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Research Article

# Connected bin packing problem on traceable graphs

A. Nejoomi and A. Dolati*

**Abstract**

We consider a new extension of the bin packing problem in which a set of connectivity constraints should be satisfied. An undirected graph with a weight function on the nodes is given. The objective is to pack all the nodes in the minimum number of unit-capacity bins, such that the induced subgraph on the set of nodes packed in each bin is connected. After analyzing some structural properties of the problem, we present a linear time approximation algorithm for this problem when the underlying graph is traceable. We show that the approximation factor of this algorithm is 2 and this factor is tight. Finally, concerning the investigated structural properties, we extend the algorithm for more general graphs. This extended algorithm also has a tight approximation factor of 2.

**AMS subject classifications (2020):** 90C27, 68Q25, 68W25.

**Keywords:** Bin Packing Problem; Connectivity; Complexity Theory; Approximation Algorithms.

## 1 Introduction

The bin packing problem (BPP) has been studied extensively in the literature due to its numerous applications and its intriguing combinatorial structure. The BPP consists of packing a given set of items with different weights into

*Corresponding author
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Ahmad Nejoomi
Department of Computer Science, Shahed University, Tehran, Iran.
e-mail: a.nejoomi@shahed.ac.ir

Ardeshir Dolati
Department of Computer Science, Shahed University, Tehran, Iran.
e-mail: dolati@shahed.ac.ir

a minimum number of same bins so that the total weight packed in any bin does not exceed the capacity. The BPP is known to be NP-hard [9].

Various extensions of the BPP have been studied in the literature. In the *BPP with conflicts* [12], any two conflict items must not be packed in a same bin. The *BPP with precedence constraints* [6] requires that the set of given precedences among the items is satisfied. The aim in *BPP with fragile objects* [2] is to pack the given weighted fragile items into the minimum number of bins, such that the total weights of each bin do not exceed the fragility of the most fragile item in the bin. Böhm et al. [3] introduced the *colored BPP* in which each item has a color. As additional condition to the capacity constraint, we require that no two items of the same color are packed into a bin consecutively.

There are huge number of algorithms for the BPP in the literature. Among these methods, some basic and simple algorithms were mostly considered by many researchers.

- The next fit (NF) algorithm. Consider the items sorted in any predefined order. In each round, if an item fits inside the current bin, then the item is packed inside it. Otherwise, the current bin is closed, a new bin is opened, and the current item is packed inside the new current bin. The running time can be bounded by $O(n\log(n))$, where $n$ is the number of items. The NF algorithm is a 2-approximation algorithm; see [13].

- The first fit (FF) algorithm. Suppose that the items are sorted in any order. In this order, the next item is always packed into the first bin, where it fits. The running time can be bounded by $O(n\log(n))$, where $n$ is the number of items. The FF algorithm is a 1.7-approximation algorithm; see [8].

- The first fit decreasing (FFD) algorithm. Sort the items in non-increasing order of their sizes and then apply the first fit algorithm. Same as FF, the running time is bounded by $O(n\log(n))$, where $n$ is the number of items. Dosa [7] showed that $FFD(I) \leq 11/9OPT(I) + 6/9$ and that this bound is tight.

In the connected variant of BPP, connected bin packing problem (CBPP), we consider the items as the nodes of a graph. For each instance of the CBPP, a graph $G = (V, E)$ and a set of infinite number of unit capacity bins are given. Each node $v \in V$ is characterized by the weight $w(v)$. Like BPP, the total weight packed in any bin could not exceed the unit. In this extension of BPP, each pair of nodes belonging to the same bin must be connected. In other words, the induced subgraph on the set of nodes packed in each bin is connected. The aim is to pack of the nodes in a minimum number of bins.

The CBPP arises in many real-world applications. When we need to pack a large number of spatial objects into a small number of contiguous regions [1], we have a CBPP.

Clearly, the basic BPP is a special case of the CBPP, in which the graph $G$ is a complete graph. Since the BPP is strongly NP-hard, by restriction, the CBPP is also strongly NP-hard. By similar conclusions, the CBPP does not admit an approximation algorithm with a ratio smaller than $\frac{3}{2}$.

In other view, the CBPP is related to the balanced connected partitioning problem [4]. Then the CBPP can be seen as a variant of *problem of partitioning a weighted graph to connected subgraphs of almost uniform size* [11]. In this problem, the aim is to partition a given node weighted graph with integer weights in a minimum (or maximum) number of connected components. When the underlying graph is a tree, then the problem could be handled by the $O(n^6)$ time algorithm [10].

The "covering problem with capacitated subtrees" [5] is another related problem, in which vertices of a graph have to be covered by rooted subtrees.

The rest of the paper is organized as follows. In Section 2, we give a formal definition of the CBPP and some complexity issues. In Section 3, we present some approximation algorithms for the CBPP on traceable graphs and some general graphs.

## 2 Preliminaries and approximation

In this Section, after a formal definition of the CBPP and preliminaries, we give some structural properties of the problem.

**Definition 1.** Let $G = (V, E)$ be an undirected graph with a weight function $w : V \to [0, 1]$ on its nodes. The aim is to pack all the nodes in the minimum number of unit capacity bins, such that the set of nodes packed in each bin induces a connected subgraph (see Figure 1, for example).

For a given CBPP (i.e., a graph $G = (V, E)$ with a weight function $w : V \to [0, 1]$) and any partition of nodes $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$, we define the merged graph $G_{\mathcal{S}} = (V_{\mathcal{S}}, E_{\mathcal{S}})$ associated with $\mathcal{S}$ as follows: We merge elements of each subset $S_i$ in the single node $v_{S_i}$ after deleting edges among them and merge the resulting multiple parallel edges in a single edge. The weight of each merged nodes, denoted by $W_{\mathcal{S}}(\cdot)$, is the sum of weights of nodes merged in it (i.e., for all $S_i \in \mathcal{S}$, $W_{\mathcal{S}}(v_{S_i}) = \sum_{v \in S_i} w(v)$).

**Theorem 1.** Suppose that a graph $G = (V, E)$ with a weight function $w : V \to [0, 1]$ is given. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ be a feasible solution and let $G_{\mathcal{B}} = (V_{\mathcal{B}}, E_{\mathcal{B}})$ be the corresponding merged graph. If for each edge $(v_i, v_j) \in E_{\mathcal{B}}$, $W_{\mathcal{B}}(v_i) + W_{\mathcal{B}}(v_j) \geq 1$ and $G_{\mathcal{B}}$ has a maximal matching of size $\lfloor \frac{m}{2} \rfloor$, then $m \leq 2m^* - 1$, where $m^*$ is the size of optimal packing.

*Proof.* Suppose that $M$ is such matching. Then

$$\text{for all } (i, j) \in M, \quad W_{\mathcal{B}}(v_i) + W_{\mathcal{B}}(v_j) \geq 1.$$

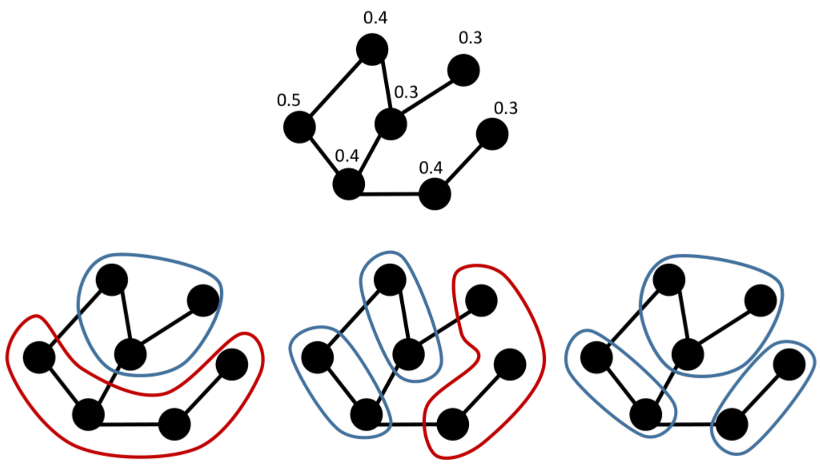


Figure 1: A CBBP example: Feasible and infeasible solutions

We have two cases:

- If $m$ is an even number, then $M$ is a maximum matching and

$$m^* \overset{\text{Trivial Upper Bound}}{\geq} \lceil \sum_{k \in V} W_B(v_k) \rceil \geq \sum_{(i,j) \in M} (W_B(v_i) + W_B(v_j)) > \frac{m}{2}.$$

- If $m$ is an odd number, then $M$ is a matching of size $\frac{m-1}{2}$ and covers $m-1$ nodes. If we denote the uncovered node by $v$, then at least $\frac{m-1}{2}$ bins are needed for packing $G - v$. Also,

$$m^* > \frac{m-1}{2} \rightarrow m^* \geq \frac{m-1}{2} + 1 \rightarrow m \leq 2m^* - 1.$$

□

Now, we investigate a trivial feasible solution to a CBPP instance that the underling graph is union of two disjoint subgraphs and a cutting edge. Suppose that two corresponding feasible solutions to CBPP's on this subgraphs are given. A trivial feasible solution to the CBPP on a hole graph is the union of the given subsolutions. In the following theorem, we prove some results for this solution.

**Theorem 2.** Suppose that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two disjoint connected graphs with weight functions $w_1$ and $w_2$ in $[0, 1]$ and that $B_1^*$ and $B_2^*$ are optimal packings for them, respectively. Let $G = (V, E)$ be a graph with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup \{(x, y)\}$, where $x \in V_1$ and $y \in V_2$. Let $B^*$ be an optimal packing for $G$. Then

$$|B^*| \geq |B_1^*| + |B_2^*| - 1.$$

*Proof.* If $x$ and $y$ are packed separately in $B^*$, then the proof is trivial. Hence we suppose that they are packed in $B_e$, where $e = (x, y)$. Let $|B^*| = k_1 + k_2 + 1$, where $k_1$ and $k_2$ are the number of bins used for packing $G_1 - B_e$ and $G_2 - B_e$, respectively. Therefore,

$$k_1 \geq |B_1^*| - 1,$$

$$k_2 \geq |B_2^*| - 1,$$

so

$$|B^*| = k_1 + k_2 + 1 \geq |B_1^*| + |B_2^*| - 1$$

□

This result can be extended as the following theorem.

**Theorem 3.** Suppose that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two disjoint connected graphs with weight functions $w_1$ and $w_2$ in $[0, 1]$ and that $B_1$ and $B_2$ are feasible packings with approximation factor of $\alpha_1$ and $\alpha_2$ for them, respectively. Let $G = (V, E)$ be any graph with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup \{(x, y)\}$, where $x \in V_1$ and $y \in V_2$. Then $B = B_1 \cup B_2$ is a feasible packing with approximation factor of $\alpha_{\max} = \max\{\alpha_1, \alpha_2\}$.

*Proof.* Let $B^*, B_1^*, B_2^*$ are optimal packings of $G, G_1, G_2$, respectively. By Theorem 2,

$$|B^*| \geq |B_1^*| + |B_2^*| - 1.$$

On the other hand, $|B_1| \leq \alpha_1 |B_1^*|$ and $|B_2| \leq \alpha_2 |B_2^*|$. Without loss of generality, suppose that $\alpha_1 \geq \alpha_2$. Then we have

$$\alpha_1 |B^*| \geq \alpha_1 |B_1^*| + \frac{\alpha_1}{\alpha_2} \alpha_2 |B_2^*| - 1.$$

Thus

$$\alpha_1 |B^*| \geq |B_1| + \frac{\alpha_1}{\alpha_2} |B_2| - 1 \geq |B_1| + |B_2| - 1.$$

□

**Corollary 1.** The complete graph $G_0 = (V_0, E_0)$ and the tree $T = (V_T, E_T)$ with weight functions $w_0$ and $w_T$ in $[0, 1]$ are given. Let $B_0$ is a packing for the corresponding BPP with weight function $w_1$ obtained from the decreasing first fit algorithm. Let $B_T$ be an optimal packing for a CBPP on $T$ and let

$G = (V, E)$ be any graph with $V = V_0 \cup V_T$ and $E = E_0 \cup E_T \cup \{(x, y)\}$, where $x \in V_0$ and $y \in V_T$. Then $B = B_0 \cup B_T$ is a feasible packing with approximation factor of $\alpha = \frac{3}{2}$.

Above theorems can be extended to any constant number of subgraphs.

**Theorem 4.** Suppose that for $i = 1, \ldots, k$, a connected graph $G_i = (V_i, E_i)$ with a weight function $w_i$ in $[0, 1]$ and an optimal packing $B_i^*$ are given. Let $G = (V, E)$ be a graph with $V = \bigcup_{i=1}^k V_i$ and $E = \bigcup_{i=1}^k E_i \cup \{E_T\}$, where $G < E_T >$ is any subtree of $G$ with exactly one node form each $V_i$. Let $B^*$ be an optimal packing for $G$. Then

$$|B^*| \geq \sum_{i=1}^{k} |B_i^*| - k + 1.$$

**Theorem 5.** Suppose that for $i = 1, \ldots, k$, a connected graph $G_i = (V_i, E_i)$ with a weight function $w_i$ in $[0, 1]$ and a feasible packing $B_i$ with approximation factor of $\alpha_i$ are given. Let $G = (V, E)$ be a graph with $V = \bigcup_{i=1}^k V_i$ and $E = \bigcup_{i=1}^k E_i \cup \{E_T\}$, where $G < E_T >$ is a subtree of $G$ with exactly one node form each $V_i$. Then $B = \bigcup_{i=1}^k B_i$ is a feasible packing with approximation factor of $\alpha_{\max} = \max_{i=1}^k \alpha_i$.

## 3 CBPP on traceable graphs

A Hamiltonian path is a path that visits each node of the graph exactly once. A traceable graph is a graph possessing a Hamiltonian path. In this section, we consider a CBPP on traceable graphs.

**Lemma 1.** Let $G = (V, E)$ be a weighted traceable graph with weight functions $w$ in $[0, 1]$ and a Hamiltonian path $P = \{v_1, v_2, \ldots, v_n\}$. We use the next fit algorithm for the BPP with items $\{v_1, v_2, \ldots, v_n\}$ in this order. The obtained packing $\mathcal{B}$ is a feasible packing for $G$ with approximation factor 2. This approximation factor is tight.

*Proof.* Obviously, $G_{\mathcal{B}}$ has a Hamiltonian path and so a maximal matching of size $\lfloor \frac{|\mathcal{B}|}{2} \rfloor$. By Theorem 1, the obtained packing $\mathcal{B}$ is a feasible packing for $G$ with approximation factor 2. Figure 2 shows a tight example in which the Hamiltonian path $\{1, 2, \ldots, 2k + 1\}$ is given, where $\epsilon > 0$ is a very small number. □

**Corollary 2.** The traceable graph $G_0 = (V_0, E_0)$ and the tree $T = (V_T, E_T)$ with weight functions $w_0$ and $w_T$ in $[0, 1]$ are given. Let $B_0$ be a packing for the CBPP obtained from next fit algorithm. Let $B_T$ be an optimal packing

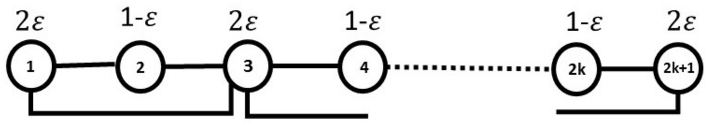


Figure 2: Tight-Example

for the CBPP on $T$ and let $G = (V, E)$ be a graph with $V = V_0 \cup V_T$ and $E = E_0 \cup E_T \cup \{(x, y)\}$, where $x \in V_0$ and $y \in V_T$. Then $B = B_1 \cup B_2$ is a feasible packing with approximation factor of $\alpha = 2$. This approximation factor is tight.

*Proof.* The proof is clear. □

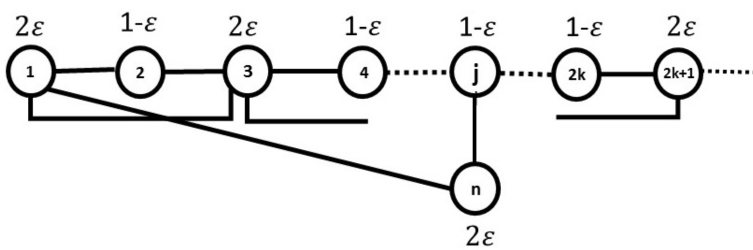


Figure 3: Tight-Example

**Lemma 2.** Suppose that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two disjoint connected traceable graphs with weight functions $w_1$ and $w_2$ in $[0, 1]$. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two feasible packings obtained by the next fit algorithm for $G_1$ and $G_2$ on their Hamiltonian path, respectively. Let $G = (V, E)$ be a graph with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup \{(x, y)\}$, where $x \in V_1$ and $y \in V_2$. If $\mathcal{B}^*$ is an optimal packing for the CBPP on $G$ and $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, then $|\mathcal{B}^*| \geq \lfloor \frac{|\mathcal{B}|}{2} \rfloor + 1$.

*Proof.* Let $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$ be optimal packings for $G_1$ and $G_2$, respectively. With a justification like Theorem 1, we could show that

$|\mathcal{B}_1^*| \geq \lfloor \frac{|\mathcal{B}_1|}{2} \rfloor + 1,$

$|\mathcal{B}_2^*| \geq \lfloor \frac{|\mathcal{B}_2|}{2} \rfloor + 1.$
So by Theorem 2, we have
$|\mathcal{B}^*| \geq |\mathcal{B}_1^*| + |\mathcal{B}_2^*| - 1 \geq \lfloor \frac{|\mathcal{B}_1|}{2} \rfloor + \lfloor \frac{|\mathcal{B}_2|}{2} \rfloor + 1 \geq \lfloor \frac{|\mathcal{B}|}{2} \rfloor + 1.$

□

**Theorem 6.** Suppose that a sequence of pair-wise disjoint connected traceable graphs $\{G_i = (V_i, E_i)\}_{i=1}^m$ is given. For each $i = 1, \ldots, m$, a weight function $w_i : V_i \to [0, 1]$ associated with graph $G_i$ and a Hamiltonian path $P_i$ is given. We assume that $\mathcal{B}_{P_i}$ is a feasible packing obtained by the next fit algorithm for $G_i$ on its Hamiltonian path $P_i$. Let $G = (V, E)$ be any graph with $V = \bigcup_{i=1}^m V_i$, for all $i$ $E_i \subset E$ and let the merged graph $G_S = (V_S, E_S)$ be a tree, where $S = \{V_1, V_2, \ldots, V_m\}$. If $\mathcal{B}^*$ is an optimal packing for the CBPP on $G$ and $\mathcal{B} = \bigcup_{i=1}^m \mathcal{B}_{P_i}$, then $|\mathcal{B}^*| \geq \lfloor \frac{|\mathcal{B}|}{2} \rfloor + 1$.

## 4 Conclusion and future works

In this paper, we considered a generalization of the classical BPP, CBPP, in which, in addition to capacity constraints, a set of connectivity constraints has to be satisfied. The CBPP is a strongly NP-hard problem.

Suppose that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two disjoint node weighted graphs and that $G = G_1 \cup G_2 \cup (x, y)$, where $x \in V_1$ *and* $y \in V_2$. In this paper, we introduced a trivial feasible solution to the CBPP on $G$ from solutions to CBPP's on $G_1$ and $G_2$ and proved some good results for the approximation factor of this solution, and we extended them. Then we presented a linear time approximation algorithm for this type of problems with approximation factor 2, when this disjoint graphs are traceable.

In perspective, our findings can be a starting point to tackle the CBPP. It would be interesting to try to design some approximation algorithms and lower bounds for the CBPP. Finally conclusion and some suggestions for more researches are given in this section.

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