{2\pi} \int_0^{\sqrt{2}}(r^3 sin(3\theta) sin(\pi r cos\theta) sin(\pi r sin\theta)drd\theta;$$

$$\beta_1 + \beta_2 + ... + \beta_{110} = \frac{2\pi}{10};$$

$$\vdots$$

$$\beta_{991} + \beta_{992} + ... + \beta_{1100} = \frac{2\pi}{10};$$

$$\alpha_n \geq 0, \quad n = 1, 2, ..., N;\ \beta_m \geq 0, \quad m = 1, 2, ..., M.$$

$$\text{(29)}$$

For a fixed $a = 10$, using HBM (Honey-Bee-Method) (see [1]) and the modified Simplex method from MATLAB 7.6, we obtained the nearly optimal artificial control (see Figure 1), shape (Figure 2), point $(0, r_{d_a}^* = 0.35)$ and the also energy value as 175.9.

Comparing with [17], we found that the optimal value of energy obtained by using shape measure method was less, while the obtained optimal regions were mostly approximately the same. Additionally, our method took less time and the obtained optimal region was independent from the number of iterations.

Example 2: In this example, we obtained the optimal value of a and ω simultaneously. As mentioned by Munch et al. (2006), in spite of the initial conditions being symmetrical, for large values of a, the obtained optimal domain might be non-symmetrical but the value of energy would be less.

Consider the two-dimensional damped wave equation (1) which is expressed on a cyclic domain with center at origin and radius as $\sqrt{2}$ in time interval $[0, 1]$. The aim was to obtain the optimal region ω, with a known area in the circle such that energy of the system was minimized in the final time. The conditions were given same as Example 1, while a was variable and supposed to be optimally determined.

With the same action as Example 1 to set up the related linear programming (29), the random search method was applied to obtain the optimal value of J_{1a}. Also, the optimal value of function J_2 was determined by using the HBM. In this manner, the obtained results were as follows:

The optimal damping coefficient $a^* = 24.3624$, $r_{d_a}^* = 0.5469$ and the value of the optimal objective function was 0.101. The optimal (artificial) control and the optimal region ω are shown in Figures 3 and 4.

In Section 5.1.4 of [17] for $T = 1$, $a = 29.09$ and by 2000 iterations, the optimal value of energy was given as 12.56. As emphasized there, for variable a, the optimal domain is completely dependent on the initial shape. In our method, despite a's being variable, the obtained optimal region was independent of the initial shape and the amount of optimal energy was considerably less, while the time consumed also decreased.

8 Conclusion

By doing an embedding process and using the property of positive Radon measures, we presented a new and very useful technique for solving the problem of minimizing the energy of a damped wave system in an unknown region. In this method, the problem was solved by a three-phase optimization search technique where the unknown damping coefficient, the region and a point of its boundary were found optimally. This method has some advantages in comparison to the method used by Munch (2009), since we did not face the difficulties mentioned there; such as level set functions being flat, divergence of the systems with respect to dispersion, and the tendency of time toward infinity when damping of numerical waves approaches zero. The most important characteristic of our shape measure method is its simplicity and its independence from the solution of the initial shape. To obtain the optimal domain, we just need to use three search techniques while solving linear programming problems. Additionally, it is necessary to emphasize that this method is much easier, linear and less time-consuming.
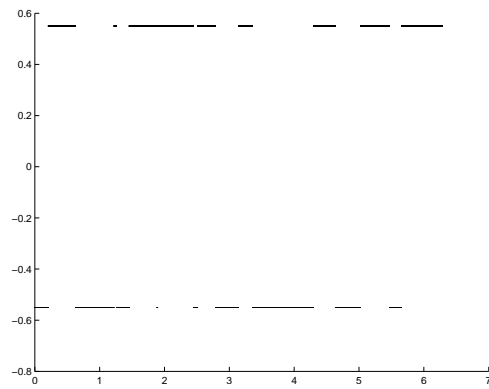
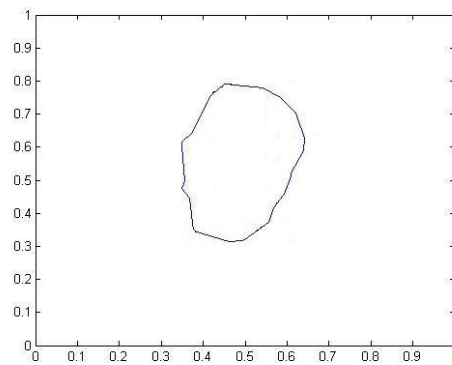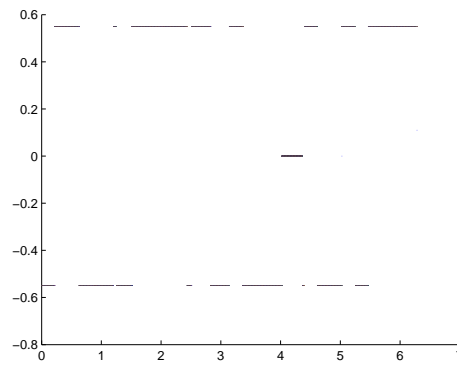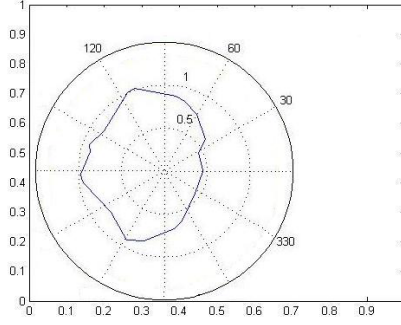Figure 1: Optimal control function in Example 1

Figure 2: Optimal domain with constant damping coefficient $a = 10$ in Example 1

Figure 3: Optimal control function in Example 2

Figure 4: Optimal region in the given domain Ω in Example 2

Appendix A. Proof of Theorem 2

To prove this theorem, first, we present the two following lemmas.

Lemma 1. Consider the linear program (27) consisting of minimizing the function $\mu \rightarrow \mu(\Theta) \equiv E(\mu, \lambda)$ over the set $Q(M_1, ..., M_7)$ of measures in $M^+(D) \times M^+(D')$ satisfying conditions (27). When $M_1, M_2, ..., M_7$ tend to infinity,

$$\eta(M_1, ..., M_7) \equiv \inf_{Q(M_1, ..., M_7)} E(\mu, \lambda)$$

tends to $\eta = \inf_Q \ E(\mu, \lambda)$. (This lemma is an extension of Proposition III.1 by Rubio (1986)).

Proof. (i) We prove, first, that the sequence $\{\eta(M_1, M_2, ..., M_7)\}$ is convergent when $M_1, ..., M_7$ tend to infinity; consider, first, the subsequence of $\eta(M_1, M_2, ..., M_7)$ in form $\{\eta(M_1, M_1, ..., M_1) : M_1 = 1, 2, ...\}$. Since $Q(M_1, ..., M_1)$ is a subset of positive Radon measures that satisfies in constraint (26). Therefore, since

$Q(1, 1, ..., 1) \supset Q(2, 2, ..., 2) \supset Q(3, 3, ..., 3) \supset ... \supset Q(M_1, ..., M_1) \supset ... \supset Q,$

$then, \eta(1, 1, ..., 1) \le \eta(2, 2, ..., 2) \le ... \le \eta(M_1, ..., M_1) \le ... \le \eta.$

This sequence is non decreasing and bounded above and hence it converges to a number $\zeta \le \eta$; thus, if $\epsilon > 0$, for $M_1 > N(\epsilon)$, we have:

$$|\eta(M_1, M_1, ..., M_1) - \zeta| < \epsilon \tag{30}$$

consider now $\eta(M_1, M_2, M_1, ..., M_1)$ for both M_1 and M_2 larger than $N(\epsilon)$. Without loss of generality, assume that $M_1 > M_2$. Then:

$$\eta(M_2, M_2, ..., M_2) \le \eta(M_1, M_2, M_1, ..., M_1) \le \eta(M_1, M_1, ..., M_1);$$

therefore,

$$\eta(M_2, M_2, ..., M_2) - \zeta \leq \eta(M_1, M_2, M_1, ..., M_1) - \zeta \leq \eta(M_1, M_1, ..., M_1) - \zeta,$$

and according to (30), we have:

$$|\eta(M_1, M_2, ..., M_1) - \zeta| \leq \epsilon. \tag{31}$$

Now consider $\eta(M_1, M_2, M_3, M_1, ..., M_1)$ for $M_1 \geqslant M_2 \geqslant M_3 \geqslant N(\epsilon)$. Then, by the same procedure, one could show that:

$$|\eta(M_1, M_2, M_3, M_1, ..., M_1) - \zeta| \leq \epsilon. \tag{32}$$

In a similar manner, for $M_1 > M_2 > M_3 > M_4 \geq N(\epsilon)$, we have:

$$\eta(M_4, M_4, ..., M_4) \leq \eta(M_1, M_4, M_4, M_4, M_1, ..., M_1)$$
$$\leq \eta(M_1, M_2, M_4, M_4, M_1, ..., M_1) \leq \eta(M_1, M_2, M_3, M_4, M_1, ..., M_1)$$
$$\leq \eta(M_1, M_2, M_3, M_1, M_1, ..., M_1),$$

and by using (30) and (32), we have:

$$|\eta(M_1, M_2, M_3, M_4, M_1, ..., M_1) - \zeta| \leq \epsilon.$$

Finally, in a similar way, for $M_1 \geq M_2 \geq ... \geq M_7 \geq N(\epsilon)$, one can show that:

$$\eta(M_7, M_7, ..., M_7) \leq \eta(M_1, M_2, ..., M_7) \leq \eta(M_1, M_2, ..., M_6, M_1),$$

and hence:

$$|\eta(M_1, M_2, ..., M_7) - \zeta| \leq \epsilon.$$

Thus, the sequence $\{\eta(M_1, M_2, ..., M_7), M_1 = 1, 2, ..., M_7 = 1, 2, ...\}$ converges to the number ζ as $M_1, ..., M_7$ tend to infinity.

(ii) We must prove now that the limit ζ equals $\eta = \inf_Q \ E(\mu, \lambda)$.
We, first, show that this limit ζ can be computed sequentially. It is known that

$$\zeta = \lim_{M_1 \to \infty} [\lim_{M_2 \to \infty} [...[\lim_{M_7 \to \infty} \eta(M_1, ..., M_7)]]...],$$

provided that $\lim_{M_7 \to \infty} \eta(M_1, ..., M_7)$ exist since ζ is a finite number. To show the existence of this, we fix $M_1, M_2, ..., M_6$ and vary M_7; since

$$Q(M_1, ..., M_6, 1) \supset Q(M_1, ..., M_6, 2) \supset ... \supset Q(M_1, ..., M_7) \supset ... \supset Q;$$

thus,

$$\eta(M_1, ..., M_6, 1) \leq \eta(M_1, ..., M_6, 2) \leq ... \leq \eta(M_1, ..., M_7) \leq ... \leq \eta.$$

For $M_6 = 1, 2, ...$ the non decreasing and bounded above sequence $\{\eta(M_1, ..., M_7), M_7 = 1, 2, ...\}$ converges to number $\zeta(M_1, ..., M_6)$. Hence, the double limit $\lim_{M_6 \to \infty} \lim_{M_7 \to \infty}$ can be computed sequentially.

Now we define

$$Q(M_1, ..., M_6) \equiv \cap_{m_7=1}^{\infty} Q(M_1, ..., M_6, M_7) \equiv Q(M_1, ..., M6, \infty).$$

For the fixed numbers $M_1, M_2, ..., M5$, since $\zeta(M_1, ..., M_6) = \lim_{M_7 \to \infty} \eta(M_1, ..., M_7) = \inf_{Q(M_1, ..., M_6)} \ E(\mu, \lambda)$, and

$$Q(M_1, ..., M_5, 1) \supset ... \supset Q(M_1, ..., M_5, M_6) \supset Q,$$

we have:

$$\zeta(M_1, ..., M_5, 1) \leq \zeta(M_1, ..., M_5, 2) \leq ... \leq \zeta(M_1, ..., M_5, M_6) \leq \eta,$$

thus $\zeta(M_1, ..., M_5, M_6)$ is convergent where $M_6 \to \infty$ and we have:

$$\zeta(M_1, ..., M_5) = \lim_{M_6 \to \infty} \zeta(M_1, ..., M_5, M_6)$$

$$= \lim_{M_6 \to \infty} [\lim_{M_7 \to \infty} \eta(M_1, ..., M_6, M_7)].$$

In a similar manner, by defining

$$Q(M_1, ..., M_5) \equiv \cap_{M_6=1}^{\infty} Q(M_1, ..., M_6),$$

we would have:

$$\zeta(M_1, ..., M_4) = \lim_{M_5 \to \infty} \zeta(M_1, ..., M_4, M_5)$$
$$= \lim_{M_5 \to \infty} [\lim_{M_6 \to \infty} [\lim_{M_7 \to \infty} \eta(M_1, ..., M_6, M_7)]].$$

therefore, in the last stage, we obtain $\lim_{M_1 \to \infty} \zeta(M_1) = \zeta$.

(iv) Regarding (i) and (ii), now we can prove $\zeta = \eta$. Let

$$P \equiv \cap_{M_1=1}^{\infty} \cap_{M_2=1}^{\infty} ... \cap_{M_6=1}^{\infty} \cap_{M_7=1}^{\infty} Q(M_1, ..., M_7),$$

then $P \supseteq Q$, since $Q(M_1, ..., M_7) \supset Q$ for all $M_1, M_2, ..., M_7$. We can show that under the conditions of the problem, $Q \supset P$; thus $Q = P$, that will finally imply

$$\zeta = \lim_{M_1 \to \infty} ...[\lim_{M_6 \to \infty} \zeta(M_1, ..., M_6)] = \inf_Q \ E(\mu, \lambda),$$

which is the contention in the theorem.

For this purpose, we prove that if $(\mu, \lambda) \in P$, then they are also in Q. For a set of total functions such as ϕ_k, $k = 1, 2, ...$, we have $\lambda(\phi_k^g) = d_{\phi_k}$, according to the definition of P. Based on the definition of total functions,

since $\phi^g = \phi_r u + \phi_\theta$ on D', $sup_{D'} |\phi_r - \phi_{r_k}|$, $sup_{D'} |\phi_\theta - \phi_{\theta_k}|$ tend to zero as k tends to infinity, where ϕ_k^g is defined in relationship (27). Therefore:

$$|\lambda(\phi^g) - d_\phi| = |\lambda(\phi^g) - d_\phi - \lambda(\phi_k^g) + d_{\phi_k}|$$
$$= |\int_{\partial\omega} (\phi_r u + \phi_\theta)d\theta - \int_{\partial\omega} (\phi_{r_k} u + \phi_{\theta_k})d\theta - (d_\phi - d_{\phi_k})|$$
$$\leq \int_{\partial\omega} |\phi_r - \phi_{r_k}||u|d\theta + \int_{\partial\omega} |\phi_\theta - \phi_{\theta_k}|d\theta + |d_\phi - d_{\phi_k}|$$
$$\leq K_1 sup_{D'} |\phi_r - \phi_{r_k}| + K_2 sup_{D'} |\phi_\theta - \phi_{\theta_k}| + K_3 sup_{D'} |\phi(r,\theta) - \phi_k(r,\theta)|,$$

tend to zero and hence $\lambda(\phi^g) = d_\phi$, i.e. $\lambda \in Q$, where K_1, K_2 and K_3 are constant numbers. Using the similar method, for functions ψ^g and υ_s, we prove the above relationship. Let $\mu \in P$, thus $\mu(F_i) = 0$ (F_i was defined in (27)), based on the definition of total functions, for every given $\varphi \in H_0^1(D)$ and $\epsilon > 0$, there are integer $N > 0$ and scalars γ_i, so that

$$sup_{[0,T]} sup_\Omega \|div(y\varphi r) - \sum_{i=1}^N \gamma_i div(y\varphi_i r)\| < \epsilon;$$

therefore, for every $F \in C(\Omega \times [0,T])$, we have

$$|\mu(F)| = |\mu(F) - \sum_{i=1}^N \gamma_i \mu(F_i)|$$
$$= |\int_0^T \int_\Omega div(y\phi r)drd\theta dt - \sum_{i=1}^N \int_0^T \int_\Omega \gamma_i div(y\varphi_i r)drd\theta dt|$$
$$\leq E_1 sup_{[0,T]} sup_\Omega \|div(y\varphi r) - \sum_{i=1}^N \gamma_i div(y\varphi_i r)\| \leq E_1 \epsilon; \ (E_1 constant).$$

Because ϵ is arbitrary, $|\mu(F)| \to 0$. Sequentially, by considering the density of the functions $f_i \in C(E)$, $h_i \in C(E')$ and $\varphi_i \in H_0^1(D)$, we use the same method in the case of functions G_j, H_j, I_j, K_j, L_i, P_i, Q_i, R_i, N_i and T_i to prove that $\mu \in Q$. Therefore, $P \subset Q$ and the proof is finished.$\square$

Lemma 2. For every $\epsilon > 0$, the problem of minimizing the function $\sum_{n=1}^N \alpha_n \Theta(Z_n)$ on the set $P(M_1, M_2, ..., M_7)^\epsilon$ described by the inequalities (34) has a solution for sufficiently large $N = N(\epsilon)$. The solution satisfies:

$$\eta(M_1, ..., M_7) + \rho(\epsilon) \leq \sum_{n=1}^N \alpha_n \Theta(Z_n) \leq \eta(M_1, ..., M_7) + \epsilon, \qquad (33)$$

where $\rho(\epsilon)$ tends to zero as ϵ tends to zero.

$$-\epsilon \leq \sum_{m=1}^{m=M} \beta_m \phi_k^g(z_m) - d_{\phi_k} \leq \epsilon, \qquad\qquad k = 1, 2, ..., M_1;$$

$$-\epsilon \leq \sum_{m=1}^{m=M} \beta_m \psi_l^g(z_m) - 0 \leq \epsilon, \qquad\qquad l = 1, 2, ..., M_2;$$

$$-\epsilon \leq \sum_{m=1}^{m=M} \beta_m \nu_s(z_m) - a_s \leq \epsilon, \qquad\qquad s = 1, 2, ..., M_3;$$

$$-\epsilon \leq \sum_{m=1}^{m=M} \beta_m \frac{1}{2} r_m^2 - (\frac{1}{T}) L \sum_{n=1}^{n=N} \alpha_n r_n \leq \epsilon,$$

$$-\epsilon \leq \sum_{n=1}^{n=N} \alpha_n F_i(Z_n) - 0 \leq \epsilon, \qquad\qquad i = 1, 2, ..., M_4;$$

$$-\epsilon \leq \sum_{n=1}^{n=N} \alpha_n G_j(Z_n) - \sum_{n=1}^{n=N} \alpha_n H_j(Z_n) - 0 \leq \epsilon, \qquad j = 1, 2, ..., M_5;$$

$$-\epsilon \leq \sum_{n=1}^{n=N} \alpha_n I_j(Z_n) - \sum_{n=1}^{n=N} \alpha_n K_j(Z_n) - 0 \leq \epsilon, \qquad j = 1, 2, ..., M_6;$$

$$-\epsilon \leq \sum_{n=1}^{n=N} \alpha_n[L_i(Z_n) - P_i(Z_n) - Q_i(Z_n) - R_i(Z_n) + N_i(Z_n)]$$
$$+ \sum_{m=1}^{m=M} \beta_m T_i(z_m) - \Phi_i \leq \epsilon, \qquad\qquad i = 1, 2, ..., M_7;$$

$$\alpha_n \geq 0, \quad n = 1, 2, ..., N; \quad \beta_m \geq 0, \quad m = 1, 2, ..., M.$$

$$(34)$$

This lemma is the developed Theorem III.1 from Rubio (1986).

Proof. (i) Given $\epsilon > 0$, a set

$$\{z^k, k = 1, 2, ..., M_1 + M_2 + M_3, Z^h, h = 1, 2, ..., M_4 + M_5 + M_6 + M_7\} \quad (35)$$

can be introduced, as in the proof of Proposition III.3 of [20], so that inequalities (34) are satisfied. For sufficiently large N and M, the set

$$w_{N,M} = \{z_i : i = 1, 2, ..., N, Z_j : j = 1, 2, ..., M\} \subset w$$

will contain (37); thus, the set $P(M_1, M_2, ..., M_7)^\epsilon$ is nonempty for such values of N and M, since the N-tuple $\{\beta_k^*, k = 1, 2, ..., M_1 + M_2 + M_3, 0, 0, ..., 0\}$ and M-tuple $\{\alpha_l^*, l = 1, 2, ..., M_4 + ... + M_7, 0, 0, ..., 0\}$ are in this set. From the first set of inequalities of (34), by $z_m = 1$ for all $m = 1, 2, ..., M$, we have $-\epsilon \leq \sum_{m=1}^M \beta_m - \Delta t \leq \epsilon$ and from the fifth set of inequalities for $F_i(Z_n) = 1$, we have $-\epsilon \leq \sum_{n=1}^N \alpha_n \leq \epsilon$; this set of N and M-tuples with nonnegative entries is bounded and also closed; thus, it is compact and the linear function $\sum_{n=1}^N \alpha_n \Theta(Z_n)$ attains its minimum over this set. Hence, we have:

$$min \quad \sum_{n=1}^N \alpha_n \Theta(Z_n) \leq \sum_{k=1}^{M_1+...+M_7} \alpha_k \Theta(Z^k) \leq \eta(M_1, ..., M_7) + \epsilon; \quad (36)$$

therefore, one of the inequalities of (33) has been proved. For the other, let us define $Q(M_1, ..., M_7)^\epsilon$, by using equations (26) as follow:

$$Q(M_1, ..., M_7)^\epsilon = \{\mu \in M^+(D), \lambda \in M^+(D^{'}) || \lambda(\phi_k^g(\theta, r, u)) - d_{\phi_k}| \leq \epsilon,$$

$$k = 1, 2, ..., M_1; \quad |\lambda(\psi_l^g(\theta, r, u)) - 0| \leq \epsilon, \ l = 1, 2, ..., M_2; ...,$$

$$|\mu(L_i) - \mu(P_i) - \mu(Q_i) - \mu(R_i) + \lambda(T_i) + \mu(N_i) - \Phi_i| \leq \epsilon, \quad i = 1, 2, ..., M_7\}.$$

Then, the set of measures of the type $\mu = \sum_{n=1}^{N} \alpha_n \delta(Z_n)$ and $\lambda = \sum_{m=1}^{M} \beta_m \delta(z_m)$ with the coefficients α_n and β_m in the set $P(M_1, ..., M_7)^\epsilon$, is a subset of $Q(M_1, ..., M_7)^\epsilon$. Thus,

$$min \sum_{n=1}^{N} \alpha_n \Theta(Z_n) \geq min\mu(\Theta), \qquad (37)$$

where the minimum in the left-hand side of this inequality is over the set $P(M_1, ..., M_7)^\epsilon$ and the one in right-hand side is over $Q(M_1, ..., M_7)^\epsilon$. Also, $Q(M_1, ..., M_7) = \cap_{\epsilon>0} Q(M_1, ..., M_7)^\epsilon$, and

$$Q(M_1, ..., M_7)^{\epsilon_1} \supset Q(M_1, ..., M_7)^{\epsilon_2} \quad if \quad \epsilon_1 > \epsilon_2. \qquad (38)$$

Let $\eta(M_1, ..., M_7, \epsilon)$ be the infimum of $\mu(\Theta)$ over the set of measures $Q(M_1, ..., M_7)^\epsilon$. Then, by (40), we have $\eta(M_1, ..., M_7, \epsilon_1) \leq \eta(M_1, ..., M_7, \epsilon_2)$, if $\epsilon_1 > \epsilon_2$.

It is sufficient for our purposes to consider a sequence of values of $\epsilon = 1/p$ where $p = 1, 2, ...$ Then,

$$\eta(M_1, ..., M_7, 1) \leq \eta(M_1, ..., M_7, \frac{1}{2}) \leq ... \leq \eta(M_1, ..., M_7, \frac{1}{p}) \leq ... \leq \eta,$$

the sequence $\{\eta(M_1, ..., M_7, \frac{1}{p})\}$ is non decreasing and bounded above. Therefore, it converges to a number $\gamma(M_1, ..., M_7)$ satisfying

$$\gamma(M_1, ..., M_7) = \lim_{p \to \infty} \eta(M_1, ..., M_7, \frac{1}{p}) = \inf_{Q(M_1, ..., M_7)} \mu(\Theta) = \eta(M_1, ..., M_7).$$

Thus

$$\rho(\epsilon) \equiv \eta(M_1, ..., M_7, \epsilon) - \eta(M_1, ..., M_7) \qquad (39)$$

tends to zero as ϵ tends to zero; it follows from (39) and (41) that

$$min \sum_{n=1}^{N} \alpha_n \Theta(Z_n) \geq min\mu(\Theta) = \eta(M_1, ..., M_7) + \rho(\epsilon),$$

where the left-hand minimum is over the set $P(M_1, ..., M_7)^\epsilon$ and the right-hand one is over $Q(M_1, ..., M_7)^\epsilon$. Now we prove Theorem 2 as follow:

Theorem 2. *If the used minimization techniques in Step 3 of the above algorithm are convergent, then, the algorithm converges to the optimal solution of (1) when $M, N, M_1, M_2, ..., M_7$ tend to infinity.*

Proof. To demonstrate the proof, we have used the proof by contradiction. Let $(\alpha^*, \beta^*, a^*, r_d^*)$ be the minimizer of $E(\alpha, \beta, a, r_d)$ but (ω^*, a^*) is not the minimizer of $E(\omega, a, t)$, this means that the algorithm does not converge to

the solution of (1). Thus, there is (ω', a') such that

$$E(\omega', a', T) < E(\omega^*, a^*, T). \tag{40}$$

According to the Riesz representation theorem and considering one to one transformation of problem (1), with objective function defined in relationship (2), to problem (26), there are unique measures $\mu'(\Theta)$ and $\mu^*(\Theta)$ corresponding to $E(\omega', a', T)$ and $E(\omega^*, a^*, T)$, where:

$$\mu'(\Theta) \equiv E(\mu', \lambda') < \mu^*(\Theta) \equiv E(\mu^*, \lambda^*).$$

According to Lemma 1, we have:

$$\eta(M_1, ..., M_7) = \inf_{Q(M_1,...,M_7)} \mu(\Theta) \to \eta = \inf_Q \mu(\Theta) = \inf E(\mu, \lambda),$$

and according to Lemma 2,

$$\eta(M_1, ..., M_7) + \rho(\epsilon) \leq \sum_{n=1}^{N} \alpha_n \Theta(Z_n) \leq \eta(M_1, ..., M_7) + \epsilon,$$

therefore,

$$\mu'(\Theta) = \inf_Q \mu(\Theta) \equiv inf\ E(\mu, \lambda) = E(\mu', \lambda') < E(\mu^*, \lambda^*)$$

and

$$\eta' + \rho(\epsilon) \leq \sum_{n=1}^{N} \alpha_n' \Theta(Z_n) \equiv E(\alpha', \beta', a', r_{d_a}') \leq \eta' + \epsilon,$$

according to the above relationships:

$$\sum_{n=1}^{N} \alpha_n' \Theta(Z_n) < \sum_{n=1}^{N} \alpha_n^* \Theta(Z_n)$$

thus according to (42), we have:

$$E(\alpha', \beta', a', r_d') < E(\alpha^*, \beta^*, a^*, r_d^*).$$

This is in contradiction with what we supposed at the beginning, thus, (ω^*, a^*) is minimizer of problem (1).

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یک روش خطی برای طراحی بهینه چشمه میرایی دارای اتلاف داخلی

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چکیده : یک سیستم موج میرای دو بعدی را در نظر بگیرید، که دارای اتلاف انرژی درزیرمجموعه‌ای مجهول از دامنه موج با پارامتر میرایی مجهول می‌باشد. هدف حل مساله بد وضع طراحی شکل، شامل بهینه‌سازی شکل این زیرمجموعه جهت کمینه کردن انرژی سیستم در یک زمان معین می‌باشد. با استفاده از یک الگوریتم جدید براساس روش نشاندن، برای اولین بار، معادلات سیستم را در قالب تغییراتی نوشته؛ پس از آن، با انتقال مساله به مختصات قطبی و تعریف دو اندازه رادون مثبت، مساله را در فضای اندازه‌ها نمایش می‌دهیم. در این روش، مساله طراحی شکل بهینه به یک مساله برنامه‌ریزی خطی نامتناهی تبدیل می‌شود که وجود جواب آن تضمین شده است. دراین مرحله، با استفاده از دو گام تقریب، جواب بهینه (کنترل بهینه، ناحیه بهینه، پارامتر میرایی بهینه و انرژی بهینه) با یک روش جستجوی سه مرحله‌ای مشخص می‌گردد. به منظور مقایسه این روش جدید با دیگر روش ها شبیه سازی عددی نیزآورده شده است. .

کلمات کلیدی : معادله موج میرا؛ کنترل اتلاف؛ اندازه رادون؛ روش جستجو؛ بهینه‌سازی شکل.