Iranian Journal of Numerical Analysis and Optimization
Vol. ??, No. ??, ??, pp ??
https://doi.org/??
https://ijnao.um.ac.ir/

Research Article

Jarratt and Jarratt-variant families of iterative schemes for scalar and system of nonlinear equations

O. Ogbereyivwe*, E. J. Atajeromavwo and S. S. Umar

Abstract

This manuscript puts forward two new generalized families of Jarratt's iterative schemes for deciding the solution of scalar and systems of nonlinear equations. The schemes involve weight functions that are based on bi-variate rational approximation polynomial of degree two in both its numerator and denominator. The convergence study conducted on the

*Corresponding author

Received ??? ; revised ??? ; accepted ???

Oghovese Ogbereyivwe
Department of Mathematics and Statistics, Delta State University of Science and Technology, Delta State, Nigeria. e-mail: ogho2015@gmail.com, oghovese.ogbereyivwe@dspz.edu.ng

Edafe John Atajeromavwo
Department of Software Engineering, Delta State University of Science and Technology, Delta State, Nigeria. e-mail: edafejohn2006@yahoo.com, edafejohn72@gmail.com

Salisu Shehu Umar
Department of Statistics, Federal Polytechnic, Auchi, Nigeria. e-mail: ssumar85@yahoo.com

How to cite this article
Ogbereyivwe, O., Atajeromavwo, E.J. and Umar, S.S., Jarratt and Jarratt-variant families of iterative schemes for scalar and system of nonlinear equations. *Iran. J. Numer. Anal. Optim.*, ??; ??(??): ??-??. ??

schemes, indicated that they have convergence order (CO) four in scalar space and retain the same number of CO in vector space. The numerical experiments conducted on the schemes when used to decide the solutions of some real-life nonlinear models show that they are good challengers of some well-known and robust existing iterative schemes.

AMS subject classifications (2020): Primary 65H05; Secondary 41A25.

Keywords: Jarratt scheme, Iterative scheme, Rational approximation polynomial, Nonlinear equation, System of nonlinear equation.

1 Introduction

A challenging task in the area of numerical computing is that of determining the solution of a scalar nonlinear equation $\Gamma(x) = 0$ and the system of nonlinear equation $\Gamma(X) = \mathbf{0}$. This is because, plethora of real life situations or systems, are often modeled into either scalar nonlinear or system of nonlinear equations. For instance, More [8] presented a compiled physical problems that are mostly modeled in the form of system of nonlinear equations. Problems in chemical equilibrium were modeled into nonlinear equations and studied by Shacham and Kehat [15]. Systems of nonlinear equations were applied to modeled problems in kinematics syntheses, neurophysiology, economics, and combustion engineering, which were considered and studied by Grosan and Abraham [3]. The nonlinear global positioning system problems modeled into nonlinear systems, were extensively exploited and solved by Yaseen, Zafar, and Alsulami [19]. More literature on physical occurrences or phenomena that are modeled into systems of nonlinear equations can be found in [6, 7, 9].

In order to have better insight of the problems that are modeled into nonlinear equations, the solutions of the models are usually desired to be known. Unfortunately, there is no general algebraic formula for solving all types of nonlinear equations. Consequent upon this, iterative schemes are designed to fill this gap. While many researchers have developed several schemes for dealing with the solutions of scalar nonlinear or system of nonlinear equations, some of the schemes suffered some setbacks such as low convergence order (CO) and efficiency, non-convergence, divergence from true solution, breakdown, or inability to directly extend the schemes for solving scalar nonlinear equations to solving multidimensional nonlinear system and still conserving their CO.

The Jarratt scheme (JS) [4] presented as

$$x_{i+1} = x_i - \left[\frac{\Gamma'(x_i) + 3\Gamma'(y_i)}{6\Gamma'(y_i) - 2\Gamma'(x_i)}\right] \frac{\Gamma(x_i)}{\Gamma'(x_i)}, \qquad i = 0, 1, 2, \ldots, \tag{1}$$

where $y_i = x_i - \frac{2}{3}\frac{\Gamma(x_i)}{\Gamma'(x_i)}$ is one of the famous classical iterative scheme that is efficient, converges to solution with high CO, and can directly be extended to multidimensional form and still retain its CO. Many JS families have been put forward in literature. One of such families is that due to Behl, Kanwar, and Sharma [1] and presented as follows:

$$x_{i+1} = x_i - \frac{\left[(q_1^2 - 22q_1q_2 - 27q_2^2)\Gamma'(x_i) + 3(q_1^2 + 10q_1q_2 + 5q_2^2)\Gamma'(y_i)\right]\Gamma(x_i)}{2\left[q_1\Gamma'(x_i) + 3q_2\Gamma'(y_i)\right]\left[3(q_1+q_2)\Gamma'(y_i) - (q_1 + 5q_2)\Gamma'(x_i)\right]}, \tag{2}$$

where $q_i \in \Re$ such that $q_1 \neq q_2$ and $q_1 \neq -3q_2$. Kanwar, Kumar, and Behl [5] modified the family of the iterative scheme in (2) to deal with problems in multivariate case. Their scheme reads as follows:

$$\begin{aligned} Y_i &= X_i - \frac{2}{3}M(X_i) \\ X_{i+1} &= X_i - \frac{1}{2}\left[(q_1 I + 3q_2 N(X_i))\left(3(q_1+q_2)N(X_i) - (q_1+5q_2)I\right)\right]^{-1} \\ &\quad \times \left[(q_1^2 - 22q_1q_2 - 27q_2^2)I + 3(q_1^2 + 10q_1q_2 + 5q_2^2)N(X_i)\right]M(X_i), \\ N(X_i) &= \Gamma'(X_i)^{-1}\Gamma'(Y_i), \qquad M(X_i) = \Gamma'(X_i)^{-1}\Gamma'(X_i), \end{aligned} \tag{3}$$

where I is an identity matrix corresponding to the nonlinear system dimension.

In [2], another JS family was designed and analyzed using complex dynamics techniques to investigate whether behind the JS in the developed family. There are other JS variants that are computationally better in stability and numerical performance. After analysis, their work concluded that JS performed best among all the concrete members of the developed family.

Deriving motivation from the schemes put forward in (2) and (3), we propose two new families of Jarratt and Jarratt-variant schemes for solving nonlinear equations in this paper. The new families of schemes were derived based on the bi-variate rational approximation polynomial of degree two in both its numerator and denominator. The derived families of the schemes are of high CO and optimal as conjectured by Traub [18].

This paper is arranged as follows: Section 2 is meant for the development of the two families and their theoretical convergence analysis. Furthermore, some typical members of the families are noted in this section. Section 3 represents the extension of the families to multivariate cases and their convergence criteria. In Section 4, the application of some identified members of the developed families of iterative schemes to solve some real-life models is presented. Finally, the concluding remarks are made in Section 5.

2 The iterative schemes

We begin by acknowledging the following fundamental concepts.

Definition 1. [18, 10] Let x_* be the solution of $\Gamma(x) = 0$ and $d_i = |x_i - x_*|$ the error at ith iteration point of an iterative scheme $\psi(x_i)$. If an equation of the form $d_{i+1} = \Omega d_i^{\eta} + O(d_i^{\eta+1})$ can be derived from $\psi(x_i)$ via the Taylor expansions of the functions $\Gamma(\cdot)$ and functions derivatives $\Gamma'(\cdot)$ it contains, then d_{i+1} is known as the iterative scheme asymptotic error equation, η is CO, and Ω is asymptotic constant.

Definition 2. [18, 10] The efficiency of an iterative scheme $\psi(x_i)$ is the value $\eta^{\frac{1}{T}}$, where T is the sum of all different functions assessment in one iteration cycle.

Definition 3. [14] The computational CO (η_{coc}) of an iterative scheme $\psi(x_i)$ is the value obtained by using the formula

$$\eta_{coc} \approx \frac{\log \omega_1}{\log \omega_2}, \tag{4}$$

where $\omega_1 = \left|\frac{\Gamma(x_i)}{\Gamma(x_{i-1})}\right|$, $\omega_2 = \left|\frac{\Gamma(x_{i-1})}{\Gamma(x_{i-2})}\right|$, and x_{i-2}, x_{i-1} together with x_i are last three consecutive iteration results of $\psi(x_i)$.

The formulas in Definitions 1, 2, and 3 can easily be modified to handle cases in vector form.

2.1 The Jarratt's family (JF)

We consider the weighted scheme presented as follows:

$$x_{i+1} = x_i - P_{2,2}\left[\Gamma'(x_i), \Gamma'(y_i)\right] \frac{\Gamma(x_i)}{\Gamma'(x_i)}, \tag{5}$$

where

$$\begin{aligned} &P_{2,2}\left[\Gamma'(x_i), \Gamma'(y_i)\right] \\ &\quad = \frac{a_2\left[\Gamma'(x_i) - \Gamma'(y_i)\right]^2 + \Gamma'(y_i)\left[a_1\left(\Gamma'(x_i) - \Gamma'(y_i)\right) + \Gamma'(y_i)\right]}{a_4\left[\Gamma'(x_i) - \Gamma'(y_i)\right]^2 + \Gamma'(y_i)\left[a_3\left(\Gamma'(x_i) - \Gamma'(y_i)\right) + \Gamma'(y_i)\right]} \end{aligned} \tag{6}$$

is a bi-variate rational approximation polynomial of degree two in its numerator and denominator, $a_i (i = 1, 2, 3, 4) \in \Re$ are free parameters such that not all $a_i = 0$ and $y = x_i - \frac{2}{3}\frac{\Gamma(x_i)}{\Gamma'(x_i)}$. To determine the contribution of the scheme (5), some conditions are set on the parameters so as to ensure its convergence when implemented to solve scalar nonlinear equations.

Theorem 1. Suppose that $\Gamma : D \subset \Re \to \Re$ is a scalar function that is sufficiently differentiable in D such that $\Gamma'(\cdot) \neq 0$ in D. Furthermore, let $x_* \in D$ be a simple solution of Γ. If x_0 is in the neighborhood of x_*, then the sequence of solution approximations $\{x_i\}_{i\geq 0}, (x_n \in D)$ produced by the family of the scheme in (5), will converge to x_* with CO four subject to the conditions that $a_3 = \frac{4a_1-3}{4}$ and $a_4 = \frac{3-12a_1+16a_2}{16}$.

Proof. Applying the Taylor's expansion on $\Gamma(x)$ and $\Gamma'(x)$ around the solution x_* and setting $x = x_i$, then

$$\Gamma\left(x_i\right) = \Gamma'\left(x_*\right)\left[d_i + \sum_{n=2}^{4} c_n d_i^n + O\left(d_i^5\right)\right], \qquad i{=}0, 1, 2, \ldots, \tag{7}$$

and

$$\Gamma'\left(x_i\right) = \Gamma'\left(x_*\right)\left[1 + \sum_{n=2}^{4} n c_n d_i^{n-1} + O\left(d_i^5\right)\right], \tag{8}$$

holds for $c_n = \frac{1}{n!}\frac{\Gamma^{(n)}(x_*)}{\Gamma'(x_*)}, n \geq 2$.

Using the expansions in (7) and (8), the expression for y_i is as follows:

$$y_i = x_i - \frac{2}{3}\frac{\Gamma(x_i)}{\Gamma'(x_i)} = \frac{1}{3}d_i + \frac{2c_2}{3}d_i^2 - \frac{3(c_2^2 - c_3)}{3}d_i^3 + \frac{2(4c_2^2 - 7c_2c_3 + 3c_4)}{3}d_i^4 + \frac{4(4c_2^4 - 10c_2^2c_3 + 3c_3^2 + 5c_2c_4 - 2c_5)}{3}d_i^5 + O(d_i^6). \tag{9}$$

Now, the Taylor's expansion for $\Gamma(y_i)$ is

$$\Gamma(y_i) = \frac{1}{3}d_i + \frac{7c_2}{9}d_i^2 + \left(\frac{8c_2^2}{9} + \frac{37c_3}{27}\right)d_i^3 + \frac{2(10c_2^3 - 16c_2c_3 + 9c_4)}{9}d_i^4 + \frac{4\left(12c_2^4 - 27c_2^2c_3 + 8c_3^2 + 12c_2c_4 - 6c_5\right)}{9}d_i^5 + O(d_i^6), \tag{10}$$

and for $\Gamma'(y_i)$, follow the next equation:

$$\Gamma'(y_i) = 1 + \frac{2}{3}d_i + \frac{4c_2^2 - c_3}{3}d_i^2 + \left(\frac{-8c_2^3}{3} + 4c_2c_3\right)d_i^3 + \frac{4(4c_2^4 - 8c_2c_3 + 2c_3^2 + 3c_2c_4)}{3}d_i^4 + \frac{4\left(8c_2^5 - 20c_2^3c_3 + 9c_2c_3^2 + 10c_2^2c_4 - 3c_3c_4 - 4c_2c_5\right)}{3}d_i^5 + O(d_i^6). \tag{11}$$

The use of the expansions in (7), (8), and (11), enables the deduction of the expansion of the the weight function $P_{2,2}\left[\Gamma'(x_i), \Gamma'(y_i)\right]$ in (6) as follows:

$$\begin{aligned} &P_{2,2}\left[\Gamma'(x_i), \Gamma'(y_i)\right]\frac{\Gamma(x_i)}{\Gamma'(x_i)} \\ &= d_i + \left(\frac{-3 + 4a_1 - 4a_3}{3}\right)d_i^2 \\ &\quad + \frac{1}{9}\begin{pmatrix} 2(9 + 8a_a + 16a_3 + 8a_3^3 - 8a_1(2 + a_3) - 8a_4)c_2^2 \\ + 3(-3 + 4a_1 - 4a_3)c_3 \end{pmatrix} d_i^3 \\ &\quad + \frac{1}{27}\begin{pmatrix} 4(-27 - 49a_3 - 52a_3^2 - 16a_3^3 - 4a_2(13 + 4a_3) \\ + \cdots + 27(-3 + 4a_1 - 4a_3)c_4 \end{pmatrix} d_i^4 \\ &\quad + \frac{1}{81}\left((8(81 + 127a_3 + 210a_3^2 + 144a_3^3 + 32a_3^4 - \cdots + 9(4 - 5a_1 + 5a_3)c_5\right)d_i^5 \\ &\quad + O(d_i^6). \end{aligned} \tag{12}$$

The substitution of (12) into (5) yields

$$
\begin{aligned}
x_{i+1} =& x_i - P_{2,2}\left[\Gamma'(x_i), \Gamma'(y_i)\right] \frac{\Gamma(x_i)}{\Gamma'(x_i)} \\
=& x_* - d_i - d_i + \left(\frac{-3+4a_1-4a_3}{3}\right) d_i^2 \\
&+ \left(\frac{2((9+8a_2+16a_3+8a_3^3-8a_1(2+a_3)-8a_4)c_2^2+3(-3+4a_1-4a_3)c_3}{9}\right) d_i^3 \\
&+ \left(\frac{(4(-27-49a_3-52a_3^2-16a_3^3-\cdots+27(-3+4a_1-4a_3)c_4}{27}\right) d_i^4 \\
&+ \left(\frac{(8(81+127a_3+210a_3^2+144a_3^3-\cdots+9(4-5a_1+5a_3)c_5}{81}\right) d_i^5 + O(d_i^6).
\end{aligned} \tag{13}
$$

The expression in (13) will reduce to equation of order four if the next set of equations hold:

$$
\begin{aligned}
&-3+4a_1-4a_3 = 0, \\
&9+8a_2+16a_3+8a_3^3-8a_1(2+a_3)-8a_4 = 0.
\end{aligned} \tag{14}
$$

The equations in (14) are satisfied when

$$
\begin{aligned}
a_3 &= \frac{4a_1-3}{4}, \\
a_4 &= \frac{3-12a_1+16a_2}{16}.
\end{aligned} \tag{15}
$$

Consequently, by the application of (15) in (13), it reduces to

$$
\begin{aligned}
x_{i+1} =& x_* + \left(\frac{(10-4a_1+16a_2)c_2^3-9c_2c_3}{9}\right) d_i^4 \\
&+ \frac{1}{108}\left(\begin{aligned} &4(-121+16a_1^2+a_1(48-64a_2)-208a_2)c_2^4 \\ &\quad -72(-13+4a_1-16a_2)c_2^2c_3-324c_2c_4+27(-8c_3^2+c_5)\end{aligned}\right) d_i^5 \\
&+ O\left(d_i^6\right).
\end{aligned} \tag{16}
$$

By Definition 1, the equation in (16) is the error equation of the scheme in (5) and therefore, has minimum CO four. □

Remark 1. When the conditions in (15) are substituted into the scheme in (5) and after some simplification, a new two-free-parameter family of the Jarratt scheme (JF) is presented as follows:

$$
x_{i+1} = x_i + P_{2,2}\left[\Gamma'(x_i), \Gamma'(y_i)\right] \frac{\Gamma(x_i)}{\Gamma'(x_i)}, \tag{17}
$$

where $P_{2,2}\left[\Gamma'(x_i),\Gamma'(y_i)\right]$ expression is

$$
\begin{aligned}
&P_{2,2}\left[\Gamma'(x_i),\Gamma'(y_i)\right]\\
&=\left[\frac{16\left(a_2\left[\Gamma'(x_i)-\Gamma'(y_i)\right]^2+\Gamma'(y_i)\left[a_1\left(\Gamma'(x_i)-\Gamma'(y_i)\right)+\Gamma'(y_i)\right]\right)}{\begin{array}{r}\left(-3+12a_1-16a_2\right)\Gamma'(x_i)^2+2\left(9-20a_1+16a_2\right)\Gamma'(x_i)\Gamma'(y_i)\\ +\left(-31+28a_1-16a_2\Gamma'(y_i)^2\right)\end{array}}\right].
\end{aligned}
\tag{18}
$$

The JF require one $\Gamma(\cdot)$ and two $\Gamma'(\cdot)$ evaluations in one cycle of iteration and by Definition 2, the family has $E.I = 1.5874$.

2.1.1 The JF Particular cases

Here, for some specific values of a_1 and a_2 imposed on the scheme in (17), a particular case of the developed JF can be put forward. Some typical cases are provided next.

Case 1: For $a_1 = \frac{1}{4}$ and $a_2 = 0$, the famous JS is rediscovered as follows:

$$x_{i+1} = x_i + \left[\frac{\Gamma'(x_i)+3\Gamma'(y_i)}{2\Gamma'(x_i)-6\Gamma'(y_i)}\right]\frac{\Gamma(x_i)}{\Gamma'(x_i)}. \tag{19}$$

Case 2: For $a_1 = \frac{3}{4}$ and $a_2 = \frac{3}{8}$, a new IS (JF_1) is put forward as follows:

$$x_{i+1} = x_i - \left[\frac{5}{8}+\frac{3\Gamma'(x_i)^2}{8\Gamma'(y_i)^2}\right]\frac{\Gamma(x_i)}{\Gamma'(x_i)}. \tag{20}$$

Case 3: For $a_1 = a_2 = 1$, a new scheme (JF_2) is formed as follows:

$$x_{i+1} = x_i - 16\left[\frac{\Gamma'(x_i)^2-\Gamma'(x_i)\Gamma'(y_i)+\Gamma'(y_i)^2}{7\Gamma'(x_i)^2-10\Gamma'(x_i)\Gamma'(y_i)+19\Gamma'(y_i)^2}\right]\frac{\Gamma(x_i)}{\Gamma'(x_i)}. \tag{21}$$

Case 4: For $a_1 = a_2 = 0$, a new scheme (JF_3) is constructed as follows:

$$x_{i+1} = x_i - \left[\frac{16\Gamma'(y_i)^2}{3\Gamma'(x_i)^2-18\Gamma'(x_i)\Gamma'(y_i)+31\Gamma'(y_i)^2}\right]\frac{\Gamma(x_i)}{\Gamma'(x_i)}. \tag{22}$$

2.2 The Jarratt-variant family (JVF)

Here an iterative scheme that is a variant of the Jarratt's scheme family put forward in (17) is presented as follows:

$$x_{i+1} = x_i - R_{2,2}\left[\Gamma'(x_i), \Gamma'(y_i)\right] \frac{\Gamma'(x_i)}{\Gamma'(x_i)}, \tag{23}$$

where

$$R_{2,2}\left[\Gamma'(x_i), \Gamma'(y_i)\right] = \left[\frac{\Gamma'(y_i)^2 + b_1\Gamma'(x_i)\Gamma'(y_i) + b_2\Gamma'(x_i)^2}{\Gamma'(y_i)^2 + b_3\Gamma'(x_i)\Gamma'(y_i) + b_4\Gamma'(x_i)^2}\right], \tag{24}$$

is a degree two rational polynomial and $b_i (i = 1, 2, 3, 4) \in \Re$ are free parameters and not all equal zero. Next, the convergence condition for the IS (23) is investigated.

Theorem 2. Under the hypothesis made on the function $\Gamma(x)$ in Theorem 1, the IS in (23) will be of CO four when $b_2 = \frac{5+b_1}{3}$, $b_3 = 2(1 + b_1)$ and $b_4 = \frac{-1-2b_1}{3}$ provided $b_1 \neq -2$.

Proof. Assume that the expansions in (7)–(11) holds. Then

$$\begin{aligned}
&R_{2,2}\left[\Gamma'(x_i), \Gamma'(y_i)\right] \frac{\Gamma'(x_i)}{\Gamma'(x_i)} \\
&= \left(\frac{1 + b_1 + b_2}{1 + b_3 + b_4}\right) d_i \\
&\quad - \frac{1}{3\theta^2}\left(3 + 7b_3 - b_2(5 + b_3 - 3b_4) + 11b_4 + b_1(-1 + 3b_3 + 7b_4)\right) d_i^2 \\
&\quad + \frac{1}{9\theta^3}\begin{pmatrix} 2((9 + 34b_3 + 33b_3^2 + 42b_4 + 90b_3b_4 + 65b_4^2 - b_2(15 + 7b_3^2 \\ + 38b_4 - 9b_4^2 + 6b_3(5 + b_4)) + b_1(-7 + 9b_3^2 - 6b_4 + 33b_4^2 \\ + b_3(-6 + 34b_4)))c_2^2 + 3(-3 - 7b_3 + b_1(1 - 3b_3 - 7b_4) \\ + b_2(5 + b_3 - 3b_4) - 11b_4)(1 + b_3 + b_4)c_3) \end{pmatrix} d_i^3 \\
&\quad + \frac{1}{27\theta^4}\begin{pmatrix} -(-4(27 + 130b_3 + 231b_3^2 + 144b_3^3 + \cdots - b_2(19 + 22b_3^3 + \cdots \\ 23b_3^2(5 + 3b_4) + \cdots + 6b_2(5 + b_3 - 3b_4) - 11b_4)(1 + b_3 + b_4)^2c_4)) \end{pmatrix} d_i^4 \\
&\quad + O(d_i^5).
\end{aligned} \tag{25}$$

where $\theta = 1 + b_3 + b_4$.

Our expectation here, is to reduce the error equation in (25) to order four. This suffices to solving the following set of equations:

$$b_1 + b_2 - b_3 - b_4 = 0,$$
$$4b_1 - b_3 + 3b_4 = -3, \tag{26}$$
$$2b_1 + 3b_4 = -1.$$

The solution set satisfying (26) is

$$b_2 = \frac{5 + b_1}{3}, \quad b_3 = 2(1 + b_1), \quad b_4 = \frac{-1 - 2b_1}{3}. \tag{27}$$

The substitution of (27) in (25) produces

$$R_{2,2}\left[\Gamma'(x_i), \Gamma'(y_i)\right] \frac{\Gamma'(x_i)}{\Gamma'(x_i)} = d_i + \left(\frac{c_2((10 + 3b_1)c_2^2 - 3(2 + b_1)c_3)}{2(2 + b_1)}\right) d_i^4 + O(d_i^5). \tag{28}$$

When (28) is substituted in (23), we have

$$x_{i+1} = x^* + \left(\frac{c_2((10 + 3b_1)c_2^2 - 3(2 + b_1)c_3)}{3(2 + b_1)}\right) d_i^4 + O(d_i^5). \tag{29}$$

From Definition 1, the equation (29) indicates that the IS in (23) has CO four. □

Remark 2. When the parameters b_2 and b_3 are as defined in (27), then the scheme in (23) will reduce to a new one-parameter JVF of schemes that are of CO four and presented as follows:

$$x_{i+1} = x_i - \left[\frac{3\Gamma'(y_i)^2 + 3b_1\Gamma'(x_i)\Gamma'(y_i) + (5 + b_1)\Gamma'(x_i)^2}{3\Gamma'(y_i)^2 + 6(1 + b_1)\Gamma'(x_i)\Gamma'(y_i) - (1 + 2b_1)\Gamma'(x_i)^2}\right] \frac{\Gamma(x_i)}{\Gamma'(x_i)}. \tag{30}$$

Again, the family in (30) requires three assessment of functions in an iteration cycle. Therefore, $E.I = 1.5874$.

2.2.1 Some JV family concrete members

Here, for specific values for b_1, a particular case of (30) can be put forward. Consider the following cases:

Case 1: Put $b_1 = -\frac{10}{3}$ in (30). A new scheme (JVF_1) of CO four is obtained as follows:

$$x_{i+1} = x_i - \left[\frac{5\Gamma'(x_i)^2 - 30\Gamma'(x_i)\Gamma'(y_i) + 9\Gamma'(y_i)^2}{17\Gamma'(x_i)^2 - 42\Gamma'(x_i)\Gamma'(y_i) + 9\Gamma'(y_i)^2}\right] \frac{\Gamma(x_i)}{\Gamma'(x_i)}. \tag{31}$$

Case 2: For $b_1 = -5$ in (30), a new scheme (JVF_2) of CO four is determined as follows:

$$x_{i+1} = x_i - \left[\frac{\Gamma'(y_i)\left(\Gamma'(y_i) - 5\Gamma'(x_i)\right)}{\Gamma'(y_i)^2 - 8\Gamma'(x_i)\Gamma'(y_i) + 3\Gamma'(x_i)^2}\right]\frac{\Gamma(x_i)}{\Gamma'(x_i)}. \tag{32}$$

3 Extension of the families to n-dimensional form

In this section, the developed JF and JVF schemes were extended to solving the n-dimensional system of nonlinear equations $\Gamma(X) = \mathbf{0}$, where $\Gamma : \Theta \subseteq \Re^n \to \Re^n$ describes the dimension of the nonlinear system. Next, we show that the JF and JVF retain their CO when extended to solve n-dimensional nonlinear systems of equations. We define $X_* = \left(x_*^{(1)}, x_*^{(2)}, \ldots, x_*^{(n)}\right)^T$ as the solution of the $\Gamma(X) = \mathbf{0}$, and for an initial iteration starting point $X_0 = \left(x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(n)}\right)^T$ close to X_*, the equivalence of the developed families of schemes in n-dimensional form are as follows:

Scheme (17) (nJF):

$$X_{i+1} = X_i - A^{-1}B\left[M(X_i)\right], \tag{33}$$

where

$$A = \begin{bmatrix} \beta_1 I \\ +2\beta_2 N(X_i) \\ +\beta_3 N(X_i)^2 \end{bmatrix}^{-1}, \qquad B = \begin{bmatrix} -16 \begin{pmatrix} a_4(I - N(X_i))^2 + \\ (a_1\left(N(X_i) - N(X_i)^2\right) \\ + \qquad N(X_i)^2) \end{pmatrix} \end{bmatrix}, \tag{34}$$

$\beta_1 = 12a_1 - 16a_2 - 3, \quad \beta_2 = -20a_1 + 16a_2 + 9, \quad \beta_3 = 28a_1 - 16a_2 - 31,$
and
Scheme (23) (nJVF):

$$X_{i+1} = X_i - G^{-1}H\left[M(X_i)\right], \tag{35}$$

where

$$G = \begin{bmatrix} 3N(X_i)^2 \\ +6(1+b_1)N(X_i) \\ -(1+2b_1)I \end{bmatrix}^{-1}, \qquad H = \begin{bmatrix} 3N(X_i)^2 \\ +3b_1 N(X_i) \\ + \ (5+b_1)I \end{bmatrix}, \tag{36}$$

respectively.

We note that the $(i+1)$th iteration error of (33) and (35) is $E_{i+1} = \Omega(E_i^{\eta}) + O\left(E_i^{\eta+1}\right)$, is known as the error equation and η is the CO. Observe that $E_i^{\eta} = \left(E_i^{(1)}, E_i^{(2)}, \ldots, E_i^{(n)}\right)^{\eta}$. Next, we prove that the families of methods in (17) and (23) retain their CO in $\Re^n$ space.

Theorem 3. Assume that $\Gamma : \Delta \subseteq \Re^n \longrightarrow \Re^n$ is Frechet differentiable and that $\Gamma'(\cdot) \neq \mathbf{0}$ in the convex set Δ contains the solution X_* of $\Gamma(X) = \mathbf{0}$. For an initial guess X_0 close to X_*, the scheme in (33) will form a sequence $\{X_i\}_{i=0}^{\infty}$ of numerical results that converges to X_* with order four.

Proof. The Taylor's expansion of $\Gamma(X)$ around X_i up to the fourth order is

$$\Gamma(X) = \Gamma(X_i) + \sum_{k=1}^{4} \frac{1}{k!}\Gamma^{(k)}(X_i)(X - X_i)^k + O\left(\|X - X_i\|^5\right), \tag{37}$$

where $\Gamma^{(k)}(\cdot)$ is the kth-Frechet derivative of $\Gamma(\cdot)$.

Let $E_i = X_i - X_*$ be error at ith iteration point. Set $X = X_*$ in (37). Then

$$\Gamma(X_i) = \sum_{k=1}^{4}\left[(-1)^{k+1}\frac{1}{k!}\Gamma^{(k)}(X_i)\left(E_i\right)^k\right] + O\left(\|E_k\|^5\right). \tag{38}$$

Pre-multiplying the equation in (38) by $\Gamma(X_i)^{-1}$, we obtain

$$M(X_i) = E_i + \sum_{k=2}^{4}\left[(-1)^{k+1}\frac{1}{k!}\Gamma(X_i)^{-1}\Gamma^{(k)}(X_i)\left(E_i\right)^k\right] + O\left(\|E_i\|^5\right). \tag{39}$$

By applying (38) into the first step of the iteration cycle $Y_i = X_i - \frac{2}{3}M(X_i)$, we get

$$Y_i - X_i = \frac{2}{3}\left(-E_i + \sum_{k=2}^{4}\left[(-1)^{k}\frac{1}{k!}\Gamma(X_i)^{-1}\Gamma^{(k)}(X_i)\left(E_i\right)^k\right] + O\left(\|E_i\|^5\right)\right). \tag{40}$$

Consequently,

$$\begin{aligned}
&(Y_k - X_i)^2 \\
&= \frac{4}{9}E_i^2 - \frac{4}{9}\Gamma(X_i)^{-1}\Gamma''(X_i)E_i^3 \\
&\quad + \frac{1}{27}\Gamma(X_i)^{-1}\left[4\Gamma'''(X_i) + 3\Gamma''(X_i)\Gamma(X_i)^{-1}\Gamma''(X_i)\right]E_i^4 + O\left(\|E_i\|^5\right),
\end{aligned} \tag{41}$$

$$(Y_i - X_i)^3 = -\frac{8}{27}E_i^3 + \frac{4}{9}\Gamma(X_i)^{-1}\Gamma''(X_i)E_i^4 + O\left(\|E_i\|^5\right), \tag{42}$$

and

$$(Y_i - X_i)^4 = -\frac{16}{81}E_i^4 + O\left(\|E_i\|^5\right). \tag{43}$$

The Taylor's expression of $\Gamma(Y_i)$ around X_i is obtained as follows:

$$\Gamma'(Y_i) = \sum_{k=1}^{4}\left[\frac{1}{k!}\Gamma^{(k)}(X_k)\,(Y_i - X_i)^{k-1}\right] + O\left(\|Y_i - X_i\|^5\right). \tag{44}$$

A little simplification on (33), while taking note that $M(X_i) = \Gamma'(X_i)\Gamma(X_i)$, we have

$$A\Gamma'(X_i)E_{i+1} = A\Gamma'(X_i)E_i - B\Gamma(X_i). \tag{45}$$

Using (38) and (44), the next expansions were obtained:

$$\begin{aligned}
&A\Gamma'(X_i)E_i\\
&= -16\Gamma'''(X_i)E_i + \frac{8}{3}(11-4a_1)\Gamma'(X_i)^2\Gamma''(X_i)E_i^2\\
&\quad + \frac{8}{9}\left(4(5a_1-2)(4+a_2)\Gamma''(X_i)^2 + (4a_1-11)\Gamma'(X_i)\Gamma'''(X_i)\right)\\
&\quad + \frac{4}{81}\begin{pmatrix} 9(28a_1-16a_2-131)\Gamma'''(X_i)^3 + 3(92a_1-32a_2\\ -161)\Gamma''(X_i)\Gamma'(X_i)\Gamma'''(X_i) + (4a_1-11)\Gamma'(X_i)^2\Gamma^{(iv)}(X_i)\end{pmatrix}E_i^4\\
&\quad + O\left(\|E_i\|^5\right)
\end{aligned} \tag{46}$$

and

$$\begin{aligned}
&B\Gamma(X_i)E_i\\
&= -16\Gamma'''(X_i)E_i + \frac{8}{3}(11-4a_1)\Gamma'(X_i)^2\Gamma''(X_i)E_i^2\\
&\quad + \frac{8}{9}\left(4(5a_1-2)(4+a_2)\Gamma''(X_i)^2 + (4a_1-11)\Gamma'(X_i)\Gamma'''(X_i)\right)\\
&\quad + \frac{2}{81}\begin{pmatrix} 108(5a_1-4a_2-6)\Gamma'''(X_i)^3 + 24(23a_1-8a_2-38)\\ \times\Gamma''(X_i)\Gamma'(X_i)\Gamma'''(X_i) + (32a_1-91)\Gamma'(X_i)^2\Gamma^{(iv)}(X_i)\end{pmatrix}E_i^4\\
&\quad + O\left(\|E_i\|^5\right).
\end{aligned} \tag{47}$$

When (46) and (47) are substituted into the right hand side of (45), we get

$$A\Gamma'(X_i)E_{i+1} = \frac{2}{27}\begin{bmatrix} 6(2a_1 - 8a_2 - 5)\Gamma'''(X_i) - \Gamma'(X_i)^2\Gamma^{(iv)}(X_i) \\ + 18\Gamma'(X_i)\Gamma''(X_i)\Gamma'''(X_i) \end{bmatrix} E_i^4 + O\left(\|E_i\|^5\right). \tag{48}$$

The error equation in (48) implies that the IS in (33) is of CO four. □

Theorem 4. Assume that the hypothesis on the function $\Gamma(X)$ holds as in Theorem 3. Then the scheme in (35) will produce sequence of numerical results that converge to X_* with CO four.

Proof. The proof follows the manner of the proof of Theorem **??**. Consequently, its error equation in $\Re^n$ space is

$$G\Gamma'(X_i)E_{i+1} = \frac{1}{54}\begin{bmatrix} 9(10 + 3b_1)\Gamma''(X_i)^3 - 18(2 + b_1)\Gamma'(X_i)\Gamma''(X_i)\Gamma'''(X_i) \\ + (2 + b_1)\Gamma'(X_i)^2\Gamma^{(iv)}(X_i) \end{bmatrix} E_i^4 + O\left(\|E_i\|^5\right). \tag{49}$$

□

4 Numerical experiments results

This section presents the experimentation of some concrete forms of the developed JF and JVF on the solution of nonlinear and systems of nonlinear equations. The numerical results obtained were compared with the results of some existing competitors that are also Jarratt's variants. To verify the theoretical CO of the JF, JV, nJF, and nJVF obtained in Sections 2 and 3, we use the computational CO (η_{coc}) given in Definition 3.

4.1 Numerical experiments with scalar nonlinear models

This subsection provides some numerical experiments conducted on the developed schemes when applied to solve some physical problems modeled into nonlinear equations.

Model application 1 (Chemical engineering, [11]). To determine the concentration of a chemical mixture at time x in a mixed reactor, it can be obtained by using the nonlinear equation

$$\Gamma_1(x) = -0.75\exp(-0.05x) + 1 = 0. \tag{50}$$

The nonlinear equation solution is $-5.7536414490\ldots$.

Model Application 2 (Chemical Reactor, [11, 12]). In a chemical reactor, the fractional species conversion x can be determined by the nonlinear equation

$$\Gamma_2(x) = \frac{8x^2\,(4-x)^2}{(2-x)(6-3x)^2} - 0.186 = 0. \tag{51}$$

The solution to the nonlinear equation is $0.2777575428\ldots$.

Model Application 3 (Projectile model, [17]). The nonlinear model governing an electron movement between two parallel plates is

$$\Gamma_3(x) = \frac{\pi}{4} + x - 0.5\cos x = 0. \tag{52}$$

Its solution is $-0.3094661392\ldots$.

Model Application 4 (Structural engineering, [17]). The nonlinear model that described stresses encountered by finite structures in underground is:

$$\Gamma_4(x) = -\frac{1}{4} + \frac{\sin x\cos x + x}{\pi} = 0. \tag{53}$$

The nonlinear model solution is $0.4160444988\ldots$.

Model Application 5 (Population model, [16, 11]). The population dynamics is described by using the differential equation

$$P'(x) = \delta P(x) + \mu, \tag{54}$$

where $P(x)$ is time x population, δ is population birth rate, and μ represent rate of immigration. The model solution is

$$P(x) = P_0\exp(\delta x) + \frac{\mu}{\delta}\left(\exp(\delta x) - 1\right), \tag{55}$$

where P_0 is initial population. A problem may arise to find to find the birth rate δ at the end of first year given that $P_0 = 100000, \mu = 435000$, and 1564000 is population added at the end of the year. This amounts to solving the nonlinear equation

$$\Gamma_5(\delta) = P(\delta) = -\frac{435000}{\delta}(\exp(\delta) - 1) - 1000000\exp(\delta) + 1564000 = 0. \quad (56)$$

The solution of the nonlinear equation in (56) is $0.1009979296\ldots$.

The obtained numerical experimentation results when the developed JF and JVF members were applied to solve the nonlinear models above were presented in Tables 1 and 2. The computation results in terms of the number of iterations (N), residual error $|\Gamma(x)|$, the total number of arithmetic operations with functions N_T, and computational order of convergence η_{coc}, were compared with results by some existing schemes taken from [1] (see (2)) using $q_1 = 0, q_2 = 0$ and denoted as (BS), while from [5](see (3)), taking $q_1 = 50, q_2 = \frac{1}{10}$ and denoted as (KS). The mpmath-PYTHON was utilized in designing computational program and $\Gamma(x_i) \leq 10^{-100}$ used as program terminal criterion. To reduce truncation error, all computations were set to 1000 decimal places accuracy. The expression $S.Te{-}U$ represents $S.T \times 10^{-U}$ where $S, T, U \in \Re$.

Table 1: Numerical results on models 1-2

Model	IS	x_0	N	N_T	$\lvert\Gamma(x_{i+1})\rvert$	η_{coc}	x_0	N	N_T	$\lvert\Gamma(x_{i+1})\rvert$	η_{coc}
	BS		5	80	3.3e-111	4.0		6	96	1.3e-323	4.0
	KS		4	84	1.5e-180	4.0		4	84	1.5e-136	4.0
	JS		4	60	2.7e-178	4.0		4	60	7.5e-134	4.0
	JF_1		4	64	8.4e-148	4.0		4	64	7.9e-104	4.0
	JF_2		4	84	5.7e-117	4.0		5	105	5.6e-295	4.0
$\Gamma_1(x)$	JF_3	5.0	4	76	1.5e-138	4.0	10.0	5	95	7.1e-307	4.0
	JVF_1		4	92	1.1e-169	4.0		4	92	5.8e-126	4.0
	JVF_2		4	80	9.3e-197	4.0		4	80	9.9e-135	4.0
	BS		5	80	2.4e-101	4.0		5	80	5.1e-101	4.0
	KS		5	105	3.3e-269	4.0		5	105	1.2e-196	4.0
	JS		5	75	5.4e-257	4.0		5	75	1.1e-193	4.0
	JF_1		5	80	2.2e-187	4.0		5	80	2.1e-147	4.0
	JF_2		5	105	1.8e-113	4.0		5	105	2.4e-103	4.0
$\Gamma_2(x)$	JF_3	0.1	5	95	5.8e-139	4.0	.78	4	76	7.6e-141	4.0
	JVF_1		5	115	1.7e-241	4.0		5	115	9.5e-173	4.0
	JVF_2		5	100	2.0e-139	4.0		5	100	2.2e-278	4.0

Table 2: Numerical results on models 3-5

Model	IS	x_0	N	N_T	$\|\Gamma(x_{i+1})\|$	η_{coc}	x_0	N	N_T	$\|\Gamma(x_{i+1})\|$	η_{coc}
	BS		4	64	1.6e-187	4.0		4	64	7.5e-154	4.0
	KS		4	84	7.8e-289	4.0		4	84	1.6e-193	4.0
	JS		4	60	2.0e-286	4.0		4	60	6.0e-193	4.0
	JF_1		4	64	2.1e-251	4.0		4	64	1.5e-179	4.0
	JF_2		4	84	4.3e-219	4.0		4	84	5.8e-162	4.0
$\Gamma_3(x)$	JF_3	-0.7	4	76	2.2e-250	4.0	0.5	4	76	3.0e-222	4.0
	JVF_1		4	92	6.2e-275	4.0		4	92	1.2e-188	4.0
	JVF_2		4	80	2.4e-304	4.0		4	80	5.9e-222	4.0
	BS		4	64	2.6e-192	4.0		5	80	1.9e-131	4.0
	KS		4	84	9.6e-203	4.0		4	84	3.8e-106	4.0
	JS		4	60	1.2e-202	4.0		4	60	2.6e-105	4.0
	JF_1		4	64	2.5e-200	4.0		5	80	1.3e-372	4.0
	JF_2		4	84	4.6e-196	4.0		5	105	2.2e-307	4.0
$\Gamma_4(x)$	JF_3	0.0	4	76	1.2e-209	4.0	0.9	5	95	1.7e-388	4.0
	JVF_1		4	92	4.9e-202	4.0		4	92	7.0e-103	4.0
	JVF_2		4	80	2.4e-202	4.0		4	80	1.5e-178	4.0
	BS		5	80	9.7e-218	4.0		6	96	2.0e-258	4.0
	KS		5	105	4.4e-391	4.0		5	105	1.7e-149	4.0
	JS		5	75	3.7e-387	4.0		5	75	2.6e-147	4.0
	JF_1		5	80	6.2e-304	4.0		6	96	1.2e-415	4.0
	JF_2		5	105	1.3e-227	4.0		6	126	1.2e-259	4.0
$\Gamma_5(x)$	JF_3	1.5	5	95	1.4e-418	4.0	3.0	5	95	1.7e-131	4.0
	JVF_1		5	115	4.2e-351	4.0		5	115	3.7e-124	4.0
	JVF_2		5	100	1.4e-132	4.0		5	100	3.8e-272	4.0

4.2 Numerical experiments with nonlinear multivariate models

This subsection put forward the experiments performed on some specific forms of the JF and JV to solve some nonlinear multivariate models. In all computational experiments, 100 digits floating point arithmetic and $\|\Gamma(X_i)\|_\infty < 10^{-25}$ as halting criteria were utilized in the computational program designed in sympy PYTHON environment. Methods performance was compared based on the number of iterations (N), function residual error $\|\Gamma(X_{i+1})\|$, computational order of convergence η_{coc}, efficiency E, and total approximate computational cost TCC. In this case, the total computational cost TCC of a scheme is estimated as follows:

$$TCC = \Gamma'_C + \Gamma_C + PMM_C + PMV_C + PSM_C + MI_C + ASM_C + ASV_C, \quad (57)$$

where $\Gamma'_C, \Gamma_C, PMM_C, PMV_C, PSM_C, MI_C, ASM_C, ASV_C$ are approximate computational cost of evaluations in the matrix Γ', vector Γ, product of matrix with matrix (PMM), product of matrix with vector (PMV), product of scalar with matrix (PSM), matrix inverse MI, addition or subtraction of matrices ASM, and addition or subtraction of vectors (ASV), respectively. Here, we assumed that the cost of evaluating arithmetic operations of functions in matrices and vectors is equal. The costs associated with these operations are presented in Table 3.

Table 3: Cost of matrices operations

Operations	Cost
Γ'	m^2
Γ	m
PMM	$2m^3 - m^2$
PMV	$2m^2 - m$
PSM	m^2
MI	$\frac{2m^3}{3}$
ASM	m^2
ASV	m

In Table 4, information on the approximate number of computational operations evaluations in one cycle of thevarious schemes are given, while Table 5 provides the respective scheme's total approximate computational cost TCC expressed as a function of the dimension m of the system of equations.

Table 4: Schemes cost of matrices operations evaluations

Methods	Γ'	Γ	PMM	PMV	PSM	MI	ASM	ASV
nBS	2	1	1	2	3	2	2	2
nKS	2	1	4	2	7	2	3	2
nJS	2	1	1	2	3	2	2	2
nJF_1	2	1	3	2	2	2	1	2
nJF_2	2	1	4	2	4	2	4	2
nJF_3	2	1	4	2	4	2	2	2
nJV_1	2	1	4	2	6	2	4	2
nJF_2	2	1	4	2	3	2	3	2

Table 5: Methods order and TCC

Methods	Order	TCC
nBS	4	$\frac{10}{3}m^3 + 11m^2 + m$
nKS	4	$\frac{28}{3}m^3 + 12m^2 + m$
nJS	4	$\frac{10}{3}m^3 + 10m^2 + m$
nJF_1	4	$\frac{22}{3}m^3 + 6m^2 + m$
nJF_2	4	$\frac{28}{3}m^3 + 10m^2 + m$
nJF_3	4	$\frac{28}{3}m^3 + 8m^2 + m$
nJV_1	4	$\frac{28}{3}m^3 + 12m^2 + m$
nJF_2	4	$\frac{28}{3}m^3 + 8m^2 + m$

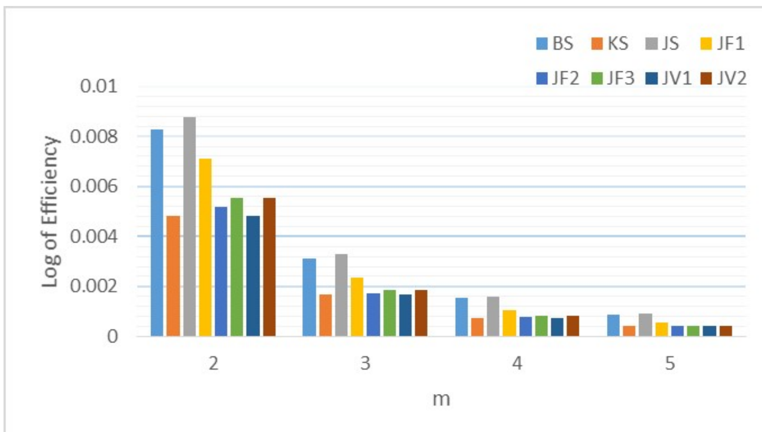

Figure 1: Schemes efficiency dynamics for $m = 2, \ldots, 5$

In line with Definition 1, we estimate the efficiencies of the schemes by $E = \eta^{\frac{1}{TCC}}$. Figures 1 and 2 show the dynamics of the schemes efficiencies as the size m changes. From Figures 1 and 2, we observe that the schemes efficiencies follow the order $JS > BS > JF_1 > JF_3 \geq JV_2 > JF_2 > JV_1 \geq KS$ when $m = 2$ to $m = 5$, and it retains this results for the case $m = 10$ to $m = 30$. The numerical results in Tables 6 and 7 verify these results. Although the compared schemes BS and KS have good efficiencies, they failed to converge to solutions for most problems tested. This is an indication that they have low stability compared to the developed schemes.

Model Application 6 (Chemical equilibrium [7]). The following set of nonlinear equations $\Gamma_6(X) = \mathbf{0}$ describes the chemical equilibrium system that involves the combination of carbon oxide (x_1), oxygen (x_2), Hydrogen

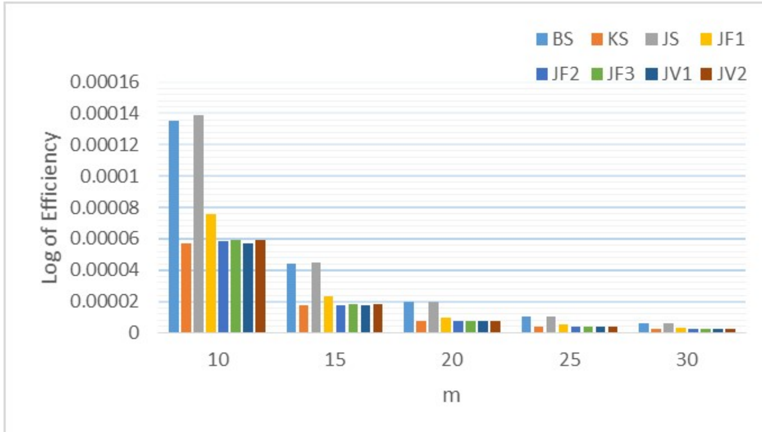

Figure 2: Schemes efficiency dynamics for $m = 10, \ldots, 30$

(x_3), Nitrogen (x_4), and x_5 a factor depicting the moles number of a product formed for each mole of consumed propane.

$$
\begin{aligned}
&x_1 + x_1x_2 - 3x_5 = 0, \\
&x_1 + 2x_1x_2 + x_2x_3^2 + \rho_8x_2 - \rho x_5 + \rho_{10}x_2^2 + \rho_7x_1x_3 + \rho_9x_2x_4 = 0, \\
&2x_2x_3^2 + 2\rho_5x_3^2 - 8x_5 + \rho_6x_3 + \rho_7x_2x_3 = 0, \\
&\rho_9x_2x_4 + 2x_4^2 - 4\rho_5 = 0, \\
&x_1(x_2+1) + \rho_{10}x_2^2 + x_2x_3^2 + \rho_8x_2 + \rho_5x_3^2 \\
&\qquad + x_4^2 - 1 + \rho_6x_3 + \rho_7x_2x_3 + \rho_9x_2x_4 = 0,
\end{aligned}
\tag{58}
$$

where $\rho = 10, \rho_{10} = 0.193, \rho_6 = \frac{0.002597}{\sqrt{40}}, \rho_7 = \frac{0.003448}{\sqrt{40}}, \rho_8 = \frac{0.00001799}{40}, \rho_9 = \frac{0.0002155}{\sqrt{40}}, \rho_{10} = \frac{0.00003846}{40}$. The model exact solution X_* in the domain $\Delta = (-1,1) \times (33.5, 35.5) \times (-1,1) \times (-0.8, 1.8) \times (-1,1)$ approximated to 20 decimal places using the initial start points $X_0^{(1)} = (0.6, 33.6, 0.6, 1.5, -0.7)^T$ and $X_0^{(2)} = (-0.2, 33.0, 0.9, 1.0, -0.3)^T$ is $X_* = (0.0031\ldots, 34.5979\ldots, 0.0650\ldots, 0.8593\ldots, 0.0369\ldots)^T$.

Model Application 7 (Combustion problem [9]). The modeled combustion problem at a temperature of 3000^0C in system of nonlinear equations $\Gamma_7(X) = \mathbf{0}$ was given as follows:

$$-(1.0e{-}15) + x_2 + 2x_6 + x_9 + 2x_{10} = 0, \quad -(3.0e{-}5) + x_3 + x_8 = 0,$$
$$-(5.0e{-}5) + x_1 + x_3 + 2x_5 + 2x_8 + x_9 + x_{10} = 0,$$
$$-(5.0e{-}5) + x_4 + 2x_7 = 0, \qquad -x_1^2 + (0.5140437e{-}7)x_5 = 0,$$
$$-2x_2^2 + (0.100006932e{-}6)x_6 = 0, \qquad -x_4^2 + (0.7816278e{-}15)x_7 = 0,$$
$$-x_1x_3 + (0.1496236e{-}6)x_8 = 0, \qquad -x_1x_2 + (0.6194411e{-}7)x_9 = 0,$$
$$-x_1x_2^2 + (0.2089296e{-}14)x_{10} = 0.$$

The solution to the model is

$X_* = (1.47e{-}7, 2.26e{-}7, 1.51e{-}5, 6.27e{-}11,$ $4.20e{-}7, 1.01e{-}6, 4.99e{-}6, 1.48e{-}5, 5.37e{-}7, 3.60e{-}6)^T$, when $X_0^{(1)} = (0.1,$ $0.4, 0.2, 0.3, 0.1, 0.6, 0.7, 0.5, 0.1, 0.4)^T$ and $X_0^{(2)} = (0.1, 0.1, 0.1, 0.1, 0.1, 0.1,$ $0.1, 0.1, 0.1, 0.1)^T$ were used as iteration starting points.

Model Application 8 (Economics model [3]). Consider the modeling in economics that can be scaled up to n-dimensions and reads as follows:

$$\sum_{i=1}^{n-1} x_i = -1, \quad \left[x_j + \sum_{i=1}^{n-j-1} x_i x_{i+j}\right] x_n = -1, \quad 1 \le j \le n-1. \tag{59}$$

Here, we set $n = 4$ to generate a system of nonlinear equations $\Gamma_8(X) = \mathbf{0}$ and used $X_0^{(1)} = \cos\left(\frac{1}{j}\right)$ and $X_0^{(2)} = \left(\frac{1}{j+4}\right)$ as starting points in two different applications. The model solutions is
$X_* = (1.4464\ldots, 1.9730\ldots, -4.4194\ldots, 0.2262\ldots)^T$.

Application 9 (Boundary value problem (BVP) [13]). Let the BVP

$$\Gamma'' + \theta^2 (\Gamma)^2 + 1 = 0, \quad \Gamma(0) = 0, \quad \Gamma(1) = 0. \tag{60}$$

The interval $[0, 1]$ is partitioned such that $x_0 = 0 < x_1 < x_2 < \cdots < x_{m-1} < x_m$, $x_{i+1} = x_i + h$, $h = \frac{1}{m}$.

Define $\Gamma_0 = \Gamma(x_0) = 0$, $\Gamma_1 = \Gamma(x_1) \cdots \Gamma_{m-1} = \Gamma(x_{m-1})$,$\Gamma_m = \Gamma(x_m) = 1$. By discretizing the problem using finite difference such that

$$\Gamma' = \frac{\Gamma_{i+1} - \Gamma_{i-1}}{2h}, \qquad \Gamma'' = \frac{\Gamma_{i+1} - 2\Gamma_i + \Gamma_{i-1}}{h^2}, \tag{61}$$

the problem becomes

$$\Gamma_{i+1} - 2\Gamma_i + \Gamma_{i-1} + \frac{\theta^2}{4}(\Gamma_{i+1} - \Gamma_{i-1}) + h^2 = 0, \quad i = 1, 2, 3, \ldots, m-1. \tag{62}$$

For $m = 10$ and $\theta = 2$, the problem was solved by using $X_0 = \{0, 0, \ldots, 0\}^T$ as initial start point and applying $\|\Gamma(X_i)\| < 10^{-5}$ as stopping condition. The problem solution is

$X_* = (0.605\ldots, 0.101\ldots, 0.128\ldots, 0.144\ldots, 0.149\ldots, 0.144\ldots, 0.128\ldots, 0.101\ldots, 0.605\ldots)^T$.

The computation results obtains on Applications 6 to 9 are presented in Tables 6 and 7.

Table 6: Numerical results on Applications 6 -7

Model	IS	X_0	N	$\|\Gamma(x_{i+1})\|$	η_{coc}	TCC	E
	nBS			Failed			
	nKS			Failed			
	nJS		3	7.4e-07	4.0	2015	1.00069
	nJF_1		4	1.1e-22	4.0	4287	1.00032
	nJF_2		4	2.6e-18	4.0	5687	1.00024
$\Gamma_6(X)$	nJF_3	$X_0^{(1)}$	4	1.0e-24	4.0	5487	1.00025
	$nJVF_1$		3	1.4e-09	4.0	4415	1.00031
	$nJVF_2$		3	9.7e-08	4.0	4115	1.00031
	nBS			Failed			
	nKS			Failed			
	nJS		4	1.7e-17	4.0	2687	1.00052
	nJF_1		4	4.2e-19	4.0	4287	1.00032
	nJF_2		4	2.0e-15	4.0	5687	1.00024
$\Gamma_6(X)$	nJF_3	$X_0^{(2)}$	4	3.8e-17	4.0	5487	1.00025
	$nJVF_1$		3	3.8e-07	4.0	4415	1.00031
	$nJVF_2$		4	4.3e-21	4.0	5487	1.00025
	nBS			Failed			
	nKS			Failed			
	nJS		31	1.8e-25	4.0	50850	1.00002726
	nJF_1		32	1.0e-25	4.0	90123	1.00001538
	nJF_2		33	3.4e-25	4.0	122045	1.00001135
$\Gamma_7(X)$	nJF_3	$X_0^{(1)}$	32	5.0e-25	4.0	115211	1.00001203
	$nJVF_1$		29	2.0e-25	4.0	110094	1.00001259
	$nJVF_2$		31	1.6e-25	4.0	111610	1.00001242

4.3 Results discussion

From the results in Tables 1, 2, 6, and 7, the new schemes put forward solved all the presented models with good precision and consistent CO. The η_{coc} obtained for all the developed schemes are in agreement with the theoretical order of convergence derived in Sections 2 and 3. On the other hand, the methods of Behl, Kanwar, and Sharma [1] (nBS) and Kanwar, Kumar, and

Table 7: Numerical results on Applications 7 -9

Model	IS	X_0	N	$\|\Gamma(x_{i+1})\|$	η_{coc}	TCC	E
	nBS			Failed			
	nKS			Failed			
	nJS		29	2.2e-25	4.0	47570	1.00002914
	nJF_1		31	2.2e-25	4.0	87306	1.00001588
	nJF_2		33	1.1e-25	4.0	122045	1.00001136
$\Gamma_7(X)$	nJF_3	$X_0^{(2)}$	31	4.6e-25	4.0	111610	1.00001242
	$nJVF_1$		26	4.6e-26	4.0	98705	1.00001404
	$nJVF_2$		29	1.4e-29	4.0	104410	1.00001328
	nBS			Failed			
	nKS			Failed			
	nJS		9	8.8e-10	4.0	3396	1.00041
	nJF_1		6	6.2e-21	4.0	3416	1.00040
	nJF_2		6	4.5e-21	4.0	4568	1.00030
$\Gamma_8(X)$	nJF_3	$X_0^{(1)}$	10	1.2e-22	4.0	7293	1.00019
	$nJVF_1$		6	2.0e-11	4.0	4760	1.00029
	$nJVF_2$		9	6.1e-07	4.0	6564	1.00021
	nBS			Failed			
	nKS			Failed			
	nJS		16	2.3e-18	4.0	6037	1.00023
	nJF_1		5	1.6e-07	4.0	2847	1.00049
	nJF_2		16	2.0e-17	4.0	12181	1.00011
$\Gamma_8(X)$	nJF_3	$X_0^{(0)}$	26	3.1e-09	4.0	18963	1.00007
	$nJVF_1$		17	1.0e-10	4.0	13487	1.00010
	$nJVF_2$		9	3.9e-07	4.0	6564	1.00021
	nBS			Failed			
	nKS		6	1.3e-05	4.0	45252	1.00003064
	nJS		6	1.3e-05	4.0	18036	1.00007687
	nJF_1		6	1.3e-05	4.0	33588	1.00004127
	nJF_2		6	1.3e-05	4.0	44289	1.00003130
$\Gamma_9(X)$	nJF_3	$X_0^{(0)}$	6	1.3e-05	4.0	43308	1.00003201
	$nJVF_1$		6	1.3e-05	4.0	45252	1.00003064
	$nJVF_2$		6	1.3e-05	4.0	43308	1.00003201

Behl [5] (nKS) failed to converge in most of the problems used for computational experiments. This is an indication that the developed families of iterative schemes are more stable. Furthermore, results show that the classical JS performed better than all its variants in the two developed families in terms of convergence and efficiency. This coincides with the findings of Cordero, Segura, and Terregros [2], that among the members of JS families, the JS is the best.

Conclusion

We have successfully developed two new families of JS and its variants to decide the solution of scalar nonlinear and system of nonlinear equations. The parameters contained in the structures of the developed families can assume any real values subject to some simple conditions to obtain infinitely many special forms that have CO four when utilized to solve problems in both scalar and vector space. In fact, the famous Jarrat IS (JS) is a special form of one of the families. The computational experiments conducted on the new schemes show that they are good competitors to some existing schemes that are also JS variants.

For further work, the complex dynamical and chaotic behavior of the developed schemes can be considered.

Acknowledgment

The authors appreciate the observations, corrections, and suggestions by the Editors and anonymous reviewers. Their contributions led to the improvement of the initial state of this manuscript.

References

[1] Behl, R., Kanwar, V. and Sharma, K.K. *Optimal equi-scaled familie of Jarratt's method for solving equations*, Int. J. Comput. Math. 90 (2013) 408–422.

[2] Cordero, A., Segura, E. and Terregrosa, J.R. *Behind Jarratt's Steps: Is Jarratt's scheme the Best version of itself?*, Discrete Dynamics in Nature and Society, 2013.

[3] Grosan, C. and Abraham, A. *A new approach for solving nonlinear equations systems*, IEEE Trans. Syst. Man Cybern. A Syst. Hum. 38 (2008) 698–714.

[4] Jarratt, P. *Some fourth-order multipoint iterative methods for solving equations*, Math. Comput. 20 (1966) 434–437.

[5] Kanwar, V., Kumar, S. and Behl, R. *Several new families of Jarratt's method for for solving systems of nonlinear equations*, Appl. Appl. Math. 8 (2013) 701–716.

[6] Lin, Y., Bao, L. and Jia, X. *Convergence analysis of a variant of the Newton method for solving nonlinear equations*, Comput. Math. Appl. 59 (2010) 2121–2127.

[7] Meintjes, K. and Morgan, A.P. *Chemical equilibrium systems as numerical test problems*, ACM Trans. Math. Softw. 16 (1990) 143–151.

[8] More, J.J. *A collection of nonlinear model problems*, Computational solution of nonlinear systems of equations (Fort Collins, CO, 1988), 723–762, Lectures in Appl. Math., 26, Amer. Math. Soc., Providence, RI, 1990.

[9] Morgan, A.P. *Solving polynomial systems using continuation for scientific and engineering problems*, Englewood Cliffs, Prentice-Hall, 1987.

[10] Ogbereyivwe, O. and Izevbizua, O. *A three-parameter class of power series based iterative method for approximation of nonlinear equations*, Iran. J. Numer. Anal. Optim. 13(2) (2023) 157–169.

[11] Ogbereyivwe, O. and Ojo-Orobosa, V. *Family of optimal two-step fourth order iterative method and its extension for solving nonlinear equations*, J. Interdiscip. Math. 24(5) (2022) 1347–1365.

[12] Ogbereyivwe, O., Umar, S.S. and Izevbizua, O. *Some high-order convergence modifications of the Housedholder method for nonlinear equations*, Commun. Nonlinear Anal. 5 (2023) 1–16.

[13] Ortega, J.M. and Rheinboldt, W.C. *Iterative solution of nonlinear equations in several variables*, Academic Press, New York, 1970.

[14] Petkovic, M.S. *Remarks on "On a general class of multipoint root-finding methods of high computational efficiency"*, SIAM J. Numer. Anal. 49(3) (2011) 1317–1319.

[15] Shacham, M. and Kehat, E. *An iteration method with memory for the solution of a non-linear equation*, Chem. Eng. Sci. 27(11) (1972) 2099–2101.

[16] Sharma, R., Kumar, S. and Singh, H. *A new class of derivative-free root solvers with increasing optimal convergence order and their complex dynamics*, SeMA J. 80 (2) (2023) 333–352.

[17] Sivakumar, P. and Jayaraman, J. *Some new higher order weighted Newton methods for solving nonlinear equations with applications*, Comput. Math. Appl. 24(2) (2023) 59.

[18] Traub. J.F. *Iterative methods for the solution of equations*, Prentice-Hall, New Jersey, 1964.

[19] Yaseen, S., Zafar, F. and Alsulami, H.A. *An efficient Jarratt-type iterative method for solving nonlinear Global Positioning System problem*, Axioms, 12(6) (2023) 562.