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Research Article

On the stabilization of a coupled fractional ordinary and partial differential equations†

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Abstract

We investigate the stabilization problem of a cascade of a fractional ordinary differential equation (FODE) and a fractional diffusion (FD) equation, where the interconnections are of Neumann type. We exploit the PDE backstepping method as a powerful tool for designing a controller to show the Mittag–Leffler stability of the FD-FODE cascade. Finally, numerical simulations are presented to verify the results.

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1 Introduction

One of the most useful approach to obtain a boundary controller is the partial differential equation (PDE) backstepping method, which was initially utilized in [5, 6] for spatial discretization case and has since been expanded for continuous case in [17] for many applications, like fluid flows [1, 2]. The PDE backstepping approach, which is an integral operator with a known

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and continuous kernel function, has been used to analyze some problems of boundary stabilization of integer order unstable heat system with boundary control law. For example, this method was used in [21] for boundary feedback stabilization of heat equation. Also, this approach was applied in [26] to investigate the boundary stabilization of a class of linear parabolic partial integro-differential equation.

Since real systems in our world are complex and display memory and genetic characteristics, they can be well characterized by fractional order's notions; see [29, 22]. On the other hand, it is confirmed that fractional calculus [29] is very effective in modeling and analysis of distributed parameter systems [8]. In the last decades, researchers have interested in investigating the stability of the fractional-order systems. In [20], the stability of the fractional-order linear system subject to input saturation has been discussed. Since most of the systems in the real world are nonlinear, the stability problem of the nonlinear systems has become an attractive issue to study. For example, in [7, 10, 18], the Mittag–Leffler stability of the fractional-order nonlinear systems for $0 < \alpha < 1$ was addressed. Authors in [13] focused on stability analysis of a class of fractional-order nonlinear systems with $0 < \alpha < 2$.

Applying the backstepping method in designing a controller for fractional ordinary differential equations (FODEs), was first used in [11]. For instance, one can refer to the work [9] in which the adaptive fractional-order backstepping was used to design an adaptive feedback control law that Mittag–Leffler stabilize the commensurate fractional-order nonlinear systems. On the other hand, some researches have been dedicated to the fractional-order PDE systems, where the backstepping method is applied to design a controller to solve the stabilization problem of the mentioned systems. However, comparing to the ordinary fractional-order systems, the stabilization problem of fractional-order PDE systems has been less investigated. For example, in [23], the stability problem of one-dimensional wave equation was discussed via boundary fractional derivative control, and in [12], the backstepping method was applied to investigate the stabilization problem of the fractional diffusion (FD) system, governed by the FD equation consisting the diffusion term, with Dirichlet or Neumann condition. In [8], the boundary feedback control problem of the FD system with mixed or Robin boundary control was addressed via the backstepping method. In [30], the backstepping method was used for the stability problem of a class of unstable time fractional diffusion equation with the Dirichlet and Neumann boundary controls.

According to the previous paragraphs, systems described by ordinary differential equations (ODEs) and also systems modeled by PDEs are common in control engineering, and many works have been dedicated to the theory of them. Recently, the coupled systems have become one of the interesting areas of study. Examples of these systems are provided in control problems of electromagnetic coupling, mechanical coupling, and chemical reaction coupling [28]. In the last decades, the stabilization problem for coupled systems

has become challenging areas. The cascade structure for the heat PDE with an ODE and also the cascade structure for a wave PDE with an ODE, when the interconnection is of Dirichlet type, were discussed in [15] and [16], respectively. Next, these results for the stability analysis of the PDE-ODE cascade when the interconnection is of Neumann type have been extended by Susto and Krstic [27]. In [28], the stabilization problem for a new cascade of PDE-ODE was studied.

To the best of our knowledge, the stabilization problem and also controller design for a cascade of an FD equation and an FODE have been less investigated. In this paper, we consider a cascade of an FODE and an FD equation, and we use an invertible integral transformation to transfer the original system to a Mittag–Leffler stable target system. Finally, we present a numerical example for verifying the results.

Notation: $L^2(0,D)$ represents the usual Lebesgue integrable functions. Let $u(.,t) \in L^2(0,D)$; then we define

$$\|u(.,t)\| = \left(\int_0^D u^2(x,t)dx \right)^{\frac{1}{2}}.$$

Also, we denote a symmetric negative definite matrix $A \in \mathbb{R}^{n\times n}$ by $A \prec 0$.

2 Preliminaries

Definition 1. [4] The Caputo fractional-order derivative is defined by

$$ {}_{t_0}^{C}D_t^{\alpha} X(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{X^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \quad (n-1 < \alpha < n) \tag{1}$$

where α is the order of fractional derivative and the gamma function Γ is defined as $\gamma(\tau) = \int_0^{\infty} t^{\tau-1} e^{-t} dt$.

Definition 2. [14] The Mittag–Leffler function is defined as

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)},$$

where $0 < \alpha < 1$. The Mittag–Leffler function with two parameters is given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}.$$

Definition 3. [19, 18](Mittag–Leffler stability) The solution of

$$ {}_{t_0}^{C}D_t^{\alpha}u(t) = f(t,u), $$

is said to be Mittag–Leffler stable if

$$ \|u(t)\| \le \left(m[u(t_0)]E_{\alpha}(-\lambda(t-t_0)^{\alpha})\right)^{b}, $$

where t_0 is the initial value of time, $\alpha \in (0,1)$, $\lambda \ge 0$, $b > 0$, $m(0) = 0$, and $m(u)$ is nonnegative and meets locally Lipschitz condition on $u \in \mathbb{B} \subset \mathbb{R}^n$ with the Lipschitz constant m_0.

Lemma 1. [3] Suppose that $u : [0,\infty) \to \mathbb{R}$ is a continuous and differentiable function. For any time $t \ge t_0 \ge 0$, one can readily show that

$$ \frac{1}{2}{}_{t_0}^{C}D_t^{\alpha}u^2(t) \le u(t){}_{t_0}^{C}D_t u(t), \quad 0 < \alpha < 1. $$

Lemma 2. [13] Assume that $X : [0,\infty) \to \mathbb{R}^n$ is a vector of differentiable function. If a continuous function $V : [t_0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ satisfies

$$ {}_{t_0}^{C}D_t^{\alpha}V(t,X(t)) \le -\gamma V(t,X(t)), $$

then

$$ V(t,X(t)) \le V(t_0,X(t_0))E_{\alpha,1}(-\gamma(t-t_0)^{\alpha}), $$

where $0 < \alpha \le 1$ and γ is a positive constant.

3 Problem statement and analysis

Consider the cascade of fractional diffusion (FD) and a fractional-order ordinary differential equation (FODE) with Caputo derivative as follows:

$$ {}_{0}^{C}D_t^{\alpha}X(t) = AX(t) + Bu_x(0,t), \tag{2} $$

$$ {}_{0}^{C}D_t^{\alpha}u(x,t) = u_{xx}(x,t), \tag{3} $$

$$ u(0,t) = 0, \tag{4} $$

$$ u(D,t) = U(t), \tag{5} $$

where $X(t) \in \mathbb{R}^n$ and $u(x,t)$ are the state of the FODE and FD, respectively, and $U(t)$ is the control input. Note that $t \ge 0$ and $x \in [0,D]$ in which $D > 0$ is the length of the PDE domain. The aim is to Mittag–Leffler stabilize the system (2)–(5).

The PDE backstepping, introduced by Krstic, is the most effective approach for boundary controller designing for the PDE systems. In this method, we use an invertible integral transformation $(X,u) \to (X,w)$ to

convert the cascade of an FD and an FODE (2)–(5) into the following target system:

$$
\begin{aligned}
{}_0^C D_t^\alpha X(t) &= (A + BK)X(t) + Bw_x(0,t), && (6)\\
{}_0^C D_t^\alpha w(x,t) &= w_{xx}(x,t), && (7)\\
w(0,t) &= 0, && (8)\\
w(D,t) &= 0. && (9)
\end{aligned}
$$

The control gain K is chosen such that the Mittag–Leffler stability of the target system is guaranteed. Since the transformation is invertible, the Mittag–Leffler stabilization of the original closed-loop system will be derived.

4 Designing the state feedback controller

We consider the following transformation:

$$
w(x,t) = u(x,t) - \int_0^x q(x,y)u(y,t)dy - \gamma(x)X(t), \tag{10}
$$

where the unknown functions $q(x,y)$ and $\gamma(x)$ should be determined to convert system (2)–(5) into the target system (6)–(9). First, we derive the kernels, and next, we prove that the target system (6)–(9) is Mittag–Leffler stable.

To determine the unknown functions, we need the first two derivatives of $w(x,t)$ with respect to x that are given by

$$
w_x(x,t) = u_x(x,t) - q(x,x)u(x,t) - \int_0^x q_x(x,y)u(y,t)dy - \gamma'(x)X(t), \tag{11}
$$

$$
\begin{aligned}
w_{xx}(x,t) = u_{xx}(x,t) - (q(x,x))'u(x,t) - q(x,x)u_x(x,t) - q_x(x,x)u(x,t)\\
- \int_0^x q_{xx}(x,y)u(y,t)dy - \gamma''(x)X(t),
\end{aligned} \tag{12}
$$

and we take the Caputo fractional derivative of $w(x,t)$ respect to t:

$$
\begin{aligned}
{}_0^C D_t^\alpha w(x,t) &= {}_0^C D_t^\alpha u(x,t) - \int_0^x q(x,y) {}_0^C D_t^\alpha u(y,t)dy - \gamma(x) {}_0^C D_t^\alpha X(t) \\
&= u_{xx}(x,t) - \int_0^x q(x,y) u_{yy}(y,t)dy - \gamma(x)(AX(t) + Bu_x(0,t)) \\
&= u_{xx}(x,t) - q(x,x)u_x(x,t) + [q(x,0) - \gamma(x)B]u_x(0,t) \\
&\quad + q_y(x,x)u(x,t) - \int_0^x q_{yy}(x,y)u(y,t)dy - \gamma(x)AX(t). \quad (13)
\end{aligned}
$$

Now we evaluate the backstepping transformation (10), and (11) in $x = 0$ and we also subtract the right-hand side of (12) from the right-hand side of (13), then

$$
\begin{aligned}
w(0,t) &= -\gamma(0)X(t), \\
w_x(0,t) &= u_x(0,t) - \gamma'(0)X(t), \\
{}_0^C D_t^\alpha w(x,t) - w_{xx}(x,t) &= 2(q'(x,x))u(x,t) + [q(x,0) - \gamma(x)B]u_x(0,t) \\
&\quad + \int_0^x [q_{xx}(x,y) - q_{yy}(x,y)]u(y,t)dy \\
&\quad + [\gamma''(x) - \gamma(x)A]X(t),
\end{aligned}
$$

in which we have used the Dirichlet boundary condition $u(0,t) = 0$. A sufficient condition for equations (7)–(9) to be held is that the unknown functions $\gamma(x)$ and $q(x,y)$ satisfying an ODE of second order and a hyperbolic PDE of second order that come, respectively,

$$\gamma''(x) = A\gamma(x), \quad (14)$$
$$\gamma(0) = 0, \quad (15)$$
$$\gamma'(0) = K, \quad (16)$$

and

$$q_{xx}(x,y) = q_{yy}(x,y), \quad (17)$$
$$q(x,x) = 0, \quad (18)$$
$$q(x,0) = \gamma(x)B. \quad (19)$$

According to [27], the solution of (14)–(16) is

$$\gamma(x) = KM(x) \quad (20)$$
$$= K \begin{bmatrix} 0 & I \end{bmatrix} e^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix} x} \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad (21)$$

and the solution to (17)–(19) is

$$\begin{aligned} q(x,y) &= s(x-y) \\ &= KM(x-y)B, \end{aligned}$$

in which we have used the functions $s(.)$ and $M(.)$ for simplifying of notation in the proof.

By [27], the backstepping transformation is invertible, and the inverse change of variables is as follows:

$$u(x,t) = w(x,t) + \int_0^x r(x,y)w(y,t)dy + \lambda(x)X(t), \tag{22}$$

where the kernel functions $r(x,y) \in \mathbb{R}$ and $\lambda(x) \in \mathbb{R}^n$ are determined as follows:

$$\lambda''(x) = (A+BK)\lambda(x), \tag{23}$$
$$\lambda(0) = 0, \tag{24}$$
$$\lambda'(0) = K, \tag{25}$$

and

$$r_{xx}(x,y) = r_{yy}(x,y), \tag{26}$$
$$r(x,x) = 0, \tag{27}$$
$$r(x,0) = \lambda(x)B. \tag{28}$$

Therefore, the solutions of (23)–(25) and (26)–(28) are, respectively,

$$\lambda(x) = KN(x),$$

and

$$r(x,y) = n(x,y),$$

where

$$\begin{aligned} N(\xi) &= \begin{bmatrix} 0 & I \end{bmatrix} e^{\begin{bmatrix} 0 & A+BK \\ I & 0 \end{bmatrix}\xi} \begin{bmatrix} I \\ 0 \end{bmatrix}, \\ n(\xi) &= KN(\xi)B. \end{aligned}$$

Also, based on relations (10) and (22), we can write:

$$w_x(x) = u_x(x) - \int_0^x s_x(x-y)u(y)dy - KM'(x)X(t), \tag{29}$$
$$u_x(x) = w_x(x) + \int_0^x n_x(x-y)w(y)dy + KN'(x)X(t). \tag{30}$$

According to [27], from (29) and (30), we have

$$\|w_x\|^2 \leq \alpha_1\|u_x\|^2 + \alpha_2\|u\|^2 + \alpha_3\|X\|^2, \tag{31}$$
$$\|u_x\|^2 \leq \beta_1\|w_x\|^2 + \beta_2\|w\|^2 + \beta_3\|X\|^2. \tag{32}$$

Now we prove that the target system (2)–(5) is Mittag–Leffler stable. Then with the help of the backstepping method and using an invertible transformation, we obtain the Mittag–Leffler stability of the original system (6)–(9). Before stating the theorem, we need to consider the following lemma.

Lemma 3. There exist positive constants μ_1 and μ_2 such that

$$\|u_x\|^2 + \|X\|^2 \leq \mu_1(\|w_x\|^2 + \|X\|^2), \tag{33}$$

and

$$\|w_x(x,0)\|^2 + \|X(0)\|^2 \leq \mu_2(\|u_x(x,0)\|^2 + \|X(0)\|^2). \tag{34}$$

Proof. By using the Poincaré inequality for $\|w\|^2$, we can write (32) in the following form:

$$\|u_x\|^2 \leq \max\left\{\beta_1 + 4D^2\beta_2, \beta_3\right\}(\|w_x\|^2 + \|X\|^2). \tag{35}$$

Let

$$\rho_1 = \max\left\{\beta_1 + 4D^2\beta_2, \beta_3\right\},$$

then, it is clear that:

$$\|u_x\|^2 + |X|^2 \leq \mu_1(\|w_x\|^2 + \|X\|^2),$$

in which $\mu_1 = \rho_1 + 1$. With the help of (31) and in a similar manner, (34) is obtained. □

Also, we consider the following assumption throughout of the paper:

(H1) We assume that system (2) is controllable.

Theorem 1. Consider the closed-loop system (2)–(5) with the control law:

$$U(t) = K\begin{bmatrix}0_n & I_n\end{bmatrix}\left\{e^{\begin{bmatrix}0_n & A\\ I_n & 0_n\end{bmatrix}D}\begin{bmatrix}I_n\\ 0_n\end{bmatrix}X(t) + \int_0^D e^{\begin{bmatrix}0_n & A\\ I_n & 0_n\end{bmatrix}(D-y)}\begin{bmatrix}I_n\\ 0_n\end{bmatrix}Bu(y,t)dy\right\}. \tag{36}$$

Assume that there exist positive constants d and β and also a symmetric positive definite matrix P, such that the control gain K satisfies in the following inequality:

$$\Omega = \begin{bmatrix} P(A+BK) + (A+BK)^T P & PB & 0 & 0\\ B^T P & -d & 0 & 0\\ 0 & 0 & -\beta & 0\\ 0 & 0 & 0 & -\beta\end{bmatrix} \prec 0. \tag{37}$$

Also, suppose that $u_x(x,0)$ is square integrable in x for any initial condition. If $w_x(.,t)$ is continuously differentiable on $t \in [0,\infty]$, then the closed-loop system under the control law (36) is Mittag–Leffler stable under the norm $(\|X\|^2 + \int_0^D u_x^2(x,t)dx)^{\frac{1}{2}}$.

Proof. We consider the following Lyapunov function:

$$V(t) = X^T PX + \frac{c}{2}\|w\|^2 + \frac{d}{2}\|w_x\|^2, \tag{38}$$

in which $c > 0$ and $d > 0$ will be chosen later. Also, $\|w(.,t)\|^2$ and $\|w_x(.,t)\|^2$ represent the notation for L_2 norms $\int_0^D w^2(x,t)dx$ and $\int_0^D w_x^2(x,t)dx$, respectively. By taking the Caputo derivative of V with respect to t, we have

$$\begin{aligned} {}_0^C D_t^\alpha V &= {}_0^C D_t^\alpha \Big(X^T PX + \frac{c}{2}\|w\|^2 + \frac{d}{2}\|w_x\|^2 \Big) \\ &= {}_0^C D_t^\alpha (X^T PX) + \frac{c}{2}\ {}_0^C D_t^\alpha \|w\|^2 + \frac{d}{2}\ {}_0^C D_t^\alpha \|w_x\|^2. \end{aligned} \tag{39}$$

According to Lemma 4 in [8], $w(x,t)$ is continuous and differentiable on $t \in [0,\infty)$. On the other hand, by the assumption, $w_x(.,t)$ is continuously differentiable, so by Lemma 1, we have

$$\begin{aligned} {}_0^C D_t^\alpha \int_0^D w^2(x,t)dx = \int_0^D {}_0^C D_t^\alpha w^2(x,t)dx &\le 2\int_0^D w(x,t)\,{}_0^C D_t^\alpha w(x,t)dx \\ &= 2\int_0^D w(x,t)w_{xx}(x,t)dx \\ &= -2\|w_x\|^2, \end{aligned}$$

in which we have used integration by parts, so:

$${}_0^C D_t^\alpha \|w\|^2 \le -2\|w_x\|. \tag{40}$$

To compute the Caputo derivative of $\|w_x\|^2$, we multiply $w_{xx}(x,t)$ by (7) and integrating from 0 to D. Then

$$\int_0^D w_{xx}\,{}_0^C D_t^\alpha w(x,t)dx = \int_0^D w_{xx}^2(x,t)dx, \tag{41}$$

and applying integration by parts to the left side of (41), we have

$$\int_0^D w_{xx}\ {}_0^C D_t^\alpha w(x,t)dx = 0 - \int_0^D w_x\ {}_0^C D_t^\alpha w_x(x,t)dx. \tag{42}$$

Because of the Dirichlet boundary condition $w(0,t) = 0$ and $w(D,t) = 0$ and based on the Caputo time fractional derivative's definition in [25], we have ${}_0^C D_t^\alpha w(0,t) = {}_0^C D_t^\alpha w(D,t) = {}_0^C D_t^\alpha 0 = 0$, (for all $t \in [0,\infty)$). By considering

(41) and (42), we have

$$\int_0^D w_x(x,t)\,{}_0^C D_t^\alpha w(x,t)dx = -\int_0^D w_{xx}^2(x,t)dx.$$

Now we can evaluate the Caputo derivative of $\|w_x\|^2$ respect to t as follows:

$$\begin{aligned}{}_0^C D_t^\alpha \|w_x\|^2 = {}_0^C D_t^\alpha \int_0^D w_x^2(x,t)dx = \int_0^D {}_0^C D_t^\alpha\ w_x^2(x,t)dx \\ \le 2\int_0^D w_x(x,t)\ {}_0^C D_t^\alpha w_x(x,t)dx = -2\|w_{xx}\|^2. \end{aligned} \tag{43}$$

On the other hand, applying Lemma 1 and using (6), it is easy to see that

$$\begin{aligned}{}_0^C D_t^\alpha X^T PX \le & X^T(t)\Big(P(A+Bk)+(A+BK)^T P\Big)X(t) \\ & + B^T PX w_x(0,t) + X^T PB w_x(0,t). \end{aligned} \tag{44}$$

Here, the goal is to establish ${}_0^C D_t^\alpha V(t) \le -\gamma V(t)$, for some positive constant γ to prove the Mittag–Leffler stability of the target system (6)–(9). Therefore, we replace relations (40)–(44) into (39), and we have

$$\begin{aligned}{}_0^C D_t^\alpha V \le & X^T(P(A+BK)+(A+BK)^T P)X \\ & + B^T PX w_x(0,t) + X^T PB w_x(0,t) - c\|w_x\|^2 - d\|w_{xx}\|^2. \end{aligned}$$

It can be shown by Agmon's inequality that for the system (6)–(9) the following inequality holds:

$$-\|w_{xx}\|^2 \le \frac{1+D}{D}\|w_x\|^2 - w_x^2(0). \tag{45}$$

Hence

$$\begin{aligned}{}_0^C D_t^\alpha V \le & X^T(P(A+BK)+(A+BK)^T P)X \\ & + B^T PX w_x(0,t) + X^T PB w_x(0,t) - (c - d\frac{1+D}{D})\|w_x\|^2 - dw_x^2(0,t). \end{aligned} \tag{46}$$

Now, we assume that

$$c - d\frac{1+D}{D} > 0. \tag{47}$$

By using Poincaré inequality, we can rewrite (46) in the following form:

$$
\begin{aligned}
{}_0^C D_t^\alpha V \leq & X^T(P(A+BK)+(A+BK)^T P)X \\
& + B^T P X w_x(0,t) + X^T P B w_x(0,t) - \frac{1}{1+4D^2}(c - d\frac{1+D}{D})\|w\|^2 \\
& - \frac{1}{1+4D^2}(c - d\frac{1+D}{D})\|w_x\|,^2 - d w_x^2(0,t). \qquad (48)
\end{aligned}
$$

We can rewrite inequality (48) as follows:

$$
{}_0^C D_t^\alpha V \leq \begin{bmatrix} X \\ w_x(0,t) \\ \|w\| \\ \|w_x\| \end{bmatrix}^T \begin{bmatrix} P(A+BK)+(A+BK)^T P & PB & 0 & 0 \\ B^T P & -d & 0 & 0 \\ 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & -\beta \end{bmatrix} \begin{bmatrix} X \\ w_x(0,t) \\ \|w\| \\ \|w_x\| \end{bmatrix}, \qquad (49)
$$

Assumption (37) and relation (49) imply that ${}_0^C D_t^\alpha V < 0$. Since $\Omega \prec 0$ and by (47), we should have

$$
\beta := \frac{1}{1+4D^2}\left(c - d\frac{1+D}{D}\right) > 0, \qquad (50)
$$

and we can conclude that

$$
{}_0^C D_t^\alpha V < 0.
$$

Let $\lambda_0 := \lambda_{\min}(-\Omega)$. Then $\lambda_0 > 0$, and for any nonzero values of $X(t)$, $w(x,t)$, and $w_x(x,t)$, we have

$$
\begin{aligned}
{}_0^C D_t^\alpha V &\leq -\lambda_0(\|X\|^2 + \|w_x(0,t)\|^2 + \|w\|^2 + \|w_x\|^2) \\
&\leq -\lambda_0(\|X\|^2 + \|w\|^2 + \|w_x\|^2). \qquad (51)
\end{aligned}
$$

Since P is a positive definite matrix, then

$$
\begin{aligned}
V &= X^T P X + \frac{c}{2}\|w\|^2 + \frac{d}{2}\|w_x\|^2 \leq \lambda_{\max}(P)\|X\|^2 + \frac{c}{2}\|w\|^2 + \frac{d}{2}\|w_x\|^2 \\
&\leq \max\left\{\lambda_{\max}(P), \frac{c}{2}, \frac{d}{2}\right\}\left(\|X\|^2 + \|w\|^2 + \|w_x\|^2\right),
\end{aligned}
$$

so we can rewrite inequality (51) in the following form:

$$
{}_0^C D_t^\alpha V \leq -\gamma V
$$

in which $\gamma = \frac{\lambda_0}{\max\left\{\lambda_{\max}(P), \frac{c}{2}, \frac{d}{2}\right\}}$. It follows from Lemma 2 that

$$
V \leq V(0) E_{\alpha,1}(-\gamma t^\alpha). \qquad (52)
$$

Therefore, we can write

$$2\lambda_{\min}(P)\|X\|^2 + d\|w_x\|^2 \leq 2X^T PX + c\|w\|^2 + d\|w_x\|^2 \leq 2V(0)E_\alpha(-\gamma t^\alpha)$$

and

$$\|X\|^2 + \|w_x\|^2 \leq \frac{2}{\min\{2\lambda_{\min}(P), d\}} V(0)E_\alpha(-\gamma t^\alpha). \tag{53}$$

On the other hand, using the Poincaré inequality in the Lyapunov function (38), we have

$$V = X^T PX + \frac{c}{2}\|w\|^2 + \frac{d}{2}\|w_x\|^2 \leq \max\left\{\lambda_{\max}(P), 2cD^2 + \frac{d}{2}\right\}(\|X\|^2 + \|w_x\|^2). \tag{54}$$

Therefore, by (54), we can write (53) in the following form:

$$\|X\|^2 + \|w_x\|^2 \leq \frac{2R}{\min\{2\lambda_{\min}(P), d\}}(\|X\|^2 + \|w_x(x,0)\|^2)E_\alpha(-\gamma t^\alpha), \tag{55}$$

in which

$$R = \max\left\{\lambda_{\max}(P), 2cD^2 + \frac{d}{2}\right\}.$$

By taking $m(u)$ in Definition 3 as
$m((\|X(0)\|, \|w_x(x,0)\|)) = \frac{2R}{\min\{2\lambda_{\min}(P),d\}}(\|X(0)\|^2 + \|w_x(x,0)\|^2)$,
it is clear that $m((\|X(0)\|, \|w_x(x,0)\|))$ is locally Lipschitz in $\|X(0)\|$ and $\|w_x(x,0)\|^2$. On the other hand, $m((\|X(0)\|, \|w_x(x,0)\|)) > 0$ for $\|X(0)\| \neq 0$ or $\|w_x(x,0)\| \neq 0$ and it is zero if and only if $\|X(0)\|$ and $\|w_x(x,0)\|$ are zero. Then by (55) and Definition 3, the target system (6)–(9) is Mittag–Leffler stable under the norm $\|X(t)\|+\|w_x(x,t)\|^2$.
Now, by Lemma 3 and relation (33), we can write (55) in the following form:

$$\|X\|^2 + \|u_x\|^2 \leq \frac{2R\mu_1}{\min\{2\lambda_{\min}(P), d\}}(\|X(0)\|^2 + \|w_x(x,0)\|^2)E_\alpha(-\gamma t^\alpha),$$

then from (34), we have

$$\|X\|^2 + \|u_x\|^2 \leq \frac{2R\mu_1\mu_2}{\min\{2\lambda_{\min}(P), d\}}(\|X(0)\|^2 + \|u_x(x,0)\|^2)E_\alpha(-\gamma t^\alpha), \tag{56}$$

which guarantees that the original system (2)–(5) is Mittag–Leffler stable. □

5 Numerical simulation

In this section, we present a numerical example to verify our theoretical results. In this example, we discuss the stability Mittag–Leffler results related to system (2)–(5).

Example 1. We consider the following unstable system:

$$
\begin{aligned}
{}_0^C D_t^\alpha X(t) &= X(t) + u_x(0,t), && (57)\\
{}_0^C D_t^\alpha u(x,t) &= u_{xx}(x,t), && (58)\\
u(0,t) &= 0, && (59)\\
u(1,t) &= U(t). && (60)
\end{aligned}
$$

Comparing with the coupled system (2)–(5), it is clear $A = 1$, $B = 1$, and $D = 1$. We assume that $\alpha = 0.75$ and $u(x,0) = 0$ and $X(0) = 1$ as initial conditions. Also, $U(t)$ is determined by relation (36) in the following form:

$$
U(t) = K\left[\sinh(1)X(t) + \int_0^1 \sinh(1-y)u(y)dy.\right] \tag{61}
$$

In order to show the stability of the closed-loop system (57)–(60) under the control law (61), we select $c = 7$ and $d = 1$ in the Lyapunov function (38) to satisfy (47) and we also choose $P = 30$. Then, we obtain the feedback gain K, with the help of CVX 1.2.1, to satisfy the stability condition:

$$
\begin{bmatrix}
P(A+BK) + (A+BK)^T P & PB & 0 & 0\\
B^T P & -d & 0 & 0\\
0 & 0 & -\beta & 0\\
0 & 0 & 0 & -\beta
\end{bmatrix} \prec 0.
$$

We get

$$
K = -16.0150. \tag{62}
$$

To figure out the state variable $X(t)$, $u(x,t)$ of the cascaded FODE-FD system (2)–(5), we have used the finite-difference approximation method described in [24] to descretize the spatial solution domain $[0, D]$ into finite numbers of $Q+1$ subintervals and the time interval $[0, T]$ into $TM+1$ grid points for some positive integers Q and M, that is, this finite difference algorithm estimates the system (2)–(5) with the special stepsize $h = \frac{1}{Q}$ and $k = \frac{1}{M}$ for x and t, respectively.

Now, we set the discretization parameters $T = 20$, $Q = 50$, and $M = 64$.

With these discretization parameters and considering (62), we can see from Figure 1 that the state of the coupled FODE-FD system, that is, $X(t)$ and $u(x,t)$, converges to zero for all $x \in [0, D]$ and implies that the closed system is Mittag–Leffler stable.

On the other hand, if one selects $K = 2$ and parameters c, d as before, then with the help of CVX 1.2.1, we found out that there is no P that sat-

isfies inequality (37) , so the stability condition (37) is not hold for $K = 2$. With the same discretization parameters, we obtain the trajectory of state $X(t)$ and $u(x,t)$ of the coupled system (57)–(60). Figure 2 shows that the cascade (57)–(60) with the boundary control law (61) is unstable.

Therefore, one can result that the Mittag–Leffler stablity of the state of the closed-loop coupled system (57)–(60) is guaranteed by choosing K satisfying the stability condition (37).

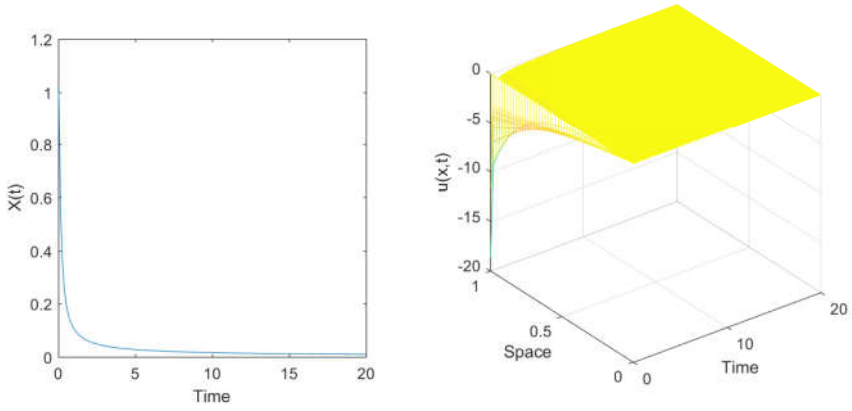

Figure 1: Evolution of state in Example 1. for $K = -16.0150$.

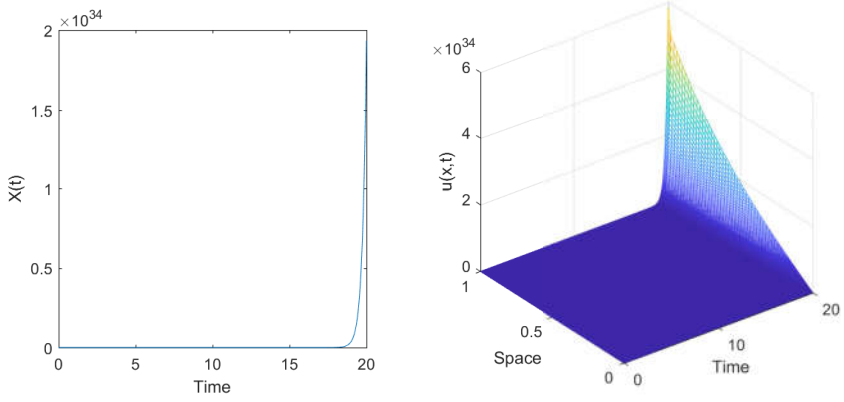

Figure 2: Evolution of state in Example 1 for $K = 2$.

6 Conclusions

In this article, we applied the backstepping method in which an invertible integral transformation is used and we designed a Dirichlet boundary feedback control to guarantee the Mittag–Leffler stability of the cascaded FD-FODE system. We presented a numerical example to confirm the obtained results. Since the stability analysis of the FODE-FD coupled system is less addressed in the literature, this paper is a beginning for the development of the stabilization problem of the cascade of a fractional-order ordinary differential equation and a fractional diffusion equation.

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