Fuzzy endpoint results for Ćirić-generalized quasicontractive fuzzy mappings

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Abstract

We introduce Ćirić-generalized quasicontractive fuzzy mappings and provide the necessary and sufficient conditions of having a unique endpoint for such mappings. Then we introduce $\beta$-$\psi$-quasicontractive fuzzy mappings, establishing an endpoint result for them. Finally, we provide some results as an application.


Keywords: Fuzzy endpoint; Ćirić-generalized; Quasicontractive fuzzy mappings; Fuzzy approximate endpoint property.

1 Introduction and preliminaries

The concept of fuzzy set was introduced initially by Zadeh [12] in 1965. In 1981, Heilpern [6] established the fuzzy contraction and proved a fuzzy fixed point theorem, which was a generalization of Nadler’s fixed point theorem for multi-valued mappings (see [9]). In 2001, Estruch and Vidal [5] utilized the result of Heilpern to fuzzy fixed point with fixed degree $\alpha$ for some $\alpha \in [0, 1]$, which was later generalized by many authors (see, for instance, [1, 3, 11]). Recently, Abbas and Turkoglu [11] proved the existence of a fuzzy fixed point for a generalized contractive fuzzy mapping. On the other hand, In 2010, Amini-Harandi [2] proved that some multi-valued mappings $T : X \to CB(X)$ have a unique endpoint if and only if they have the approximate endpoint property. Afterwards, considering the same properties, Moradi and Khojasteh [8] generalized Amini-Harandi’s result. In this paper, in the sense of [8], we prove

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that some fuzzy mappings have a unique fuzzy endpoint if and only if they have the fuzzy approximate endpoint property.

**Definition 1.** (see [6]) Let $X$ be a space of points with generic element $x$ and $I = [0, 1]$. A fuzzy set in $X$ is a function that associates any point of $X$ with a number in interval $[0, 1]$. If $A$ is a fuzzy set in $X$ and $x \in X$, then $A(x)$ is called the grade of membership of $x$ in $A$.

**Definition 2.** (see [6]) Let $(X, d)$ be a metric space and let $A$ be a fuzzy set in $X$. For $\alpha \in (0, 1]$, the $\alpha$-level set of $A$ denoted by $[A]_\alpha$, is defined as

$$[A]_\alpha = \{ x | A(x) \geq \alpha \} \text{ if } \alpha \in (0, 1]$$

and

$$[A]_0 = \{ x | A(x) > 0 \},$$

where $\overline{B}$ denotes the closure of the nonfuzzy set $B$.

**Definition 3.** (see [6]) Let $X$ be a nonempty set. For $x \in X$, we write $\{x\}$ the characteristic function of the ordinary subset $\{x\}$ of $X$. For $\alpha \in (0, 1]$, the fuzzy point $x_\alpha$ of $X$ is the fuzzy set in $X$ given by

$$x_\alpha(y) = \begin{cases} \alpha & y = x, \\ 0 & y \neq x. \end{cases}$$

Define

$$W_\alpha(X) = \{ C \in I^X : [C]_\alpha \text{ is nonempty and compact} \}.$$ 

Throughout this paper, $I^X$ denotes the collection of all fuzzy sets in $X$. For $A, B \in I^X$, it is called that $A$ is more accurate than $B$ (denoted by $A \subset B$) whenever $A(x) \leq B(x)$ for all $x \in X$. For $x \in X$, $S \subseteq X$, $A, B \in W_\alpha(X)$, and $\alpha \in (0, 1]$, we define

$$d(x, S) = \inf \{ d(x, a) : a \in S \},$$

$$p_\alpha(x, A) = \inf \{ d(x, a) : a \in [A]_\alpha \},$$

$$p_\alpha(A, B) = \inf \{ d(a, b) : a \in [A]_\alpha, b \in [B]_\alpha \},$$

$$D_\alpha(A, B) = H([A]_\alpha, [B]_\alpha) = \max \{ \sup_{x \in A} p_\alpha(x, B), \sup_{y \in B} p_\alpha(y, A) \},$$

where $H$ is the Hausdorff distance. It is easily seen that $D_\alpha$ is the Hausdorff metric on $W_\alpha(X)$ induced by the metric $d$. Hereafter, we denote by $D_\alpha(x, A)$ the amount $D_\alpha(\{x\}, A) = H(\{x\}, [A]_\alpha)$ for all $x \in X$ and $A \in W_\alpha(X)$.

**Definition 4.** (see [5]) Let $X$ be a nonempty set, let $T : X \to I^X$, and let $\alpha \in (0, 1]$. A fuzzy point $x_\alpha$ is called a fuzzy fixed point of $T$ if $x_\alpha \subset Tx$ (or equally $x \in [Tx]_\alpha$). This means that the fixed degree of $x$ is at least $\alpha$. If $\{x\} \subset Tx$, then it is called that $x$ is a fixed point of $T$. 
2 Main results

Now, we are ready to state and prove the main results of this study. Firstly, we give the following definition:

**Definition 5.** Let \( X \) be a nonempty set, let \( T : X \to I^X \), and let \( \alpha \in (0, 1] \). We say that a point \( x \in X \) is a fuzzy endpoint of \( T \) if \( \{x\} = [Tx]_{\alpha} \). This means that \( x \) is the only point in \( X \) that the fixed degree of \( x \) is at least \( \alpha \). If \( \{x\} = [Tx]_1 \), we say that \( x \) is an endpoint of \( T \).

Now, we give the following definition of fuzzy approximate endpoint property in the sense of Amini-Harandi [2].

**Definition 6.** Let \((X, d)\) be a metric space, let \( T : X \to I^X \), and let \( \alpha \in (0, 1] \). We say that \( T \) has the fuzzy approximate endpoint property whenever

\[
\inf_{x \in X} \sup_{y \in [Tx]_\alpha} d(x, y) = 0
\]

or equally

\[
\inf_{x \in X} D_\alpha(x, Tx) = 0.
\]

**Definition 7.** Let \((X, d)\) be a metric space, let \( \alpha \in (0, 1] \), and let \( T : X \to W_\alpha(X) \). We say that \( T \) is a Ćirić-generalized quasicontractive fuzzy mapping whenever there exists an upper semicontinuous (u.s.c) mapping \( \psi : [0, +\infty) \to [0, +\infty) \) such that \( \psi(t) < t \), for all \( t > 0 \) and \( \lim\inf_{t \to \infty}(t - \psi(t)) > 0 \) satisfying

\[
D_\alpha(Tx, Ty) \leq \psi(M(x, y)) \quad \text{for all } x, y \in X,
\]

where

\[
M(x, y) = \max\{d(x, y), D_\alpha(x, Tx), D_\alpha(y, Ty), D_\alpha(x, Ty), D_\alpha(y, Tx)\}.
\]

**Theorem 1.** Let \((X, d)\) be a complete metric space, let \( \alpha \in (0, 1] \), and let \( T : X \to W_\alpha(X) \) be a Ćirić-generalized quasicontractive fuzzy mapping. Then, \( T \) has a unique fuzzy endpoint if and only if \( T \) has the fuzzy approximate endpoint property.

**Proof.** If \( T \) has a fuzzy endpoint, obviously, it has the fuzzy approximate endpoint property. Conversely, let \( T \) has the fuzzy approximate endpoint property. Then, there exists a sequence \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} D_\alpha(x_n, Tx_n) = 0 \). Now for any \( n, m \in \mathbb{N} \), we have
\[ M(x_n, x_m) = \max \{ d(x_n, x_m), D_\alpha(x_n, Tx_n), \\
D_\alpha(x_n, Tx_m), D_\alpha(x_n, TTx_n), D_\alpha(x_m, Tx_n) \} \]
\[ \leq D_\alpha(x_n, Tx_n) + D_\alpha(x_m, Tx_n) + D_\alpha(Tx_n, Tx_m) \]
\[ \leq D_\alpha(x_n, Tx_n) + D_\alpha(x_m, Tx_n) + \psi(M(x_n, x_m)). \]  

Therefore, from the above inequality, we have
\[ \liminf_{n,m \to \infty} (M(x_n, x_m) - \psi(M(x_n, x_m))) = 0. \]

From the property of \( \psi \), we can conclude that \( \limsup_{n,m \to \infty} M(x_n, x_m) < \infty \).
Thus from (2) and by upper semicontinuity of \( \psi \), we have
\[ \limsup_{n,m \to \infty} M(x_n, x_m) \leq \limsup_{n,m \to \infty} \psi(M(x_n, x_m)) \]
\[ \leq \psi(\limsup_{n,m \to \infty} M(x_n, x_m)). \]

So we have \( \limsup_{n,m \to \infty} M(x_n, x_m) = 0 \) and so \( \{x_n\} \) is a Cauchy sequence.
Since \( X \) is complete, there exists \( x^* \in X \) such that \( \lim_{n \to \infty} d(x_n, x^*) = 0 \).
We shall show that \( \{x^*\} = [Tx^*_\alpha] \). To see this, we have
\[ D_\alpha(x^*, Tx^*) \leq d(x^*, x_n) + D_\alpha(x_n, Tx_n) + D_\alpha(Tx_n, Tx^*) \]
\[ \leq d(x^*, x_n) + D_\alpha(x_n, Tx_n) + \psi(M(x_n, x^*)). \]  

Limiting from both sides of (3), we get
\[ D_\alpha(x^*, Tx^*) \leq \limsup_{n \to \infty} \psi(M(x_n, x^*)). \]  

On the other hand,
\[ M(x_n, x^*) = \max \{ d(x_n, x^*), D_\alpha(x_n, Tx_n), \\
D_\alpha(x_n, Tx^*), D_\alpha(x_n, TTx_n), D_\alpha(x^*, Tx_n) \} \]
\[ \leq d(x_n, x^*) + D_\alpha(x_n, Tx_n) + D_\alpha(x^*, Tx^*), \]

which implies
\[ \limsup_{n \to \infty} M(x_n, x^*) \leq D_\alpha(x^*, Tx^*). \]  

Consequently, from right upper semicontinuity of \( \psi \), (4) and (5) yield
\[ D_\alpha(x^*, Tx^*) \leq \psi(D_\alpha(x^*, Tx^*)) \]
and so \( H(\{x^*\}, [Tx^*_\alpha]) = D_\alpha(x^*, Tx^*) = 0 \). This means that \( \{x^*\} = [Tx^*_\alpha] \).
The uniqueness of endpoint is concluded from (1).

**Definition 8.** Let \( (X, d) \) be a metric space, \( \alpha \in (0, 1] \), and \( T : X \to W_\alpha(X) \). We say that \( T \) is a Ćirić-generalized \( \beta-\psi \)-quasicontractive fuzzy mapping whenever there exists an upper semicontinuous (u.s.c) mapping \( \psi : \)
For any \( x \) there exists a sequence so that \( M(x, y) = \max\{d(x, y), D_\alpha(x, Tx), D_\alpha(y, Ty), D_\alpha(x, Ty), D_\alpha(y, Tx)\} \).

**Theorem 2.** Let \((X, d)\) be a complete metric space, let \( \alpha \in (0, 1] \), and let \( T : X \to W_\alpha(X) \) be a Ćirić-generalized \( \beta \)-\( \psi \)-quasicontractive fuzzy mapping. Moreover suppose that

(i) there exists a sequence \( \{x_n\} \) in \( X \) such that \( \beta(x_n, x_m) \geq 1 \) for all \( n, m \in \mathbb{N} \) with \( n < m \) and \( \lim_{n \to \infty} D_\alpha(x_n, Tx_n) = 0 \),

(ii) for any sequence \( \{x_n\} \) in \( X \) which \( \beta(x_n, x_m) \geq 1 \) for all \( n, m \in \mathbb{N} \) with \( n < m \) and \( x_n \to x \), we have \( \beta(x_n, x) \geq 1 \), for all \( n \in \mathbb{N} \).

Then, \( T \) has a fuzzy endpoint.

**Proof.** For any \( n, m \in \mathbb{N} \), we have

\[
M(x_n, x_m) = \max\{d(x_n, x_m), D_\alpha(x_n, Tx_n), D_\alpha(x_m, Tx_m), D_\alpha(x_n, Tx_m), D_\alpha(x_m, Tx_n)\} \\
\leq D_\alpha(x_n, Tx_n) + D_\alpha(x_m, Tx_m) + \beta(x_n, x_m)D_\alpha(Tx_n, Tx_m) \\
\leq D_\alpha(x_n, Tx_n) + D_\alpha(x_m, Tx_m) + \psi(M(x_n, x_m)).
\]

(7)

Similar to Theorem 1, we conclude that \( \limsup_{n,m \to \infty} M(x_n, x_m) = 0 \) and so \( \{x_n\} \) is a Cauchy sequence. Let \( \lim_{n \to \infty} d(x_n, x^*) = 0 \). We show that \( \{x^*\} = [Tx^*]_\alpha \). To see this, we have

\[
D_\alpha(x^*, Tx^*) \leq d(x^*, x_n) + D_\alpha(x_n, Tx_n) + \beta(x_n, x^*)D_\alpha(Tx_n, Tx^*) \\
\leq d(x^*, x_n) + D_\alpha(x_n, Tx_n) + \psi(M(x_n, x^*)).
\]

(8)

Consequently, as in Theorem 1, we obtain

\[
D_\alpha(x^*, Tx^*) \leq \psi(D_\alpha(x^*, Tx^*)),
\]

which implies \( H(\{x^*\}, [Tx^*]_\alpha) = D_\alpha(x^*, Tx^*) = 0 \). This means that \( \{x^*\} = [Tx^*]_\alpha \).

Let \( \preceq \) be the partial order on \( W_\alpha(X) \) defined by \( A \preceq B \) if and only if \( A(x) \leq B(x) \) for all \( x \in X \). In the following result, we restrict the contraction condition only for \( x, y \in X \) with \( Tx \subseteq Ty \).
**Corollary 1.** Let \((X, d)\) be a complete metric space, \(\alpha \in (0, 1]\), and \(T : X \to W_\alpha(X)\) be a fuzzy mapping such that there exists an upper semicontinuous (u.s.c) mapping \(\psi : [0, +\infty) \to [0, +\infty)\) with \(\psi(t) < t\), for all \(t > 0\) and \(\lim \inf_{t \to \infty}(t - \psi(t)) > 0\) satisfying
\[
D_\alpha(Tx, Ty) \leq \psi(M(x, y)) \quad \text{for all } x, y \in X \text{ with } Tx \subset Ty, \quad (9)
\]
where
\[
M(x, y) = \max\{d(x, y), D_\alpha(x, Tx), D_\alpha(y, Ty), D_\alpha(x, Ty), D_\alpha(y, Tx)\}.
\]
Moreover suppose that
\begin{enumerate}
  \item there exists a sequence \(\{x_n\}\) in \(X\) such that \(\{Tx_n\}\) is a nondecreasing sequence in \(W_\alpha(X)\) and \(\lim_{n \to \infty} D_\alpha(x_n, Tx_n) = 0\),
  \item for any sequence \(\{x_n\}\) in \(X\) which \(\{Tx_n\}\) is a nondecreasing sequence in \(W_\alpha(X)\) and \(x_n \to x\), we have \(Tx_n \subset Tx\), for all \(n \in \mathbb{N}\).
\end{enumerate}
Then, \(T\) has a fuzzy endpoint.

**Proof.** Define the mapping \(\beta : X \times X \to [0, \infty)\) by \(\beta(x, y) = 1\), whenever \(Tx \subset Ty\) and \(\beta(x, y) = 0\) otherwise. Then apply Theorem 2. \qed

**Corollary 2.** Let \((X, d)\) be a complete metric space, let \(x^* \in X\) be a fixed element, let \(\alpha \in (0, 1]\), and let \(T : X \to W_\alpha(X)\) be a fuzzy mapping such that there exists an upper semicontinuous (u.s.c) mapping \(\psi : [0, +\infty) \to [0, +\infty)\) with \(\psi(t) < t\), for all \(t > 0\) and \(\lim \inf_{t \to \infty}(t - \psi(t)) > 0\) satisfying
\[
D_\alpha(Tx, Ty) \leq \psi(M(x, y)) \quad \text{for all } x, y \in X \text{ with } Tx(x^*) = Ty(x^*), \quad (10)
\]
where
\[
M(x, y) = \max\{d(x, y), D_\alpha(x, Tx), D_\alpha(y, Ty), D_\alpha(x, Ty), D_\alpha(y, Tx)\}.
\]
Moreover suppose that
\begin{enumerate}
  \item there are a sequence \(\{x_n\}\) in \(X\) and \(\lambda \in [0, 1]\) such that \(Tx_n(x^*) = \lambda\) is fixed for all \(n \in \mathbb{N}\) and \(\lim_{n \to \infty} D_\alpha(x_n, Tx_n) = 0\),
  \item for any sequence \(\{x_n\}\) in \(X\) that \(Tx_n(x^*) = \lambda\) is fixed for all \(n \in \mathbb{N}\) and \(x_n \to x\), we have \(Tx(x^*) = \lambda\), for all \(n \in \mathbb{N}\).
\end{enumerate}
Then, \(T\) has a fuzzy endpoint.

**Proof.** Define the mapping \(\beta : X \times X \to [0, \infty)\) by \(\beta(x, y) = 1\), whenever \(Tx(x^*) = Ty(x^*)\) and \(\beta(x, y) = 0\) otherwise. Then applying Theorem 2 completes the proof. \qed
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References


