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Research Article

Coupled fixed points for a class of noncyclic contractions in uniformly convex Banach spaces with error estimates

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Abstract

In this paper, we introduce a noncyclic contraction pair for an ordered pair of mappings. We then deduce the existence and uniqueness of coupled fixed points by associating a noncyclic contraction with them. In addition, we obtain both a priori and a posteriori error estimates for these points in uniformly convex Banach spaces. The main result is illustrated with an example, and as an application, a cyclic version of the main theorem is derived.

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Keywords: Noncyclic contraction pair; Coupled fixed point; Coupled best proximity point; A priori and a posteriori error estimates; UC property.

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1 Introduction and preliminaries

Let $(\mathcal{Z}, \varrho)$ be a complete metric space, and let $\mathcal{K}$ and $\bar{\mathcal{K}}$ be nonempty closed subsets of $\mathcal{Z}$. A map $T : \mathcal{K} \cup \bar{\mathcal{K}} \to \mathcal{K} \cup \bar{\mathcal{K}}$ is termed a cyclic contraction map if it satisfies

- T is cyclic, that is, $T(\mathcal{K}) \subseteq \bar{\mathcal{K}}$ and $T(\bar{\mathcal{K}}) \subseteq \mathcal{K}$;
- there exists a constant $k \in (0, 1)$ such that for all $x \in \mathcal{K}$ and $\bar{x} \in \bar{\mathcal{K}}$, the following inequality holds:
$$\varrho(Tx, T\bar{x}) \leq k\varrho(x, \bar{x}) + (1 - k)\varrho(\mathcal{K}, \bar{\mathcal{K}});$$

and is termed a noncyclic contraction map if it satisfies

- T is noncyclic, that is, $T(\mathcal{K}) \subseteq \mathcal{K}$ and $T(\bar{\mathcal{K}}) \subseteq \bar{\mathcal{K}}$;
- there exists a constant $k \in (0, 1)$ such that for all $x \in \mathcal{K}$ and $\bar{x} \in \bar{\mathcal{K}}$, the following inequality holds:
$$\varrho(Tx, T\bar{x}) \leq k\varrho(x, \bar{x}) + (1 - k)\varrho(\mathcal{K}, \bar{\mathcal{K}}),$$

where $\varrho(\mathcal{K}, \bar{\mathcal{K}}) := \inf\{\varrho(x, \bar{x}) : x \in \mathcal{K}, \bar{x} \in \bar{\mathcal{K}}\}$. A point $x^* \in \mathcal{K} \cup \bar{\mathcal{K}}$ is considered a best proximity point for a cyclic map T if $\varrho(x^*, Tx^*) = \varrho(\mathcal{K}, \bar{\mathcal{K}})$ and is a fixed point for a cyclic or noncyclic map T if $Tx^* = x^*$ (see [9, 1] and their references).

The main theorems of [6, 7, 13] prove the existence of a fixed point and a best proximity point for two types of cyclic mappings. Additionally, they establish error estimates for the Picard iteration concerning these points.

Some generalizations of the concepts of a best proximity point and a fixed point are presented through the definitions in [11, 8, 3].

Definition 1 (see [11])**.** Let $\mathcal{K}$ and $\bar{\mathcal{K}}$ be nonempty subsets of the metric space $(\mathcal{Z}, \varrho)$. Let $\Gamma : \mathcal{K} \times \mathcal{K} \to \bar{\mathcal{K}}$. An ordered pair $(x^*, x^{**}) \in \mathcal{K} \times \mathcal{K}$ is called a coupled best proximity point of Γ if
$$\varrho(x^*, \Gamma(x^*, x^{**})) = \varrho(\mathcal{K}, \bar{\mathcal{K}}) \quad \text{and} \quad \varrho(x^{**}, \Gamma(x^{**}, x^*)) = \varrho(\mathcal{K}, \bar{\mathcal{K}}).$$

Definition 2 (see [3])**.** Let $\mathcal{K}$ be nonempty subset of the metric space $(\mathcal{Z}, \varrho)$. Let $\Gamma : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$. An ordered pair $(x^*, x^{**}) \in \mathcal{K} \times \mathcal{K}$ is called a coupled fixed point of Γ if $\Gamma(x^*, x^{**}) = x^*$ and $\Gamma(x^{**}, x^*) = x^{**}$.

Definition 3 (see [11])**.** Let $\mathcal{K}$ and $\bar{\mathcal{K}}$ be nonempty subsets of the metric space $(\mathcal{Z}, \varrho)$. Let $\Gamma : \mathcal{K} \times \mathcal{K} \to \bar{\mathcal{K}}$ and $\bar{\Gamma} : \bar{\mathcal{K}} \times \bar{\mathcal{K}} \to \mathcal{K}$. For any initial values $x_0, y_0 \in \mathcal{K}$, let the sequences $\{x_n\}$ and $\{y_n\}$ be defined by

$$x_{2n+1} := \Gamma(x_{2n}, y_{2n}) \in \bar{\mathcal{K}}, \qquad x_{2n+2} := \bar{\Gamma}(x_{2n+1}, y_{2n+1}) \in \mathcal{K},$$
$$y_{2n+1} := \Gamma(y_{2n}, x_{2n}) \in \bar{\mathcal{K}}, \qquad y_{2n+2} := \bar{\Gamma}(y_{2n+1}, x_{2n+1}) \in \mathcal{K},$$

for every $n \geq 0$. Also, we define $\Gamma' : \mathcal{K} \times \mathcal{K} \to \bar{\mathcal{K}}$ and $\bar{\Gamma}' : \bar{\mathcal{K}} \times \bar{\mathcal{K}} \to \mathcal{K}$ by

$$\Gamma'(x, y) = \Gamma(y, x), \text{ for any pairs } (x, y) \in \mathcal{K} \times \mathcal{K},$$
$$\bar{\Gamma}'(x, y) = \bar{\Gamma}(y, x), \text{ for any pairs } (x, y) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}.$$

Remark 1. Let $\mathcal{K}$ and $\bar{\mathcal{K}}$ be nonempty subsets of the metric space $(\mathcal{Z}, \varrho)$. Let $\Gamma : \mathcal{K} \times \mathcal{K} \to \bar{\mathcal{K}}$ and $\bar{\Gamma} : \bar{\mathcal{K}} \times \bar{\mathcal{K}} \to \mathcal{K}$. We define mappings $(\Gamma, \Gamma') : \mathcal{K} \times \mathcal{K} \to \bar{\mathcal{K}} \times \bar{\mathcal{K}}$ and $(\bar{\Gamma}, \bar{\Gamma}') : \bar{\mathcal{K}} \times \bar{\mathcal{K}} \to \mathcal{K} \times \mathcal{K}$ by

$$(\Gamma, \Gamma')(x, y) = (\Gamma(x, y), \Gamma'(x, y)) \quad \text{for} \quad (x, y) \in \mathcal{K} \times \mathcal{K}$$

and

$$(\bar{\Gamma}, \bar{\Gamma}')(x, y) = (\bar{\Gamma}(x, y), \bar{\Gamma}'(x, y)) \quad \text{for} \quad (x, y) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}.$$

Then for the sequences $\{x_n\}$ and $\{y_n\}$ defined in Definition 3, we have

$$(x_{2n+1}, y_{2n+1}) = (\Gamma, \Gamma')(x_{2n}, y_{2n}) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}},$$
$$(x_{2n+2}, y_{2n+2}) = (\bar{\Gamma}, \bar{\Gamma}')(x_{2n+1}, y_{2n+1}) \in \mathcal{K} \times \mathcal{K},$$

for every $n \geq 0$. So

$$\begin{aligned}
&(x_1, y_1) = (\Gamma, \Gamma')(x_0, y_0),\\
&(x_2, y_2) = (\bar{\Gamma}, \bar{\Gamma}')(x_1, y_1) = (\bar{\Gamma}, \bar{\Gamma}')((\Gamma, \Gamma')(x_0, y_0)) = ((\bar{\Gamma}, \bar{\Gamma}') \circ (\Gamma, \Gamma'))(x_0, y_0),\\
&(x_3, y_3) = (\Gamma, \Gamma')(x_2, y_2) = (\Gamma, \Gamma')((\bar{\Gamma}, \bar{\Gamma}')(x_1, y_1)) = ((\Gamma, \Gamma') \circ (\bar{\Gamma}, \bar{\Gamma}'))(x_1, y_1),\\
&(x_4, y_4) = (\bar{\Gamma}, \bar{\Gamma}')(x_3, y_3) = (\bar{\Gamma}, \bar{\Gamma}')((\Gamma, \Gamma')(x_2, y_2)) = ((\bar{\Gamma}, \bar{\Gamma}') \circ (\Gamma, \Gamma'))^2(x_0, y_0),\\
&\vdots\\
&(x_{2n}, y_{2n}) = ((\bar{\Gamma}, \bar{\Gamma}') \circ (\Gamma, \Gamma'))^n(x_0, y_0),\\
&(x_{2n+1}, y_{2n+1}) = ((\Gamma, \Gamma') \circ (\bar{\Gamma}, \bar{\Gamma}'))^n(x_1, y_1),
\end{aligned}$$

for every $n \in \mathbb{N} \cup \{0\}$.

We now refer to the definition of a cyclic contraction of an ordered pair defined by Sintunavarat and Kumam [11].

Definition 4 (see [11])**.** Let $\mathcal{K}$ and $\bar{\mathcal{K}}$ be nonempty subsets of the metric space $(\mathcal{Z}, \varrho)$. Let $\Gamma : \mathcal{K} \times \mathcal{K} \to \bar{\mathcal{K}}$ and $\bar{\Gamma} : \bar{\mathcal{K}} \times \bar{\mathcal{K}} \to \mathcal{K}$. The ordered pair $(\Gamma, \bar{\Gamma})$ is said to be a cyclic contraction pair if there exist nonnegative numbers λ, μ such that $\lambda + \mu < 1$, and the following inequality holds:

$$\varrho(\Gamma(x, y), \bar{\Gamma}(\bar{x}, \bar{y})) \leq \lambda\varrho(x, \bar{x}) + \mu\varrho(y, \bar{y}) + (1 - (\lambda + \mu))\varrho(\mathcal{K}, \bar{\mathcal{K}}),$$

for all $(x, y) \in \mathcal{K} \times \mathcal{K}$ and $(\bar{x}, \bar{y}) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$.

Definition 5 (see [12])**.** Let $\mathcal{K}$ and $\bar{\mathcal{K}}$ be nonempty subsets of the metric space $(\mathcal{Z}, \varrho)$. We say that the pair $(\mathcal{K}, \bar{\mathcal{K}})$ satisfies the UC property if the following condition holds: whenever sequences $\{x_n\}$ and $\{x'_n\}$ in $\mathcal{K}$ and a sequence $\{\bar{x}_n\}$ in $\bar{\mathcal{K}}$ satisfy

$$\lim_{n\to\infty} \varrho(x_n, \bar{x}_n) = \lim_{n\to\infty} \varrho(x'_n, \bar{x}_n) = \varrho(\mathcal{K}, \bar{\mathcal{K}}),$$

then it follows that $\lim_{n\to\infty} \varrho(x_n, x'_n) = 0$.

Suzuki, Kikkawa, and Vetro [12] demonstrated that if $\mathcal{K}$ and $\bar{\mathcal{K}}$ are nonempty, convex subsets of a uniformly convex Banach space $\mathcal{Z}$, then the pair $(\mathcal{K}, \bar{\mathcal{K}})$ possesses the UC property. They also established that if $\mathcal{K}$ and $\bar{\mathcal{K}}$ are nonempty subsets of a metric space $(\mathcal{Z}, \varrho)$ such that $\varrho(\mathcal{K}, \bar{\mathcal{K}}) = 0$, then the pair $(\mathcal{K}, \bar{\mathcal{K}})$ satisfies the UC property.

To obtain an error estimate for the sequence of successive iterations approximating the best proximity point, generated by a cyclic contraction, the modulus of convexity $\delta_{\mathcal{Z}}$ is used [13].

Definition 6 (see [2])**.** A Banach space $\mathcal{Z}$ is said to be uniformly convex if there exists a strictly increasing function $\delta_{\mathcal{Z}} : [0, 2] \to [0, 1]$ such that the following implication holds for all $x, x', \bar{x} \in \mathcal{Z}$, $\mathcal{R} > 0$ and $r \in [0, 2\mathcal{R}]$:

$$\left.\begin{array}{r} \|x - \bar{x}\| \leq \mathcal{R} \\ \|x' - \bar{x}\| \leq \mathcal{R} \\ \|x - x'\| \geq r \end{array}\right\} \quad \Rightarrow \quad \|\frac{x + x'}{2} - \bar{x}\| \leq \big(1 - \delta_{\mathcal{Z}}(\frac{r}{\mathcal{R}})\big)\mathcal{R}. \tag{1}$$

In this paper, we demonstrate that a noncyclic contraction pair $(\Gamma, \bar{\Gamma})$ can be formulated as a noncyclic contraction within the space $\mathcal{Z}^2 = \mathcal{Z} \times \mathcal{Z}$ through the adoption of a suitable metric. This formulation facilitates the derivation of our main results concerning the existence and uniqueness of coupled fixed points for noncyclic contraction pairs, as well as the estimation of a priori and a posteriori error bounds.

2 Auxiliary results based on property UC

For a complete metric space $(\mathcal{Z},\varrho)$, we define $\mathcal{Z}^2=\mathcal{Z}\times\mathcal{Z}$ and $\varrho^*:\mathcal{Z}^2\times\mathcal{Z}^2\to\mathbb{R}^+$ such that $\varrho^*((x,y),(\bar{x},\bar{y}))=\max\{\varrho(x,\bar{x}),\varrho(y,\bar{y})\}$. It can easily be seen that the completeness of the metric space $(\mathcal{Z}^2,\varrho^*)$ follows from the completeness of $(\mathcal{Z},\varrho)$. On the normed space $(\mathcal{Z},\|\cdot\|)$, we use $\|\cdot\|^*$ to denote the norm on $\mathcal{Z}^2$, defined as $\|(x,y)\|^*=\max\{\|x\|,\|y\|\}$, so we have $\|(x,y)-(\bar{x},\bar{y})\|^*=\max\{\|x-\bar{x}\|,\|y-\bar{y}\|\}$.

Let $\mathcal{K}$ and $\bar{\mathcal{K}}$ be nonempty subsets of a metric space $(\mathcal{Z},\varrho)$. Then

$$\begin{aligned}
&\varrho^*(\mathcal{K}\times\mathcal{K},\bar{\mathcal{K}}\times\bar{\mathcal{K}})\\
&\quad=\inf\{\varrho^*((x,y),(\bar{x},\bar{y})):(x,y)\in\mathcal{K}\times\mathcal{K},(\bar{x},\bar{y})\in\bar{\mathcal{K}}\times\bar{\mathcal{K}}\}\\
&\quad\le\inf\{\varrho^*((x,x),(\bar{x},\bar{x})):x\in\mathcal{K},\bar{x}\in\bar{\mathcal{K}}\}\\
&\quad=\inf\{\varrho(x,\bar{x}):(x,\bar{x})\in\mathcal{K}\times\bar{\mathcal{K}}\}\\
&\quad=\varrho(\mathcal{K},\bar{\mathcal{K}})\\
&\quad=\inf\{\varrho(x,\bar{x}):(x,\bar{x})\in\mathcal{K}\times\bar{\mathcal{K}}\}\\
&\quad\le\inf\{\max\{\varrho(x,\bar{x}),\varrho(y,\bar{y})\}:(x,\bar{x})\in\mathcal{K}\times\bar{\mathcal{K}},(y,\bar{y})\in\mathcal{K}\times\bar{\mathcal{K}}\}\\
&\quad=\inf\{\varrho^*((x,y),(\bar{x},\bar{y})):(x,y)\in\mathcal{K}\times\mathcal{K},(\bar{x},\bar{y})\in\bar{\mathcal{K}}\times\bar{\mathcal{K}}\}\\
&\quad=\varrho^*(\mathcal{K}\times\mathcal{K},\bar{\mathcal{K}}\times\bar{\mathcal{K}}).
\end{aligned}$$

So we obtain

$$\varrho^*(\mathcal{K}\times\mathcal{K},\bar{\mathcal{K}}\times\bar{\mathcal{K}})=\varrho(\mathcal{K},\bar{\mathcal{K}}).\tag{2}$$

Definition 7. Let $\mathcal{K}$ and $\bar{\mathcal{K}}$ be nonempty subsets of a Banach space $(\mathcal{Z},\|\cdot\|)$ such that $(\mathcal{K},\bar{\mathcal{K}})$ has the UC property and $\varrho:=\varrho(\mathcal{K},\bar{\mathcal{K}})>0$. If there exist $C>0$ and $p>0$ such that the following implication holds for all $x,x'\in\mathcal{K}$, $\bar{x}\in\bar{\mathcal{K}}$ and $\mathcal{R}\ge\varrho$:

$$\left.\begin{array}{r}\|x-\bar{x}\|\le\mathcal{R},\\ \|x'-\bar{x}\|\le\mathcal{R},\end{array}\right\}\quad\Rightarrow\quad\|x-x'\|\le\mathcal{R}\sqrt[p]{\frac{\mathcal{R}-\varrho}{\mathcal{R}C}},\tag{3}$$

then, we say that $(\mathcal{K},\bar{\mathcal{K}})$ has property UC with convexity approximation coefficients C and p.

Note that in the previous definition, since $(\mathcal{K},\bar{\mathcal{K}})$ has the UC property, relation (3) clearly holds when $\mathcal{R}=\varrho$. Therefore, to prove that the definition is satisfied, it is enough to show that relation (3) holds for $\mathcal{R}>\varrho$.

Moreover, [10, Theorem 2] can be stated as follows for sets $\mathcal{K}$ and $\bar{\mathcal{K}}$ satisfying the property UC with convexity approximation coefficients C and p, using the same proof as provided by the original authors.

Theorem 1. Suppose that $\mathcal{K}$ and $\bar{\mathcal{K}}$ are nonempty, closed subsets of a Banach space $(\mathcal{Z}, \|\cdot\|)$ such that $(\mathcal{K}, \bar{\mathcal{K}})$ and $(\bar{\mathcal{K}}, \mathcal{K})$ have the UC property and $\varrho := \varrho(\mathcal{K}, \bar{\mathcal{K}}) > 0$. Let $T : \mathcal{K} \cup \bar{\mathcal{K}} \to \mathcal{K} \cup \bar{\mathcal{K}}$ be a noncyclic contraction map; that is,

$$\varrho(Tx, T\bar{x}) \leq k\varrho(x, \bar{x}) + (1-k)\varrho(\mathcal{K}, \bar{\mathcal{K}}),$$

for all $x \in \mathcal{K}$ and $\bar{x} \in \bar{\mathcal{K}}$ and some $k \in [0, 1)$. Then

(i) T has a unique optimal pair of fixed points $(x^*, \bar{x}^*) \in \mathcal{K} \times \bar{\mathcal{K}}$;

(ii) for every $x_0 \in \mathcal{K}$ and $\bar{x}_0 \in \bar{\mathcal{K}}$ the sequences $\{x_n\} = \{T^n x_0\}$ and $\{\bar{x}_n\} = \{T^n \bar{x}_0\}$ converge to x^* and $\bar{x}^*$, respectively.

Furthermore, if there exist $C > 0$ and $p > 0$ such that $(\mathcal{K}, \bar{\mathcal{K}})$ has the UC property with convexity approximation coefficients C and p, then

(iii) the following priori error estimate holds

$$\|x^* - T^m x_0\| \leq \frac{\Delta_{x_0, \bar{x}_0}}{1 - \sqrt[p]{k}} \sqrt[p]{\frac{\Delta_{x_0, \bar{x}_0} - \varrho}{C\varrho}} (\sqrt[p]{k})^m;$$

(iv) the following posteriori error estimate holds

$$\|T^n x_0 - x^*\| \leq \frac{\Delta_{x_n, \bar{x}_n}}{1 - \sqrt[p]{k}} \sqrt[p]{\frac{\Delta_{x_n, \bar{x}_n} - \varrho}{C\varrho}};$$

where $\Delta_{x, \bar{x}} := \max\{\|x - \bar{x}\|, \|Tx - \bar{x}\|\}$, for every $x \in \mathcal{K}$ and $\bar{x} \in \bar{\mathcal{K}}$.

The following lemma is necessary to obtain the coupled version of Theorem 1 in the next section.

Lemma 1. Suppose that $\mathcal{K}$ and $\bar{\mathcal{K}}$ are nonempty, closed, and convex subsets of a uniformly convex Banach space $(\mathcal{Z}, \|\cdot\|)$ such that $\varrho := \varrho(\mathcal{K}, \bar{\mathcal{K}}) > 0$. If $\delta_{\mathcal{Z}}(\epsilon) \geq C\epsilon^p$ for some $C > 0$, $p > 0$ and every $\epsilon \in (0, 2]$, then the pair $(\mathcal{K} \times \mathcal{K}, \bar{\mathcal{K}} \times \bar{\mathcal{K}})$ as subsets of the Banach space $(\mathcal{Z}^2, \|\cdot\|^*)$, has property UC with convexity approximation coefficients C and p.

Proof. It is easily proven that the pair $(\mathcal{K}\times\mathcal{K},\bar{\mathcal{K}}\times\bar{\mathcal{K}})$ as subsets of the Banach space $(\mathcal{Z}^2,\|\cdot\|^*)$, has property UC. Since $(\mathcal{Z},\|\cdot\|)$ is a uniformly convex Banach space, then there exists a strictly increasing function $\delta_{\mathcal{Z}}:[0,2]\to[0,1]$ such that the implication (1) holds for all $x,x',\bar{x}\in\mathcal{Z}$, $\mathcal{R}>0$ and $r\in[0,2\mathcal{R}]$. Let $(x,y),(x',y')\in\mathcal{K}\times\mathcal{K}$, $(\bar{x},\bar{y})\in\bar{\mathcal{K}}\times\bar{\mathcal{K}}$, $\mathcal{R}>\varrho$ and

$$\begin{cases} \|(x,y)-(\bar{x},\bar{y})\|^*\leq\mathcal{R}, \\ \|(x',y')-(\bar{x},\bar{y})\|^*\leq\mathcal{R}. \end{cases}$$

So we obtain

$$\begin{cases} \|x-\bar{x}\|\leq\mathcal{R}, \\ \|x'-\bar{x}\|\leq\mathcal{R}, \end{cases} \qquad \textit{and} \qquad \begin{cases} \|y-\bar{y}\|\leq\mathcal{R}, \\ \|y'-\bar{y}\|\leq\mathcal{R}. \end{cases}$$

Since $(\mathcal{Z},\|\cdot\|)$ is a uniformly convex Banach space, then

$$\varrho\leq\|\frac{x+x'}{2}-\bar{x}\|\leq(1-\delta_{\mathcal{Z}}(\frac{\|x-x'\|}{\mathcal{R}}))\mathcal{R}$$

and

$$\varrho\leq\|\frac{y+y'}{2}-\bar{y}\|\leq(1-\delta_{\mathcal{Z}}(\frac{\|y-y'\|}{\mathcal{R}}))\mathcal{R}.$$

It follows from the inequality $\delta_{\mathcal{Z}}(\epsilon)\geq C\epsilon^p$ that $\delta_{\mathcal{Z}}^{-1}(\epsilon)\leq\left(\frac{\epsilon}{C}\right)^{\frac{1}{p}}$ for every $\epsilon\in(0,2]$. Since $0<\frac{\mathcal{R}-\varrho}{\mathcal{R}}<1$, then we get

$$\|x-x'\|\leq\delta_{\mathcal{Z}}^{-1}(\frac{\mathcal{R}-\varrho}{\mathcal{R}})\mathcal{R}\leq\mathcal{R}\sqrt[p]{\frac{\mathcal{R}-\varrho}{\mathcal{R}C}}$$

and

$$\|y-y'\|\leq\delta_{\mathcal{Z}}^{-1}(\frac{\mathcal{R}-\varrho}{\mathcal{R}})\mathcal{R}\leq\mathcal{R}\sqrt[p]{\frac{\mathcal{R}-\varrho}{\mathcal{R}C}}.$$

Therefore

$$\|(x,y)-(x',y')\|^*=\max\{\|x-x'\|,\|y-y'\|\}\leq\mathcal{R}\sqrt[p]{\frac{\mathcal{R}-\varrho}{\mathcal{R}C}}.$$

□

3 Coupled fixed points for noncyclic contraction pair

We define a noncyclic contraction for an ordered pair of mappings as follows.

Definition 8. Let $\mathcal{K}$ and $\bar{\mathcal{K}}$ be nonempty subsets of the metric space $(\mathcal{Z}, \varrho)$. Let $\Gamma : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ and $\bar{\Gamma} : \bar{\mathcal{K}} \times \bar{\mathcal{K}} \to \bar{\mathcal{K}}$. The ordered pair $(\Gamma, \bar{\Gamma})$ is said to be a noncyclic contraction pair if there exist nonnegative numbers λ, μ such that $\lambda + \mu < 1$, and the following inequality holds:

$$\varrho(\Gamma(x, y), \bar{\Gamma}(\bar{x}, \bar{y})) \leq \lambda \varrho(x, \bar{x}) + \mu \varrho(y, \bar{y}) + (1 - (\lambda + \mu))\varrho(\mathcal{K}, \bar{\mathcal{K}}),$$

for all $(x, y) \in \mathcal{K} \times \mathcal{K}$ and $(\bar{x}, \bar{y}) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$.

For any initial values $x_0, y_0 \in \mathcal{K}$ and $\bar{x}_0, \bar{y}_0 \in \bar{\mathcal{K}}$, let the sequences $\{x_n\}$, $\{y_n\}$, $\{\bar{x}_n\}$, and $\{\bar{y}_n\}$ be defined by

$$\begin{aligned} x_{n+1} &:= \Gamma(x_n, y_n) \in \mathcal{K}, \qquad & \bar{x}_{n+1} &:= \bar{\Gamma}(\bar{x}_n, \bar{y}_n) \in \bar{\mathcal{K}}, \\ y_{n+1} &:= \Gamma(y_n, x_n) \in \mathcal{K}, \qquad & \bar{y}_{n+1} &:= \bar{\Gamma}(\bar{y}_n, \bar{x}_n) \in \bar{\mathcal{K}}, \end{aligned}$$

for every $n \geq 0$. From Remark 1,

$$\begin{aligned} (x_{n+1}, y_{n+1}) &= (\Gamma, \Gamma')(x_n, y_n) \in \mathcal{K} \times \mathcal{K}, \\ (\bar{x}_{n+1}, \bar{y}_{n+1}) &= (\bar{\Gamma}, \bar{\Gamma}')(\bar{x}_n, \bar{y}_n) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}, \end{aligned}$$

and

$$\begin{aligned} \{(x_n, y_n)\} &= \{(\Gamma, \Gamma')^n(x_0, y_0)\}, \\ \{(\bar{x}_n, \bar{y}_n)\} &= \{(\bar{\Gamma}, \bar{\Gamma}')^n(\bar{x}_0, \bar{y}_0)\}, \end{aligned}$$

for every $n \geq 0$.

We continue this section by presenting two practical lemmas that will be useful in subsequent analyses.

Lemma 2. Let $\mathcal{K}$ and $\bar{\mathcal{K}}$ be nonempty closed subsets of a complete metric space $(\mathcal{Z}, \varrho)$ and let $\Gamma : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ and $\bar{\Gamma} : \bar{\mathcal{K}} \times \bar{\mathcal{K}} \to \bar{\mathcal{K}}$. Suppose that the ordered pair $(\Gamma, \bar{\Gamma})$ is a noncyclic contraction pair. Then the noncyclic mapping $T : (\mathcal{K} \times \mathcal{K}) \cup (\bar{\mathcal{K}} \times \bar{\mathcal{K}}) \to (\mathcal{K} \times \mathcal{K}) \cup (\bar{\mathcal{K}} \times \bar{\mathcal{K}})$, defined by

$$T(x, y) = \begin{cases} (\Gamma, \Gamma')(x, y) & \text{if } (x, y) \in \mathcal{K} \times \mathcal{K}, \\ (\bar{\Gamma}, \bar{\Gamma}')(x, y) & \text{if } (x, y) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}, \end{cases}$$

is a noncyclic contraction in the complete metric spaces $(\mathcal{Z}^2, \varrho^*)$ with constant $\lambda + \mu$.

Proof. Consider the fixed but arbitrary points $(x, y) \in \mathcal{K} \times \mathcal{K}$ and $(\bar{x}, \bar{y}) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$. The ordered pair $(\Gamma, \bar{\Gamma})$ is a noncyclic contraction pair, so we have

$$\begin{aligned}
\varrho(\Gamma(x,y), \bar{\Gamma}(\bar{x},\bar{y})) &\leq \lambda\varrho(x,\bar{x}) + \mu\varrho(y,\bar{y}) + (1-(\lambda+\mu))\varrho(\mathcal{K},\bar{\mathcal{K}}) \\
&\leq \lambda \max\{\varrho(x,\bar{x}), \varrho(y,\bar{y})\} + \mu \max\{\varrho(x,\bar{x}), \varrho(y,\bar{y})\} \\
&\quad + (1-(\lambda+\mu))\varrho(\mathcal{K},\bar{\mathcal{K}}) \\
&= \lambda\varrho^*((x,y),(\bar{x},\bar{y})) + \mu\varrho^*((x,y),(\bar{x},\bar{y})) \\
&\quad + (1-(\lambda+\mu))\varrho(\mathcal{K},\bar{\mathcal{K}}) \\
&= (\lambda+\mu)\varrho^*((x,y),(\bar{x},\bar{y})) + (1-(\lambda+\mu))\varrho(\mathcal{K},\bar{\mathcal{K}}).
\end{aligned}$$

From the previous relation, for $(y, x) \in \mathcal{K} \times \mathcal{K}$ and $(\bar{y}, \bar{x}) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$, we have

$$\begin{aligned}
\varrho(\Gamma(y,x), \bar{\Gamma}(\bar{y},\bar{x})) &\leq (\lambda+\mu)\varrho^*((y,x),(\bar{y},\bar{x})) + (1-(\lambda+\mu))\varrho(\mathcal{K},\bar{\mathcal{K}}) \\
&= (\lambda+\mu)\varrho^*((x,y),(\bar{x},\bar{y})) + (1-(\lambda+\mu))\varrho(\mathcal{K},\bar{\mathcal{K}}).
\end{aligned}$$

Based on the symbols introduced in Remark 1, we establish the following relationship:

$$\begin{aligned}
&\varrho^*((\Gamma,\Gamma')(x,y), (\bar{\Gamma},\bar{\Gamma}')(\bar{x},\bar{y})) \\
&\qquad = \max\{\varrho(\Gamma(x,y), \bar{\Gamma}(\bar{x},\bar{y})), \varrho(\Gamma(y,x), \bar{\Gamma}(\bar{y},\bar{x}))\}.
\end{aligned}$$

From (2) we have $\varrho^*(\mathcal{K} \times \mathcal{K}, \bar{\mathcal{K}} \times \bar{\mathcal{K}}) = \varrho(\mathcal{K}, \bar{\mathcal{K}})$, combining these relations yield the inequality:

$$\begin{aligned}
\varrho^*(T(x,y), T(\bar{x},\bar{y})) &= \varrho^*((\Gamma,\Gamma')(x,y), (\bar{\Gamma},\bar{\Gamma}')(\bar{x},\bar{y})) \\
&\leq (\lambda+\mu)\varrho^*((x,y),(\bar{x},\bar{y})) \\
&\quad + (1-(\lambda+\mu))\varrho^*(\mathcal{K}\times\mathcal{K}, \bar{\mathcal{K}}\times\bar{\mathcal{K}}).
\end{aligned}$$

This demonstrates that T is a noncyclic contraction in the complete metric space $(\mathcal{Z}^2, \varrho^*)$. □

Lemma 3. Suppose that $\mathcal{K}$ and $\bar{\mathcal{K}}$ are nonempty subsets of a metric space $(\mathcal{Z}, \varrho)$ such that $(\mathcal{K}, \bar{\mathcal{K}})$ and $(\bar{\mathcal{K}}, \mathcal{K})$ have the *UC* property and $\varrho := \varrho(\mathcal{K}, \bar{\mathcal{K}}) > 0$. Let $\Gamma : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ and $\bar{\Gamma} : \bar{\mathcal{K}} \times \bar{\mathcal{K}} \to \bar{\mathcal{K}}$ be mappings such that the ordered pair $(\Gamma, \bar{\Gamma})$ constitutes a noncyclic contraction pair. If Γ possesses a coupled fixed point $(x^*, x^{**}) \in \mathcal{K} \times \mathcal{K}$ and $\bar{\Gamma}$ possesses a coupled fixed point $(\bar{x}^*, \bar{x}^{**}) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$, then it follows that $x^* = x^{**}$ and $\bar{x}^* = \bar{x}^{**}$ and $\varrho(x^*, \bar{x}^*) = \varrho(\mathcal{K}, \bar{\mathcal{K}})$.

Proof. Since Γ possesses a coupled fixed point $(x^*, x^{**}) \in \mathcal{K} \times \mathcal{K}$ and $\bar{\Gamma}$ possesses a coupled fixed point $(\bar{x}^*, \bar{x}^{**}) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$, then

$$\varrho(x^*, \bar{x}^*) + \varrho(x^{**}, \bar{x}^{**}) = \varrho(\Gamma(x^*, x^{**}), \bar{\Gamma}(\bar{x}^*, \bar{x}^{**})) + \varrho(\Gamma(x^{**}, x^*), \bar{\Gamma}(\bar{x}^{**}, \bar{x}^*))$$

and

$$\varrho(x^*, \bar{x}^{**}) + \varrho(x^{**}, \bar{x}^*) = \varrho(\Gamma(x^*, x^{**}), \bar{\Gamma}(\bar{x}^{**}, \bar{x}^*)) + \varrho(\Gamma(x^{**}, x^*), \bar{\Gamma}(\bar{x}^*, \bar{x}^{**})).$$

Given that $(\Gamma, \bar{\Gamma})$ is a noncyclic contraction pair, we obtain the following inequalities from the above relations:

$$\begin{aligned}
&\varrho(x^*, \bar{x}^*) + \varrho(x^{**}, \bar{x}^{**}) \\
&\qquad \le \lambda\varrho(x^*, \bar{x}^*) + \mu\varrho(x^{**}, \bar{x}^{**}) + (1 - (\lambda + \mu))\varrho \\
&\qquad\quad + \lambda\varrho(x^{**}, \bar{x}^{**}) + \mu\varrho(x^*, \bar{x}^*) + (1 - (\lambda + \mu))\varrho \\
&\qquad = (\lambda + \mu)(\varrho(x^*, \bar{x}^*) + \varrho(x^{**}, \bar{x}^{**})) + 2(1 - (\lambda + \mu))\varrho
\end{aligned}$$

and

$$\begin{aligned}
&\varrho(x^*, \bar{x}^{**}) + \varrho(x^{**}, \bar{x}^*) \\
&\qquad \le \lambda\varrho(x^*, \bar{x}^{**}) + \mu\varrho(x^{**}, \bar{x}^*) + (1 - (\lambda + \mu))\varrho \\
&\qquad\quad + \lambda\varrho(x^{**}, \bar{x}^*) + \mu\varrho(x^*, \bar{x}^{**}) + (1 - (\lambda + \mu))\varrho \\
&\qquad = (\lambda + \mu)(\varrho(x^*, \bar{x}^{**}) + \varrho(x^{**}, \bar{x}^*)) + 2(1 - (\lambda + \mu))\varrho.
\end{aligned}$$

These imply that

$$(1 - (\lambda + \mu))(\varrho(x^*, \bar{x}^*) + \varrho(x^{**}, \bar{x}^{**})) \le 2(1 - (\lambda + \mu))\varrho$$

and

$$(1 - (\lambda + \mu))(\varrho(x^*, \bar{x}^{**}) + \varrho(x^{**}, \bar{x}^*)) \le 2(1 - (\lambda + \mu))\varrho.$$

Hence

$$\varrho(x^*, \bar{x}^*) + \varrho(x^{**}, \bar{x}^{**}) \le 2\varrho = 2\varrho(\mathcal{K}, \bar{\mathcal{K}})$$

and

$$\varrho(x^*, \bar{x}^{**}) + \varrho(x^{**}, \bar{x}^*) \leq 2\varrho = 2\varrho(\mathcal{K}, \bar{\mathcal{K}}).$$

Since $\varrho(x^*, \bar{x}^*) \geq \varrho(\mathcal{K}, \bar{\mathcal{K}})$, $\varrho(x^{**}, \bar{x}^{**}) \geq \varrho(\mathcal{K}, \bar{\mathcal{K}})$, $\varrho(x^*, \bar{x}^{**}) \geq \varrho(\mathcal{K}, \bar{\mathcal{K}})$, and $\varrho(x^{**}, \bar{x}^*) \geq \varrho(\mathcal{K}, \bar{\mathcal{K}})$, then

$$\varrho(x^*, \bar{x}^*) = \varrho(\mathcal{K}, \bar{\mathcal{K}}) \tag{4}$$

and

$$\varrho(x^*, \bar{x}^{**}) = \varrho(x^{**}, \bar{x}^*) = \varrho(\mathcal{K}, \bar{\mathcal{K}}). \tag{5}$$

Since $(\mathcal{K}, \bar{\mathcal{K}})$ and $(\bar{\mathcal{K}}, \mathcal{K})$ satisfy the UC property, then from (4) and (5), we obtain $\bar{x}^* = \bar{x}^{**}$ and $x^* = x^{**}$, respectively. □

We now have the necessary tools to present our main result.

Theorem 2. Suppose that $\mathcal{K}$ and $\bar{\mathcal{K}}$ are nonempty, closed, and convex subsets of a uniformly convex Banach space $(\mathcal{Z}, \|\cdot\|)$ with $\varrho := \varrho(\mathcal{K}, \bar{\mathcal{K}}) > 0$. Let $\Gamma : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$, $\bar{\Gamma} : \bar{\mathcal{K}} \times \bar{\mathcal{K}} \to \bar{\mathcal{K}}$ such that the ordered pair $(\Gamma, \bar{\Gamma})$ is a noncyclic contraction pair. Then

(i) Γ has a unique coupled fixed point $(x^*, x^*) \in \mathcal{K} \times \mathcal{K}$ and $\bar{\Gamma}$ has a unique coupled fixed point $(\bar{x}^*, \bar{x}^*) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$. Moreover, $\varrho(x^*, \bar{x}^*) = \varrho(\mathcal{K}, \bar{\mathcal{K}})$;

(ii) for every $(x_0, y_0) \in \mathcal{K} \times \mathcal{K}$ and $(\bar{x}_0, \bar{y}_0) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$, the sequences $\{(x_n, y_n)\} = \{(\Gamma, \Gamma')^n(x_0, y_0)\}$ and $\{(\bar{x}_n, \bar{y}_n)\} = \{(\bar{\Gamma}, \bar{\Gamma}')^n(\bar{x}_0, \bar{y}_0)\}$ converge to (x^*, x^*) and $(\bar{x}^*, \bar{x}^*)$, respectively.

Furthermore, if $\delta_{\mathcal{Z}}(\epsilon) \geq C\epsilon^p$ for some $C > 0$, $p > 0$ and every $\epsilon \in (0, 2]$, then for every $n \geq 0$ we have

(iii) the following priori error estimate holds

$$\max\{\|x^* - x_n\|, \|x^* - y_n\|\} \leq \frac{\Delta_0}{1 - \sqrt[p]{\lambda + \mu}} \sqrt[p]{\frac{\Delta_0 - \varrho}{C\varrho}} (\sqrt[p]{\lambda + \mu})^n;$$

(iv) the following posteriori error estimate holds

$$\max\{\|x_n - x^*\|, \|y_n - x^*\|\} \leq \frac{\Delta_n}{1 - \sqrt[p]{\lambda + \mu}} \sqrt[p]{\frac{\Delta_n - \varrho}{C\varrho}},$$

where

$$\Delta_n = \max\left\{\|(x_n, y_n) - (\bar{x}_n, \bar{y}_n)\|^*, \|(\Gamma, \Gamma')(x_n, y_n) - (\bar{x}_n, \bar{y}_n)\|^*\right\},$$

or, equality

$$\Delta_n = \max\left\{\|x_n - \bar{x}_n\|, \|y_n - \bar{y}_n\|, \|x_{n+1} - \bar{x}_n\|, \|y_{n+1} - \bar{y}_n\|\right\}.$$

Proof. Note that $\mathcal{K} \times \mathcal{K}$ and $\bar{\mathcal{K}} \times \bar{\mathcal{K}}$ are nonempty, closed, and convex subsets of the Banach space $(\mathcal{Z}^2, \|\cdot\|^*)$ and that $T : (\mathcal{K} \times \mathcal{K}) \cup (\bar{\mathcal{K}} \times \bar{\mathcal{K}}) \to (\mathcal{K} \times \mathcal{K}) \cup (\bar{\mathcal{K}} \times \bar{\mathcal{K}})$ defined by

$$T(x, y) = \begin{cases} (\Gamma, \Gamma')(x, y) & \text{if } (x, y) \in \mathcal{K} \times \mathcal{K}, \\ (\bar{\Gamma}, \bar{\Gamma}')(x, y) & \text{if } (x, y) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}, \end{cases}$$

from Lemma 2, is a noncyclic contraction with constant $\lambda + \mu$.

(i) From Theorem 1(i), there exists a unique point $(x^*, x^{**}) \in \mathcal{K} \times \mathcal{K}$, which is a fixed point for the map T that is $(\Gamma, \Gamma')(x^*, x^{**}) = (x^*, x^{**})$. This implies that

$$\Gamma(x^*, x^{**}) = x^* \quad \text{and} \quad \Gamma(x^{**}, x^*) = x^{**}.$$

Conversely, since every coupled fixed point of Γ is a fixed point of T, it follows that (x^*, x^{**}) is the unique coupled fixed point of Γ in $\mathcal{K} \times \mathcal{K}$. Similarly, we can show that $\bar{\Gamma}$ has a unique coupled fixed point $(\bar{x}^*, \bar{x}^{**}) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$. Lemma 3 requires that $\varrho(x^*, \bar{x}^*) = \varrho(\mathcal{K}, \bar{\mathcal{K}})$, $x^* = x^{**}$ and $\bar{x}^* = \bar{x}^{**}$.

(ii) From Theorem 1(ii), for every $(x_0, y_0) \in \mathcal{K} \times \mathcal{K}$ and $(\bar{x}_0, \bar{y}_0) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$ the sequences $\{(x_n, y_n)\} = \{T^n(x_0, y_0)\} = \{(\Gamma, \Gamma')^n(x_0, y_0)\}$ and $\{(\bar{x}_n, \bar{y}_n)\} = \{T^n(\bar{x}_0, \bar{y}_0)\} = \{(\bar{\Gamma}, \bar{\Gamma}')^n(\bar{x}_0, \bar{y}_0)\}$, converge to (x^*, x^*) and $(\bar{x}^*, \bar{x}^*)$, respectively.

From Theorem 1, it follows that for every $n \geq 0$, we have

$$\begin{aligned}
&\Delta_{(x_n, y_n), (\bar{x}_n, \bar{y}_n)} \\
&= \max\left\{\|(x_n, y_n) - (\bar{x}_n, \bar{y}_n)\|^*, \|T(x_n, y_n) - (\bar{x}_n, \bar{y}_n)\|^*\right\} \\
&= \max\left\{\|(x_n, y_n) - (\bar{x}_n, \bar{y}_n)\|^*, \|(\Gamma, \Gamma')(x_n, y_n) - (\bar{x}_n, \bar{y}_n)\|^*\right\} \\
&= \max\left\{\|x_n - \bar{x}_n\|, \|y_n - \bar{y}_n\|, \|\Gamma(x_n, y_n) - \bar{x}_n\|, \|\Gamma(y_n, x_n) - \bar{y}_n\|\right\} \\
&= \max\left\{\|x_n - \bar{x}_n\|, \|y_n - \bar{y}_n\|, \|x_{n+1} - \bar{x}_n\|, \|y_{n+1} - \bar{y}_n\|\right\} = \Delta_n.
\end{aligned}$$

In particular

$$\Delta_{(x_0, y_0), (\bar{x}_0, \bar{y}_0)} = \max\left\{\|x_0 - \bar{x}_0\|, \|y_0 - \bar{y}_0\|, \|x_1 - \bar{x}_0\|, \|y_1 - \bar{y}_0\|\right\} = \Delta_0.$$

On the other hand,

$$\|(x^*,x^*)-T^n(x_0,y_0)\|^*=\max\{\|x^*-x_n\|,\|x^*-y_n\|\}.$$

Moreover, T is a noncyclic contraction with constant $\lambda+\mu$. These observations, together with Theorem 1(iii)-(iv), complete the proof of parts (iii) and (iv). $\square$

We illustrate Theorem 2 with an example.

Example 1. Consider the real number space $\mathbb{R}$, equipped with the standard absolute value norm, $|\cdot|$. Then, we have $\delta_{\mathbb{R}}(\epsilon)=\frac{\varepsilon}{2}$ (see [4]). So $C=\frac{1}{2}$ and $p=1$.

Define the sets $\mathcal{K}$ to be the closed interval $[1,2]$ and $\bar{\mathcal{K}}$ as the closed interval $[-2,-1]$. Let $\Gamma:\mathcal{K}\times\mathcal{K}\to\mathcal{K}$ and $\bar{\Gamma}:\bar{\mathcal{K}}\times\bar{\mathcal{K}}\to\bar{\mathcal{K}}$ be defined as follows:

$$\Gamma(x,y):=\frac{x+2y+5}{8}\quad\text{and}\quad\bar{\Gamma}(\bar{x},\bar{y}):=\frac{\bar{x}+2\bar{y}-5}{8}.$$

It is easy to observe that $x=x^*,y=x^{**},\bar{x}=\bar{x}^*,\bar{y}=\bar{x}^{**}$ in $\mathcal{K}\cup\bar{\mathcal{K}}$ constitute a solution of the system

$$\begin{cases} 7x-2y=5,\\ 7y-2x=5,\\ 7\bar{x}-2\bar{y}=-5,\\ 7\bar{y}-2\bar{x}=-5,\end{cases}$$

if and only if

$$\Gamma(x^*,x^{**})=x^*,\ \Gamma(x^{**},x^*)=x^{**},\ \bar{\Gamma}(\bar{x}^*,\bar{x}^{**})=\bar{x}^*\quad\text{and}\quad\bar{\Gamma}(\bar{x}^{**},\bar{x}^*)=\bar{x}^{**}.$$

Note that

$$\begin{aligned}\varrho(\Gamma(x,y),\bar{\Gamma}(\bar{x},\bar{y}))&=|\frac{x+2y+5}{8}-\frac{\bar{x}+2\bar{y}-5}{8}|\\ &=|\frac{1}{8}(x-\bar{x})+\frac{2}{8}(y-\bar{y})+\frac{5}{8}\times 2|\\ &\le\frac{1}{8}|x-\bar{x}|+\frac{2}{8}|y-\bar{y}|+(1-\frac{3}{8})\times\varrho(\mathcal{K},\bar{\mathcal{K}}).\end{aligned}$$

Hence $(\Gamma,\bar{\Gamma})$ is a noncyclic contraction pair. From Theorem 2(i), Γ has a unique coupled fixed point $(x^*,x^*)=(1,1)\in\mathcal{K}\times\mathcal{K}$ and $\bar{\Gamma}$ has a unique coupled fixed point $(\bar{x}^*,\bar{x}^*)=(-1,-1)\in\bar{\mathcal{K}}\times\bar{\mathcal{K}}$. For all $(x_0,y_0)\in\mathcal{K}\times\mathcal{K}$ and $(\bar{x}_0,\bar{y}_0)\in\bar{\mathcal{K}}\times\bar{\mathcal{K}}$ the sequences $\{(x_n,y_n)\}=\{(\Gamma,\Gamma')^n(x_0,y_0)\}$ and $\{(\bar{x}_n,\bar{y}_n)\}=\{(\bar{\Gamma},\bar{\Gamma}')^n(\bar{x}_0,\bar{y}_0)\}$ converge to $(1,1)$ and $(-1,-1)$, respectively. See Table 1.

Table 1: Convergence of the sequences in Example 1

	x_n	y_n	$\bar{x}_n$	$\bar{y}_n$
$n=0$	1.35	1.45	-1.8	-1.35
$n=1$	1.15625	1.14375	-1.1875	-1.24375
$n=2$	1.05546875	1.05703125	-1.084375	-1.07734375
$n=3$	1.0211914063	1.0209960938	-1.0298828125	-1.0307617188
$n=4$	1.0078979492	1.0079223633	-1.0114257813	-1.0113159180
$n=5$	1.0029678345	1.0029647827	-1.0042572021	-1.0042709351
$n=6$	1.0011121750	1.0011125565	-1.0015998840	-1.0015981674
$n=7$	1.0004171610	1.0004171133	-1.0005995274	-1.0005997419
$n=8$	1.0001564234	1.0001564294	-1.0002248764	-1.0002248496
$n=9$	1.0000586603	1.0000586595	-1.0000843219	-1.0000843253
$n=10$	1.0000219974	1.0000219975	-1.0000316216	-1.0000316211

A priori error estimate holds

$$\begin{aligned}\max\{\|x^*-x_n\|,\|x^*-y_n\|\} &\leq \frac{\Delta_0}{1-\sqrt[p]{\lambda+\mu}}\sqrt[p]{\frac{\Delta_0-\varrho}{C\varrho}}(\sqrt[p]{\lambda+\mu})^n\\ &= 5.796\times(\frac{3}{8})^n.\end{aligned}$$

Therefore after ten iterations, the derived error bound guarantees that the priori error estimate is less than 0.0004.

A posteriori error estimate holds

$$\max\{\|x_n-x^*\|,\|y_n-x^*\|\} \leq \frac{\Delta_n}{1-\sqrt[p]{\lambda+\mu}}\sqrt[p]{\frac{\Delta_n-\varrho}{C\varrho}} = \frac{8}{5}\Delta_n(\Delta_n-2).$$

Therefore after ten iterations, the derived error bound guarantees that the posteriori error estimate is less than 0.0002.

As a consequence of Theorem 2, we obtain its cyclic version. The statements of this theorem are similar to [5, Theorem 3], but with new error bounds. The subsequent lemma will help us in the proof.

Lemma 4. Suppose that $\mathcal{K}$ and $\bar{\mathcal{K}}$ are nonempty subsets of the metric space $(\mathcal{Z},\varrho)$. Let $\Gamma:\mathcal{K}\times\mathcal{K}\to\bar{\mathcal{K}}$, $\bar{\Gamma}:\bar{\mathcal{K}}\times\bar{\mathcal{K}}\to\mathcal{K}$ such that the ordered pair $(\Gamma,\bar{\Gamma})$ is a cyclic contraction pair with nonnegative constants λ and μ in which $\lambda+\mu<1$. Then

(i) $(\bar{\Gamma}\circ(\Gamma,\Gamma'))'=\bar{\Gamma}'\circ(\Gamma,\Gamma')$ and $(\Gamma\circ(\bar{\Gamma},\bar{\Gamma}'))'=\Gamma'\circ(\bar{\Gamma},\bar{\Gamma}')$;

(ii) $(\bar{\Gamma} \circ (\Gamma, \Gamma'), \Gamma \circ (\bar{\Gamma}, \bar{\Gamma}'))$ is a noncyclic contraction pair with constants $\lambda^2 + \mu^2$ and $2\lambda\mu$.

Proof. From the definitions of Γ and $\bar{\Gamma}$, the composition $\bar{\Gamma} \circ (\Gamma, \Gamma')$ defines a mapping from $\mathcal{K} \times \mathcal{K}$ to $\mathcal{K}$, while $\Gamma \circ (\bar{\Gamma}, \bar{\Gamma}')$ defines a mapping from $\bar{\mathcal{K}} \times \bar{\mathcal{K}}$ to $\bar{\mathcal{K}}$.

(i) For all $(x, y) \in \mathcal{K} \times \mathcal{K}$, we have

$$\begin{aligned}(\bar{\Gamma} \circ (\Gamma, \Gamma'))'(x, y) &= (\bar{\Gamma} \circ (\Gamma, \Gamma'))(y, x) \\ &= \bar{\Gamma}(\Gamma(y, x), \Gamma(x, y)) \\ &= \bar{\Gamma}'(\Gamma(x, y), \Gamma'(x, y)) \\ &= (\bar{\Gamma}' \circ (\Gamma, \Gamma'))(x, y).\end{aligned}$$

So $(\bar{\Gamma} \circ (\Gamma, \Gamma'))' = \bar{\Gamma}' \circ (\Gamma, \Gamma')$. Similarly we can show that $(\Gamma \circ (\bar{\Gamma}, \bar{\Gamma}'))' = \Gamma' \circ (\bar{\Gamma}, \bar{\Gamma}')$.

(ii) The ordered pair $(\Gamma, \bar{\Gamma})$ is a cyclic contraction pair with constants λ and μ, so for all $(x, y) \in \mathcal{K} \times \mathcal{K}$ and $(\bar{x}, \bar{y}) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$, we have

$$\begin{aligned}&\varrho((\bar{\Gamma} \circ (\Gamma, \Gamma'))(x, y), (\Gamma \circ (\bar{\Gamma}, \bar{\Gamma}'))(\bar{x}, \bar{y})) \\ &\quad= \varrho((\Gamma \circ (\bar{\Gamma}, \bar{\Gamma}'))(\bar{x}, \bar{y}), (\bar{\Gamma} \circ (\Gamma, \Gamma'))(x, y)) \\ &\quad\le \lambda\varrho(\bar{\Gamma}(\bar{x}, \bar{y}), \Gamma(x, y)) + \mu\varrho(\bar{\Gamma}(\bar{y}, \bar{x}), \Gamma(y, x)) + (1 - (\lambda + \mu))\varrho(\mathcal{K}, \bar{\mathcal{K}}) \\ &\quad= \lambda\varrho(\Gamma(x, y), \bar{\Gamma}(\bar{x}, \bar{y})) + \mu\varrho(\Gamma(y, x), \bar{\Gamma}(\bar{y}, \bar{x})) + (1 - (\lambda + \mu))\varrho(\mathcal{K}, \bar{\mathcal{K}}) \\ &\quad\le \lambda^2\varrho(x, \bar{x}) + \lambda\mu\varrho(y, \bar{y}) + \lambda(1 - (\lambda + \mu))\varrho(\mathcal{K}, \bar{\mathcal{K}}) + \mu\lambda\varrho(y, \bar{y}) \\ &\qquad+ \mu^2\varrho(x, \bar{x}) + \mu(1 - (\lambda + \mu))\varrho(\mathcal{K}, \bar{\mathcal{K}}) + (1 - (\lambda + \mu))\varrho(\mathcal{K}, \bar{\mathcal{K}}) \\ &\quad= (\lambda^2 + \mu^2)\varrho(x, \bar{x}) + 2\lambda\mu\varrho(y, \bar{y}) + (1 - (\lambda + \mu)^2)\varrho(\mathcal{K}, \bar{\mathcal{K}}).\end{aligned}$$

Hence $(\bar{\Gamma} \circ (\Gamma, \Gamma'), \Gamma \circ (\bar{\Gamma}, \bar{\Gamma}'))$ is a noncyclic contraction pair with constants $\lambda^2 + \mu^2$ and $2\lambda\mu$. $\square$

Corollary 1. Suppose that $\mathcal{K}$ and $\bar{\mathcal{K}}$ are nonempty, closed, and convex subsets of a uniformly convex Banach space $(\mathcal{Z}, \|\cdot\|)$ such that $\varrho := \varrho(\mathcal{K}, \bar{\mathcal{K}}) > 0$. Let $\Gamma : \mathcal{K} \times \mathcal{K} \to \bar{\mathcal{K}}$, $\bar{\Gamma} : \bar{\mathcal{K}} \times \bar{\mathcal{K}} \to \mathcal{K}$ such that the ordered pair $(\Gamma, \bar{\Gamma})$ is a cyclic contraction pair. Then

(i) Γ has a unique coupled best proximity point $(x^*, x^*) \in \mathcal{K} \times \mathcal{K}$ and $\bar{\Gamma}$ has a unique coupled best proximity point $(\bar{x}^*, \bar{x}^*) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$, that is,

$$\|x^* - \Gamma(x^*, x^*)\| = \varrho \quad \text{and} \quad \|\bar{x}^* - \bar{\Gamma}(\bar{x}^*, \bar{x}^*)\| = \varrho.$$

Moreover,

$$\Gamma(x^*, x^*) = \bar{x}^*, \quad \bar{\Gamma}(\bar{x}^*, \bar{x}^*) = x^*, \quad \|x^* - \bar{x}^*\| = \varrho;$$

(ii) For any arbitrary point (x_0, y_0), let the sequences $\{x_n\}$ and $\{y_n\}$ defined as in Definition 3. Then

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} y_{2n} = x^*, \quad \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} y_{2n+1} = \bar{x}^*.$$

Furthermore, if $\delta_{\mathcal{Z}}(\epsilon) \geq C\epsilon^p$ for some $C > 0$, $p > 0$ and every $\epsilon \in (0, 2]$, then

(iii) the following priori error estimate holds

$$\max\{\|x^* - x_{2m}\|, \|x^* - y_{2m}\|\} \leq \Delta_0 \sqrt[p]{\frac{\Delta_0 - \varrho}{C\varrho}} \frac{(\sqrt[p]{\lambda+\mu})^{2m}}{1 - \sqrt[p]{(\lambda+\mu)^2}};$$

(iv) the following posteriori error estimate holds

$$\max\{\|x^* - x_{2n}\|, \|x^* - y_{2n}\|\} \leq \frac{\Delta_n}{1 - \sqrt[p]{(\lambda+\mu)^2}} \sqrt[p]{\frac{\Delta_n - \varrho}{C\varrho}},$$

where

$$\Delta_0 = \max\left\{\|(x_0, y_0) - (x_1, y_1)\|^*, \|(x_2, y_2) - (x_1, y_1)\|^*\right\}$$

and

$$\begin{aligned}\Delta_n = \max\big\{&\|(x_{2n}, y_{2n}) - (x_{2n+1}, y_{2n+1})\|^*,\\ &\|(x_{2n+2}, y_{2n+2}) - (x_{2n+1}, y_{2n+1})\|^*\big\}.\end{aligned}$$

Proof. (i) From Lemma 4, $(\bar{\Gamma} \circ (\Gamma, \Gamma'), \Gamma \circ (\bar{\Gamma}, \bar{\Gamma}'))$ is a noncyclic contraction pair. So, from Theorem 2(i), $\bar{\Gamma} \circ (\Gamma, \Gamma')$ has a unique coupled fixed point $(x^*, x^*) \in \mathcal{K} \times \mathcal{K}$ and $\Gamma \circ (\bar{\Gamma}, \bar{\Gamma}')$ has a unique coupled fixed point $(\bar{x}^*, \bar{x}^*) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$ such that $\|x^* - \bar{x}^*\| = \varrho(\mathcal{K}, \bar{\mathcal{K}})$. On the other hand, $(\Gamma, \bar{\Gamma})$ is a cyclic contraction pair, so we have

$$\begin{aligned}\|\Gamma(x^*, x^*) - x^*\| &= \|\Gamma(x^*, x^*) - (\bar{\Gamma} \circ (\Gamma, \Gamma'))(x^*, x^*)\|\\ &\leq (\lambda+\mu)\|x^* - \Gamma(x^*, x^*)\| + (1 - (\lambda+\mu))\varrho.\end{aligned}$$

This implies $\|\Gamma(x^*, x^*) - x^*\| = \varrho(\mathcal{K}, \bar{\mathcal{K}})$. Since $\|x^* - \bar{x}^*\| = \varrho(\mathcal{K}, \bar{\mathcal{K}})$, then from Property UC of $(\bar{\mathcal{K}}, \mathcal{K})$, we get $\Gamma(x^*, x^*) = \bar{x}^*$. The proof of the other cases is similar.

(ii) From Theorem 2(ii), for every $(x_0, y_0) \in \mathcal{K} \times \mathcal{K}$ and $(\bar{x}_0, \bar{y}_0) := (\Gamma, \Gamma')(x_0, y_0) = (x_1, y_1) \in \bar{\mathcal{K}} \times \bar{\mathcal{K}}$ the sequences

$$\{(\bar{\Gamma} \circ (\Gamma, \Gamma'), \bar{\Gamma}' \circ (\Gamma, \Gamma'))^n (x_0, y_0)\}$$

and

$$\{(\Gamma \circ (\bar{\Gamma}, \bar{\Gamma}'), \Gamma' \circ (\bar{\Gamma}, \bar{\Gamma}'))^n (x_1, y_1)\}$$

converge to (x^*, x^*) and $(\bar{x}^*, \bar{x}^*)$, respectively. Indeed from Remark 1, we know

$$\begin{aligned}\{(\bar{\Gamma} \circ (\Gamma, \Gamma'), \bar{\Gamma}' \circ (\Gamma, \Gamma'))^n (x_0, y_0)\} &= \{((\bar{\Gamma}, \bar{\Gamma}') \circ (\Gamma, \Gamma'))^n (x_0, y_0)\} \\ &= \{(x_{2n}, y_{2n})\}\end{aligned}$$

and

$$\begin{aligned}\{(\Gamma \circ (\bar{\Gamma}, \bar{\Gamma}'), \Gamma' \circ (\bar{\Gamma}, \bar{\Gamma}'))^n (x_1, y_1)\} &= \{((\Gamma, \Gamma') \circ (\bar{\Gamma}, \bar{\Gamma}'))^n (x_1, y_1)\} \\ &= \{(x_{2n+1}, y_{2n+1})\},\end{aligned}$$

which completes the proof of (ii).

(iii) In Theorem 2(iii), let $(\bar{x}_0, \bar{y}_0) := (\Gamma, \Gamma')(x_0, y_0) = (x_1, y_1)$. Then

$$\begin{aligned}\Delta_0 = \max\big\{&\|(x_0, y_0) - (x_1, y_1)\|^*, \\ &\|(\bar{\Gamma} \circ (\Gamma, \Gamma'), \bar{\Gamma}' \circ (\Gamma, \Gamma'))(x_0, y_0) - (x_1, y_1)\|^*\big\} \\ = \max\big\{&\|(x_0, y_0) - (x_1, y_1)\|^*, \|((\bar{\Gamma}, \bar{\Gamma}') \circ (\Gamma, \Gamma'))(x_0, y_0) - (x_1, y_1)\|^*\big\} \\ = \max\big\{&\|(x_0, y_0) - (x_1, y_1)\|^*, \|(x_2, y_2) - (x_1, y_1)\|^*\big\}.\end{aligned}$$

Since from Lemma 4, $(\bar{\Gamma} \circ (\Gamma, \Gamma'), \Gamma \circ (\bar{\Gamma}, \bar{\Gamma}'))$ is a noncyclic contraction pair with constants $\lambda^2 + \mu^2$ and $2\lambda\mu$ and $\lambda^2 + \mu^2 + 2\lambda\mu = (\lambda + \mu)^2$, then the following priori error estimate holds

$$\begin{aligned}&\max\{\|x^* - x_{2m}\|, \|x^* - y_{2m}\|\} \\ &\quad = \|(x^*, x^*) - ((\bar{\Gamma}, \bar{\Gamma}') \circ (\Gamma, \Gamma'))^m (x_0, y_0)\|^* \\ &\quad = \|(x^*, x^*) - (\bar{\Gamma} \circ (\Gamma, \Gamma'), \bar{\Gamma}' \circ (\Gamma, \Gamma'))^m (x_0, y_0)\|^* \\ &\quad \leq \frac{\Delta_0}{1 - \sqrt[p]{(\lambda + \mu)^2}} \sqrt[p]{\frac{\Delta_0 - \varrho}{C\varrho}} (\sqrt[p]{\lambda + \mu})^{2m}.\end{aligned}$$

(iv) In Theorem 2(iv), let $(\bar{x}_0, \bar{y}_0) := (\Gamma, \Gamma')(x_0, y_0) = (x_1, y_1)$. Then

$$\begin{aligned}\Delta_n &= \Delta_{(\bar{\Gamma} \circ (\Gamma, \Gamma'), \bar{\Gamma}' \circ (\Gamma, \Gamma'))^n (x_0, y_0), (\Gamma \circ (\bar{\Gamma}, \bar{\Gamma}'), \Gamma' \circ (\bar{\Gamma}, \bar{\Gamma}'))^n (x_1, y_1)} \\ &= \Delta_{((\bar{\Gamma}, \bar{\Gamma}') \circ (\Gamma, \Gamma'))^n (x_0, y_0), ((\Gamma, \Gamma') \circ (\bar{\Gamma}, \bar{\Gamma}'))^n (x_1, y_1)} \\ &= \Delta_{(x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})} \\ &= \max\big\{\|(x_{2n}, y_{2n}) - (x_{2n+1}, y_{2n+1})\|^*,\end{aligned}$$

$$
\begin{aligned}
&\qquad \|(\bar{\Gamma}\circ(\Gamma,\Gamma'),\bar{\Gamma}'\circ(\Gamma,\Gamma'))(x_{2n},y_{2n})-(x_{2n+1},y_{2n+1})\|^*\big\}\\
&=\max\big\{\|(x_{2n},y_{2n})-(x_{2n+1},y_{2n+1})\|^*,\\
&\qquad \|((\bar{\Gamma},\bar{\Gamma}')\circ(\Gamma,\Gamma'))(x_{2n},y_{2n})-(x_{2n+1},y_{2n+1})\|^*\big\}\\
&=\max\big\{\|(x_{2n},y_{2n})-(x_{2n+1},y_{2n+1})\|^*,\\
&\qquad \|(x_{2n+2},y_{2n+2})-(x_{2n+1},y_{2n+1})\|^*\big\}.
\end{aligned}
$$

Since from Lemma 4, $(\bar{\Gamma}\circ(\Gamma,\Gamma'),\Gamma\circ(\bar{\Gamma},\bar{\Gamma}'))$ is a noncyclic contraction pair with constants $\lambda^2+\mu^2$ and $2\lambda\mu$ and $\lambda^2+\mu^2+2\lambda\mu=(\lambda+\mu)^2$, then the following posteriori error estimate holds

$$
\begin{aligned}
\max\{\|x^*-x_{2n}\|,\|x^*-y_{2n}\|\} &= \|((\bar{\Gamma},\bar{\Gamma}')\circ(\Gamma,\Gamma'))^n(x_0,y_0)-(x^*,x^*)\|^*\\
&\le \frac{\Delta_n}{1-\sqrt[p]{(\lambda+\mu)^2}}.\sqrt[p]{\frac{\Delta_n-\varrho}{C\varrho}}.
\end{aligned}
$$

□

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Conflicts of Interest

All authors jointly worked on the results and they read and approved the final manuscript, and also declare that they have no competing interests.

References

[1] Eldred, A. and Veeramani, P. *Existence and convergence of best proximity points*, J. Math. Anal. Appl. 323(2) (2006), 1001–1006.

[2] Goebel, K. and Kirk, W.A. *Topics in metric fixed point theory*, Cambridge University Press, 1990.

[3] Guo, D. and Lakshmikantham, V. *Coupled fixed points of nonlinear operators with applications*, Nonlinear Anal. 11(5) (1987), 623–632.

[4] Ilchev, A. *On an application of coupled best proximity points theorems for solving systems of linear equations*, AIP Conf. Proc. 2048 (2018), 050003.

[5] Ilchev, A. and Zlatanov, B. *Error estimates for approximation of coupled best proximity points for cyclic contractive maps*, Appl. Math. Comput. 290 (2016), 412–425.

[6] Kirk, W., Srinivasan, P. and Veeramani, P. *Fixed points for mappings satisfying cyclical contractive conditions*, Fixed Point Theory, 4(1) (2003), 79–89.

[7] Petric, M. and Zlatanov, B. *Fixed point theorems of Kannan type for cyclical contractive condition*, in: Proceedings of the Conferences Organized by Faculty of Mathematics and Informatics, REMIA, Plovdiv University, Plovdiv, Bulgaria, 2010, 187–194.

[8] Puthusseria, A. and Ramesh Kumar, D. *Some coupled fixed point theorems for* (Ψ, Φ)*-contraction with applications to fractals*, Filomat, 38(26) (2024), 9305–9320.

[9] Safari-Hafshejani, A. *The existence of best proximity points for generalized cyclic quasi-contractions in metric spaces with the UC and ultrametric properties*, Fixed Point Theory, 23(2) (2022), 507–518.

[10] Safari-Hafshejani, A. *Error estimates for approximating fixed points and best proximity points for noncyclic and cyclic contraction mappings*, Iran. J. Numer. Anal. Optim. 13(3) (2023), 385–396.

[11] Sintunavarat, W. and Kumam, P. *Coupled best proximity point theorem in metric spaces*, Fixed Point Theory Appl. 2012(1) (2012), Article 93.

[12] Suzuki, T., Kikkawa, M. and Vetro, C. *The existence of best proximity points in metric spaces with the property UC*, Nonlinear Anal. 71 (2009), 2918–2926.

[13] Zlatanov, B. *Error estimates for approximating best proximity points for cyclic contractive maps*, Carpathian J. Math. 32(2) (2016), 265–270.