Iranian Journal of Numerical Analysis and Optimization
Vol. 12, No. 3 (Special Issue), 2022, pp 704–718
DOI:10.22067/ijnao.2022.77745.1163
https://ijnao.um.ac.ir/

How to cite this article
Research Article

A new iteration method for solving space fractional coupled nonlinear Schrödinger equations

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Abstract

A linearly implicit difference scheme for the space fractional coupled nonlinear Schrödinger equation is proposed. The resulting coefficient matrix of the discretized linear system consists of the sum of a complex scaled identity and a symmetric positive definite, diagonal-plus-Toeplitz, matrix. An efficient block Gauss–Seidel over-relaxation (BGSOR) method has been established to solve the discretized linear system. It is worth noting that the proposed method solves the linear equations without the need for any system solution, which is beneficial for reducing computational cost and memory requirements. Theoretical analysis implies that the BGSOR method is convergent under a suitable condition. Moreover, an appropriate approach to compute the optimal parameter in the BGSOR method is exploited. Finally, the theoretical analysis is validated by some numerical experiments.

AMS subject classifications (2020): 65F10, 81Q05, 81V99.

Keywords: The space fractional Schrödinger equations, Toeplitz matrix, Block Gauss-Seidel over-relaxation method, Convergence analysis.

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Received 17 July 2022; revised 24 September 2022; accepted 25 September 2022

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1 Introduction

The Schrödinger equation is a crucial equation in quantum mechanics, a science that studies submicroscopic phenomena. It can arise from the Brownian path integral. In [6], the path integral method to the Lévy-α process was generalized, and the space fractional equations were derived.

Consider the space fractional coupled nonlinear Schrödinger (CNLS) equations

$$\begin{cases} \imath \mathrm{u}_t + \xi(-\Delta)^{\frac{\alpha}{2}}\mathrm{u} + \eta\left(|\mathrm{u}|^2 + \theta|\mathrm{v}|^2\right)\mathrm{u} = 0, \\ \imath \mathrm{v}_t + \xi(-\Delta)^{\frac{\alpha}{2}}\mathrm{v} + \eta\left(|\mathrm{v}|^2 + \theta|\mathrm{u}|^2\right)\mathrm{v} = 0, \end{cases} \qquad a_1 \le x \le a_2, \quad 0 < t < T. \tag{1}$$

Given the conditions of the initial boundary value as follows:

$$\begin{aligned} \mathrm{u}(x,0) =& \mathrm{u}_0(x), \qquad \mathrm{v}(x,0) = \mathrm{v}_0(x), \qquad a_1 \le x \le a_2, \\ \mathrm{v}(a_1,t) =& \mathrm{u}(a_2,t) = 0, \quad \mathrm{v}(a_1,t) = \mathrm{v}(a_2,t) = 0, \qquad 0 \le t \le T, \end{aligned}$$

where $\imath$ is the imaginary unit, $\xi > 0$, $\eta > 0$, $\theta \ge 0$ are some constants, and $1 < \alpha < 2$. In [5], the fractional Laplacian was designated as

$$(-\Delta)^{\frac{\alpha}{2}}\mathrm{u}(x,t) = \mathscr{H}^{-1}\left(|\phi|^{\alpha}\mathscr{H}(\mathrm{u}(x,t))\right),$$

in which $\mathscr{H}$ stands for the Fourier transform applied to the spatial variable x. Assuming that ${}_{-\infty}D_x^{\alpha}\mathrm{u}(x,t)$ and ${}_xD_{+\infty}^{\alpha}\mathrm{u}(x,t)$ are the left and right Riemann–Liouville fractional derivatives of order $\alpha \in \mathbb{R}^+$ given by

$${}_{-\infty}D_x^{\alpha}\mathrm{u}(x,t) = \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_{-\infty}^{x}(x-\mu)^{n-1-\alpha}\mathrm{u}(\mu,t)d\mu,$$

$${}_xD_{+\infty}^{\alpha}\mathrm{u}(x,t) = \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_{x}^{+\infty}(\mu-x)^{n-1-\alpha}\mathrm{u}(\mu,t)d\mu,$$

respectively, the Riesz fractional derivative can be calculated as

$$\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}\mathrm{u}(x,t) = -(-\Delta)^{\frac{\alpha}{2}}\mathrm{u}(x,t) = -\frac{1}{2\cos\frac{\pi\alpha}{2}}\left[{}_{-\infty}D_x^{\alpha}\mathrm{u}(x,t) + {}_xD_{+\infty}^{\alpha}\mathrm{u}(x,t)\right].$$

In general, analyzing and understanding the behavior of the exact solutions of the space fractional CNLS equations is so challenging. In recent years, some numerical methods have been proposed to solve the CNLS equations. The difference method [12, 13, 11], the Crank–Nickelson scheme [1], and the collocation method [2] have been presented to solve the CNLS equations.

The discretization of the CNLS equations leads to the solution of the complex symmetric linear systems. The coefficient matrix consists of the sum of the symmetric positive definite, diagonal-plus-Toeplitz, matrix and the complex identity scaled matrix. Recently, Dai and Wu [4] developed a suited 2×2

linear system and employed the block Gauss–Seidel (BGS) iteration scheme to solve the resulting linear systems. Then they analyzed the convergence of the BGS scheme for the corresponding 2×2 linear system. In this work, we establish a fast block Gauss–Seidel over-relaxation (BGSOR) scheme for solving the two-by-two linear system that arises from the discretization of CNLS equations. Notably, the new method allows the corresponding systems to be solved without the need to compute the inverse of the coefficient matrices. Moreover, it should be pointed out that the BGS method can be regarded as a special case of the new method when the relaxation parameter is set to be one.

The arrangement of this work is as follows. In Section 2, the model problem will be studied, and a linearly implicit difference technique will be presented. Application, convergence theory, and finding the optimal parameter for the BGSOR method are proposed in Section 3. Section 4 is devoted to giving some numerical examinations. In Section 5, we finally made some conclusions.

2 Model problem and a linearly implicit difference scheme

The domain $\Omega = (a_1, a_2)\times(0, T)$ is divided into a uniform grid of mesh points (x_j, t_k), where

$$x_j = a_1 + jh, \quad h = \frac{a_2 - a_1}{m+1}, \quad 0 \le j \le m+1,$$

and

$$t_k = k\tau, \quad \tau = \frac{T}{n}, \quad 0 \le k \le n.$$

At grid points, the values of functions $\mathrm{u}(x,t), \mathrm{v}(x,t)$ are, respectively, denoted by $\mathrm{u}_j^k = \mathrm{u}(x_j, t_k)$, $\mathrm{v}_j^k = \mathrm{v}(x_j, t_k)$, and $\mathscr{U}_j^k, \mathscr{V}_j^k$ are the approximate solutions of (1).

The following equation gives a discrete approximation to $\frac{\partial^\alpha}{\partial|x|^\alpha}\mathrm{u}(x,t)$ [10]:

$$\frac{\partial^\alpha}{\partial|x|^\alpha}\mathrm{u}(x_j,t_k) = -\frac{\Psi_\alpha}{h^\alpha}\left[\sum_{l=0}^{\infty}\tilde{w}_k^{(\alpha)}\mathrm{u}(x_{j-l+1},t_k)+\sum_{l=0}^{\infty}\tilde{w}_k^{(\alpha)}\mathrm{u}(x_{j+l-1},t_k)\right]+\mathcal{O}(h^2), \tag{2}$$

where $\Psi_\alpha = \frac{1}{2\cos(\frac{\pi\alpha}{2})}$ and $\{\tilde{w}_k^\alpha\}$ is defined as follows:

$$\tilde{w}_0^{(\alpha)} = \frac{\alpha}{2}g_0^{(\alpha)}, \qquad \tilde{w}_l^{(\alpha)} = \frac{\alpha}{2}g_l^{(\alpha)} + \left(1-\frac{\alpha}{2}\right)g_{l-1}^{(\alpha)}, \qquad l \ge 1,$$

$$g_0^{(\alpha)} = 1, \quad g_l^{(\alpha)} = \left(1-\frac{\alpha+1}{l}\right)g_{l-1}^{(\alpha)}, \qquad l = 1, 2, \ldots.$$

Ortigueira [7] proposed the following fractional central difference operator:

$$\Delta_h^\alpha \mathrm{u}(x) = \sum_{l=-\infty}^{\infty} \hat{g}_l^{(\alpha)} \mathrm{u}(x - lh),$$

where

$$\hat{g}_l^{(\alpha)} = \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2} - l + 1)\Gamma(\frac{\alpha}{2} + l + 1)}.$$

As stated in [7], the coefficient $\{\hat{g}_l^{(\alpha)}\}$ satisfies

$$\left| 2\sin\left(\frac{x}{2}\right) \right|^2 = \sum_{l=-\infty}^{\infty} \hat{g}_l^{(\alpha)} e^{\imath l x}, \qquad x \in \mathbb{R}.$$

When $\alpha > -1$, the recursive relations for $\{\hat{g}_l^{(\alpha)}\}$ are as follows:

$$\hat{g}_0^{(\alpha)} = \frac{\Gamma(\alpha+1)}{\Gamma^2(\alpha/2+1)}, \qquad \hat{g}_l^{(\alpha)} = \left(1 - \frac{\alpha+1}{\alpha/2+l}\right)\hat{g}_{l-1}^{(\alpha)}, \qquad l \geq 1;$$
$$\hat{g}_{-l}^{(\alpha)} = \hat{g}_l^{(\alpha)}, \qquad l \geq 1.$$

Lemma 1. [10] Assume that $u(x) \in \mathbb{C}^5(\mathbb{R})$ and that its all derivatives of order up to 5 belong to $L^1(\mathbb{R})$. Then, it holds

$$-\frac{\Delta_h^\alpha \mathrm{u}(x)}{h^\alpha} = \frac{\partial^\alpha \mathrm{u}(x)}{\partial |x|^\alpha} + \mathcal{O}(h^2). \tag{3}$$

From Lemma 1, it follows that

$$(-\Delta)^{\frac{\alpha}{2}} \mathrm{u}(x_j, t_k) = \frac{\Delta_h^\alpha \mathrm{u}(x)}{h^\alpha} + \mathcal{O}(h^2) = \frac{1}{h^\alpha} \sum_{l=1}^{M} \hat{g}_{j-l}^{(\alpha)}(x_j, t_k) + \mathcal{O}(h^2).$$

Now, we consider the following numerical scheme for solving (1) [12]:

$$\begin{aligned}
&\imath \frac{\mathscr{U}_j^{k+1} - \mathscr{U}_j^{k-1}}{2\tau} + \frac{\gamma}{h^\alpha} \sum_{l=1}^{m} \hat{g}_{j-l}^{(\alpha)} \left(\frac{\mathscr{U}_l^{k+1} + \mathscr{U}_l^{k-1}}{2} \right) + \rho\big(|\mathscr{U}_j^k|^2 + \beta|\mathscr{V}_j^k|^2\big) \\
&\qquad + \frac{\mathscr{U}_l^{k+1} + \mathscr{U}_l^{k-1}}{2} = 0, \\
&\imath \frac{\mathscr{V}_j^{k+1} - \mathscr{V}_j^{k-1}}{2\tau} + \frac{\gamma}{h^\alpha} \sum_{l=1}^{m} \hat{g}_{j-l}^{(\alpha)} \left(\frac{\mathscr{V}_l^{k+1} + \mathscr{V}_l^{k-1}}{2} \right) + \rho\big(|\mathscr{V}_j^k|^2 + \beta|\mathscr{U}_j^k|^2\big) \\
&\qquad + \frac{\mathscr{V}_l^{k+1} + \mathscr{V}_l^{k-1}}{2} = 0,
\end{aligned} \tag{4}$$

where $1 \le j \le m,\ 1 \le k \le n-1$. Another scheme should be provided for the numerical solution at $k = 1$. We consider the following scheme for this purpose (see [3]):

$$
\begin{aligned}
&\imath\frac{\mathscr{U}_j^1 - \mathscr{U}_j^0}{\tau} + \frac{\gamma}{h^\alpha}\sum_{l=1}^{m}\hat{g}_{j-l}^{(\alpha)}\mathscr{U}_l^{(1)} + \rho\big(|\mathscr{U}_j^0|^2 + \beta|\mathscr{V}_j^0|^2\big)\mathscr{U}_j^1 = 0,\\
&\imath\frac{\mathscr{V}_j^1 - \mathscr{V}_j^0}{\tau} + \frac{\gamma}{h^\alpha}\sum_{l=1}^{m}\hat{g}_{j-l}^{(\alpha)}\mathscr{V}_l^1 + \rho\big(|\mathscr{V}_j^0|^2 + \beta|\mathscr{U}_j^0|^2\big)\mathscr{V}_j^{(1)} = 0,\\
&\imath\frac{\mathscr{U}_j^1 - \mathscr{U}_j^0}{\tau} + \frac{\gamma}{h^\alpha}\sum_{l=1}^{m}\hat{g}_{j-l}^{(\alpha)}\left(\frac{\mathscr{U}_l^1 + \mathscr{U}_l^0}{2}\right)\\
&+ \rho\left(\frac{3}{2}|\mathscr{U}_j^1|^2 - \frac{1}{2}|\mathscr{U}_j^0|^2 + \beta\left(\frac{3}{2}|\mathscr{V}_j^{(1)}|^2 - \frac{1}{2}|\mathscr{V}_j^0|^2\right)\right)\frac{\mathscr{U}_j^1 + \mathscr{U}_j^0}{2} = 0,\\
&\imath\frac{\mathscr{V}_j^1 - \mathscr{V}_j^0}{\tau} + \frac{\gamma}{h^\alpha}\sum_{l=1}^{m}\hat{g}_{j-l}^{(\alpha)}\left(\frac{\mathscr{V}_l^1 + \mathscr{V}_l^0}{2}\right)\\
&+ \rho\left(\frac{3}{2}|\mathscr{V}_j^1|^2 - \frac{1}{2}|\mathscr{V}_j^0|^2 + \beta\left(\frac{3}{2}|\mathscr{U}_j^1|^2 - \frac{1}{2}|\mathscr{U}_j^0|^2\right)\right)\frac{\mathscr{V}_j^1 + \mathscr{V}_j^0}{2} = 0.
\end{aligned}
$$

The structure of the first and second difference equations in (4) is the same. Set

$$
\begin{aligned}
\mathscr{U}^{k+1} =& [\mathscr{U}_1^{k+1}, \dots, \mathscr{U}_m^{k+1}]^T, \quad b^{k+1} = [b_1^{k+1}, \dots, b_m^{k+1}]^T,\\
\mu =& \frac{\xi\tau}{h^\alpha}, \quad d_j^{k+1} = \eta\tau\big(|\mathscr{U}_j^k|^2 + \beta|\mathscr{V}_j^k|^2\big), \quad D^{k+1} = \operatorname{diag}(d_1^{k+1}, \dots, d_m^{k+1}).
\end{aligned}
$$

So, at each time step, we need to solve the following systems of linear equations:

$$
\begin{aligned}
A^{k+1}\mathscr{U}^{k+1} &= b^{k+1}, \qquad 1 \le k \le n-1,\\
B^{k+1}\mathscr{V}^{k+1} &= c^{k+1}, \qquad 1 \le k \le n-1,
\end{aligned}
\tag{5}
$$

where $A^{k+1} = T + D^{k+1} + \imath I$ and b^{k+1} is as follows:

$$
b^{k+1} = \begin{pmatrix}
\imath\mathscr{U}_1^{k-1} - \mu\sum_{l=1}^{m}\hat{g}_{1-l}^{(\alpha)}\mathscr{U}_l^{k-1} - d_1^{k+1}\mathscr{U}_1^{k-1}\\
\imath\mathscr{U}_2^{k-1} - \mu\sum_{l=1}^{m}\hat{g}_{2-l}^{(\alpha)}\mathscr{U}_l^{k-1} - d_2^{k+1}\mathscr{U}_2^{k-1}\\
\vdots\\
\imath\mathscr{U}_{m-1}^{k-1} - \mu\sum_{l=1}^{m}\hat{g}_{m-1-l}^{(\alpha)}\mathscr{U}_l^{k-1} - d_{m-1}^{k+1}\mathscr{U}_{m-1}^{k-1}\\
\imath\mathscr{U}_m^{k-1} - \mu\sum_{l=1}^{m}\hat{g}_{m-l}^{(\alpha)}\mathscr{U}_l^{k-1} - d_m^{k+1}\mathscr{U}_m^{k-1}
\end{pmatrix}.
$$

Moreover, T is the Toeplitz matrix, which has the following structure:

$$T=\mu\begin{pmatrix}\hat{g}_0^{(\alpha)} & \hat{g}_{-1}^{(\alpha)} & \cdots & \hat{g}_{2-m}^{(\alpha)} & \hat{g}_{1-m}^{(\alpha)}\\ \hat{g}_1^{(\alpha)} & \hat{g}_0^{(\alpha)} & \cdots & \hat{g}_{3-m}^{(\alpha)} & \hat{g}_{2-m}^{(\alpha)}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \hat{g}_{m-2}^{(\alpha)} & \hat{g}_{m-3}^{(\alpha)} & \cdots & \hat{g}_0^{(\alpha)} & \hat{g}_{-1}^{(\alpha)}\\ \hat{g}_{m-1}^{(\alpha)} & \hat{g}_{m-2}^{(\alpha)} & \cdots & \hat{g}_1^{(\alpha)} & \hat{g}_0^{(\alpha)}\end{pmatrix}. \tag{6}$$

Also, it should be noted that B^{k+1} and c^{k+1} can be obtained by changing the roles of $\mathscr{U}$ and $\mathscr{V}$ in A^{k+1} and b^{k+1}.

3 The BGSOR iteration method

To establish the BGSOR iteration method, we need to give some preliminaries. Let us first consider the iterative solution of the linear equation

$$A\mathscr{U}=b, \tag{7}$$

in which $A\in\mathbb{C}^{\ell\times\ell}$ is a nonsingular complex symmetric matrix as follows:

$$A=T+D+\imath I,$$

where T is the symmetric positive definite (SPD) and Toeplitz matrix designated in (6), $D=\operatorname{diag}(d_1,d_2,\ldots,d_\ell)$ with $d_i\geq 0$, $i=1,2,\ldots,\ell$, is the diagonal matrix, $U,b\in\mathbb{C}^{\ell}$. Let $U=x+\imath y$ and $b=f+\imath g$ be complex vectors, where $y,z,p,q\in\mathbb{R}^{\ell}$. So, the system can be rewritten as a particular form, namely,

$$\mathscr{A}\mathbf{x}\equiv\begin{pmatrix}-I & Q\\ Q & I\end{pmatrix}\begin{pmatrix}y\\ x\end{pmatrix}=\begin{pmatrix}f\\ g\end{pmatrix}\equiv\mathscr{P}, \tag{8}$$

where $Q=D+T$. We are now in a position to design a new method for solving (8).

To introduce the BGSOR iteration method, we consider the following decomposition for the coefficient matrix (8):

$$\mathscr{A}=(\omega\mathscr{D}-\mathscr{E})-(\mathscr{E}^T-(1-\omega)\mathscr{D})=:\mathscr{M}-\mathscr{N}, \tag{9}$$

where

$$\mathscr{D}=\begin{pmatrix}-I & 0\\ 0 & I\end{pmatrix},\qquad \mathscr{E}=\begin{pmatrix}0 & 0\\ -Q & 0\end{pmatrix},$$

and ω is a positive parameter, which is known as the relaxation parameter. Using the decomposition (9), the BGSOR iteration method is stated as

$$\mathscr{M}\mathbf{z}^{(k+1)}=\mathscr{N}\mathbf{z}^{(k)}+\mathscr{P},\quad k=0,1,2,\ldots,$$

where $\mathscr{M}$ and $\mathscr{N}$ are defined as (9), and $\mathbf{z}^{(k)} = (\mathbf{y}^{(k)}; \mathbf{x}^{(k)})$. Note that $\mathbf{y}^{(k)}$ and $\mathbf{x}^{(k)}$ are two M-vectors that stand for the iterations. Also, $\mathbf{z}^{(0)}$ is an arbitrary initial guess. Thereupon, the iterations take the following procedure:

$$\begin{cases} \mathbf{y}^{(k+1)} = \frac{1}{\omega}\left((\omega-1)\mathbf{y}^{(k)} + Q\mathbf{x}^{(k)} - f\right), \\ \mathbf{x}^{(k+1)} = \frac{1}{\omega}\left((\omega-1)\mathbf{x}^{(k)} + g - Q\mathbf{y}^{(k+1)}\right). \end{cases} \tag{10}$$

As can be seen, there is not any system solution in each iteration, and only two matrix-vector multiplication are needed. This can be very important because the new scheme requires insignificant computational efforts and just contains the matrix-vector multiplications. Furthermore, if $\omega = 1$, then the iteration scheme (10) reduces to

$$\begin{cases} \mathbf{y}^{(k+1)} = Q\mathbf{x}^{(k)} - f, \\ \mathbf{x}^{(k+1)} = g - Q\mathbf{y}^{(k+1)}, \end{cases} \tag{11}$$

which is presented in [4] and known as the BGS iteration method. Therefore, the BGS iteration method is a special case of the BGSOR iteration method.

Next, we investigate the convergence of the BGSOR method for solving (8), and then we obtain the optimal value of the relaxation parameter ω. In the following, we recall a result that will be useful later.

Lemma 2. [14] Suppose that the quadratic equation $x^2 - px + q = 0$, where p and q are real numbers. Both roots of the equation are less than one in modulus if and only if $|q| < 1$ and $|p| < 1 + q$.

Theorem 1. Consider $A = D + T + \imath I \in \mathbb{R}^{\ell\times\ell}$ as a matrix, where D and T are diagonal and Toeplitz SPD matrices, respectively. The necessary and sufficient condition for convergence of the BGSOR iteration method to the solution of (8) for any initial guess, is

$$\omega > \frac{1 + \mu_{\max}(Q)}{2},$$

where $\mu_{\max}(Q)$ is the largest eigenvalue of Q.

Proof. Let λ be an eigenvalue of the iteration matrix $\mathscr{G} = \mathscr{M}^{-1}\mathscr{N}$, and let $\mathbf{x} = [\mathbf{u}; \mathbf{v}]$ be the corresponding eigenvector. Without loss of generality, let $\lambda \neq 0$. Then

$$(\mathscr{D} - \omega\mathscr{E})^{-1}(\mathscr{E}^T - (1-\alpha\mathscr{D}))\mathbf{x} = \lambda\mathbf{x},$$

equivalently,

$$(1-\omega)\mathbf{u} - Q\mathbf{v} = -\lambda\omega\mathbf{u}, \tag{12}$$

$$(\omega-1)\mathbf{v} = \lambda(Q\mathbf{u} + \omega\mathbf{v}). \tag{13}$$

We can derive from (12) and the positive definiteness of Q that

$$\mathbf{v} = ((\lambda - 1)\omega + 1)Q^{-1}\mathbf{u}. \tag{14}$$

Substituting (14) into (13), gives

$$-\lambda Q^2\mathbf{u} = ((\lambda - 1)\omega + 1)^2\mathbf{u}. \tag{15}$$

This shows that if μ is an eigenvalue of Q, then

$$\lambda\mu^2 = -\left((\lambda - 1)\,\omega + 1\right)^2 \tag{16}$$

$$= -\left(\lambda^2\omega^2 + 2\omega(1 - \omega)\lambda + (\omega - 1)^2\right). \tag{17}$$

From (17), we get

$$\lambda^2 - \left(\frac{2\omega^2 - 2\omega - \mu^2}{\omega^2}\right)\lambda + (\frac{\omega - 1}{\omega})^2 = 0. \tag{18}$$

Now it follows from Lemma 2 that $|\lambda| < 1$ if and only if

$$|\omega - 1| < \omega,$$
$$|2\omega^2 - 2\omega - \mu^2|7 < 2\omega^2 - 2\omega + 1.$$

It is straightforward to see that $|\omega - 1| < \omega$ is equivalent to $\omega > \frac{1}{2}$. By some easy manipulations, we can observe, whenever

$$(2\omega - 1)^2 > \mu^2, \tag{19}$$

the second inequality holds. The inequality (19) is ensured, if

$$|2\omega - 1| > \mu \quad \text{or} \quad |2\omega - 1| < -\mu,$$

equivalently,

$$\omega < \frac{1 - \mu}{2} \quad \text{or} \quad \omega > \frac{1 + \mu}{2}. \tag{20}$$

Evidently, the first inequality of (20) cannot be true. On the other hand, holding the second inequality of (20) ensures $\omega > \frac{1}{2}$, and then it completes the proof. □

In the following, we would like to find the optimal value of the relaxation parameter ω, denoted by ω^*. To do so, ω^* should be computed to minimize the spectral radius of the iteration matrix of the BGSOR method, that is,

$$\rho\left(\mathscr{G}_{\omega^*}\right) = \arg \min_{\omega > \frac{1+\mu_{\max}(Q)}{2}} \rho\left(\mathscr{G}_{\omega}\right).$$

To compute the optimal value of w, we state and prove the next theorem.

Theorem 2. Assume that the hypothesis of Theorem 1 are met. Then the optimal value of the relaxation parameter and the corresponding optimal

convergence factor in the BGSOR iteration method are as follows:

$$\omega^* = \frac{1}{2}\left(1 + \sqrt{1+\rho^2(Q)}\right), \tag{21}$$

and

$$\rho\left(\mathscr{G}_{\omega^*}\right) = 1 - \frac{1}{\omega^*} = \left(\frac{\rho(Q)}{1+\sqrt{1+\rho^2(Q)}}\right)^2.$$

Proof. If λ is an eigenvalue of the iteration matrix $\mathscr{G}_\omega$, then $\lambda < 0$ or $\lambda \in \mathbb{C} \setminus \mathbb{R}$, according to (16). First, we consider the case $\lambda < 0$. So, there exists an eigenvalue μ of Q such that (18) holds true. The discriminant of this quadratic equation is

$$\Delta = \left(\frac{2\omega^2 - 2\omega - \mu^2}{\omega^2}\right)^2 - 4\left(\frac{\omega-1}{\omega}\right)^2,$$

and the roots of (18) are as follows:

$$\lambda_{1,2}(\omega) = \frac{2\omega^2 - 2\omega - \mu^2}{2\omega^2} \pm \frac{\sqrt{\Delta}}{2}.$$

From (16), we get

$$(\lambda - 1)\omega + 1 = \pm\mu\sqrt{-\lambda}. \tag{22}$$

Set

$$\begin{aligned} f_\omega(\lambda) =& (\lambda-1)\omega + 1 = \omega\lambda + 1 - \omega, \\ g(\lambda) =& \pm\,\mu\sqrt{-\lambda}. \end{aligned}$$

Clearly, the function f_ω passes through the point $(1,1)$, that is, $f_\omega(1) = 1$ and the slope of $f_\omega(\lambda)$ is ω. Figure 1 displays the points of intersections of the functions $f_\omega(\lambda)$ and $g(\lambda)$ for an arbitrary value of ω. This figure shows that by increasing ω, the maximum of absolute values of the abscissas of the points of intersection of the functions $f_\omega(\lambda)$ and $g(\lambda)$, that is, $\max\{\lambda_1, \lambda_2\}$, decrease, while $f_\omega(\lambda)$ gets tangent to $g(\lambda)$. In the tangent case, we have $\lambda_1 = \lambda_2$, and it indicates that $\Delta = 0$. From $\Delta = 0$, it is straightforward to verify that $\mu = 0$ or $4\omega^2 - 4\omega - \mu^2 = 0$. The case $\mu = 0$ is impossible, because of the positive definiteness of Q. Thus, $4\omega^2 - 4\omega - \mu^2 = 0$. This quadratic equation has two roots, as follows:

$$\omega_\pm = \frac{1}{2}\left(1 \pm \sqrt{1+\mu^2}\right).$$

Due to the condition $\omega > \frac{1+\mu_{\max}(Q)}{2}$, ω_- is not acceptable. So, we consider

$$\omega_+ = \frac{1}{2}\left(1 + \sqrt{1+\mu^2}\right),$$

and in this case, we have

$$\lambda_1 = \lambda_2 = \lambda_+ = \frac{1}{\omega_+} - 1.$$

Now suppose that $\omega > \omega_+$. In this case, the roots of the quadratic equation (18) are complex and conjugate, which are as follows:

$$\lambda_{1,2}(\omega) = \frac{-2\omega^2 + 2\omega + \mu^2}{2\omega^2} \pm \imath \frac{\sqrt{\Delta'}}{2},$$

where

$$\Delta' = 4\left(\frac{\omega - 1}{\omega}\right)^2 - \left(\frac{2\omega^2 - 2\omega - \mu^2}{\omega^2}\right)^2.$$

Then

$$|\lambda_{1,2}| = 1 - \frac{1}{\omega}.$$

By recalling that $\omega > \omega_+$ and having in mind that $w_+ > 1$, we have

$$1 - \frac{1}{\omega_+} < 1 - \frac{1}{\omega},$$

and this shows that ω_+ is the best choice for ω. On the other hand, the curve $g(\lambda) = \pm\rho(Q)\sqrt{-\lambda}$ serves an upper bound for each curve as $\pm\mu\sqrt{-\lambda}$, where $0 \leq \mu \leq \rho(Q)$. Summarizing the above results, we see that

$$\rho\left(\mathscr{G}_{\omega^*}\right) = \min_{\omega} \max_{\omega > \frac{1+\mu_{\max}}{2}} |1 - \frac{1}{\omega}| = 1 - \frac{1}{\omega^*} = \left(\frac{\rho(Q)}{1 + \sqrt{1 + \rho^2(Q)}}\right)^2,$$

where ω^* was considered as in (21). □

Remark 1. In Theorem 2, for computing ω^*, we need to compute $\rho(q)$. One may use a few iterations of the power method to compute $\lambda_{\max}(Q)$. On the other hand, because of positive definiteness of Q, we have

$$\rho(Q) = \lambda_{\max}(Q) = \|Q\|_2.$$

So, we can compute $\|Q\|_2$ instead of $\rho(Q)$. In practice, the `normest` command of MATLAB can be used to compute an estimation of $\|Q\|_2$.

4 Numerical experiments

This section is devoted to numerical experiments to evaluate the effectiveness of the BGSOR iteration scheme for solving linear systems (8). The numerical results of the proposed method are compared with those of the GMRES [8, 9]

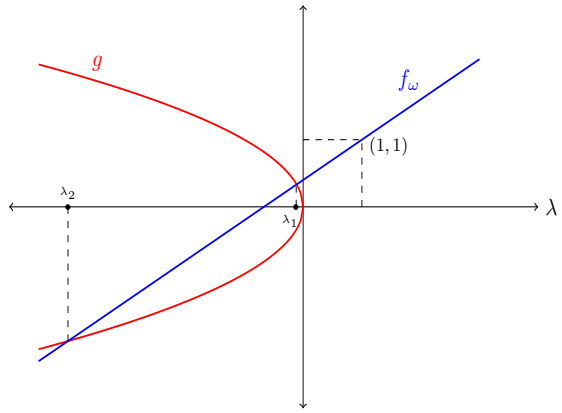

Figure 1: The graph of the functions $f_\omega(\lambda)$ and $g(\lambda)$.

and the BGS methods. In all the test problems, we use the restart version of GMRES with a restarting number 10. The initial guess is assumed to be a random vector, and iterations are terminated when

$$Res = \frac{\|r_k\|_2}{\|r_0\|_2} < 10^{-9},$$

where $r_k = \mathscr{P} - \mathscr{A}\mathbf{z}^{(k)}$ is the residual at the kth iteration or if the maximum number of iterations $maxit = 1000$ is exceeded. The terms "IT" and "CPU" in the tables refer to the total number of iterations and the elapsed CPU time in seconds for convergence, respectively. We comment that five runs were performed for each test, and then the average of CPU times and iterations are reported (The average of the iteration numbers were rounded). For the BGSOR method, the optimal parameter is computed according to the rule (21). The numerical results were carried out under MATLAB-R2017 on a laptop running Windows 10 and an Intel (R) Core(TM) i5-8265U CPU @ 1.60 GHz 8 GB.

Example 1. Let $\theta = 0$. The system (1) is then decoupled and becomes

$$\imath \mathrm{u}_t + (-\Delta)^{\frac{\alpha}{2}}\mathrm{u} + 2|\mathrm{u}|^2\mathrm{u} = 0,$$

when the initial value

$$\mathrm{u}(x, 0) = \mathrm{sech}(x) \cdot \exp(2\imath x),$$

is applied. In this example, the original problem was truncated in $[-20, 20]$. Set $u(-20, t) = u(20, t) = 0$. For this problem, we choose the parameters $\xi = 1.3$ and $\eta = 1.2$.

Table 1: The optimal parameters ω^* for BGSOR method with $\alpha = 1.3$ and $n = 4m$ at $t = 2$ for Example 1.

ℓ	800	1600	3200	6400
ω^*	1.002	1.004	1.006	1.009

We set $m = 800, 1600, 3200, 6400$ and examine two values of α, $\alpha = 1.3, 1.6$. When $\alpha = 1.3$, we set $n = 4m$; otherwise, we choose $n = 6m$. The optimal values of the relaxation parameter in the BGSOR method for $\alpha = 1.3$ are given in Table 1, and the ones for $\alpha = 1.6$ are given in Table 3.

In Tables 2 and 4, we have listed the numerical results at $t = 2$. From these tables, we observe that the BGSOR method is superior to the examined methods in terms of both the iterations and the elapsed CPU times.

Table 2: Numerical results with $\alpha = 1.3$ and $n = 4m$ at $t = 2$ for Example 1.

Method	ℓ	800	1600	3200	6400
BGSOR	IT	5	5	5	5
	CPU	0.016	0.051	0.171	0.955
BGS	IT	5	6	6	7
	CPU	0.018	0.072	0.228	1.705
GMRES(10)	IT	6	7	7	7
	CPU	0.080	0.112	0.352	3.610

Example 2. For the following coupled system with $\theta \neq 0$:

$$\begin{cases} \imath \mathrm{u}_t + (-\Delta)^{\frac{\alpha}{2}} \mathrm{u} + 2\left(|\mathrm{u}|^2 + |\mathrm{v}|^2\right)\mathrm{u} = 0, \\ \imath \mathrm{v}_t + (-\Delta)^{\frac{\alpha}{2}} \mathrm{v} + 2\left(|\mathrm{v}|^2 + |\mathrm{v}|^2\right)\mathrm{v} = 0, \end{cases} \quad -20 \leq x \leq 20, 0 < t \leq 2. \tag{23}$$

We will use

$$\begin{cases} \mathrm{u}(x,0) = \operatorname{sech}(x + D_0) \cdot \exp(\imath \mathrm{v}_0 x), \quad \mathrm{v}(x,0) = \operatorname{sech}(x - D_0) \cdot \exp(-\imath \mathrm{v}_0 x), \\ \mathrm{u}(-20,0) = \mathrm{u}(20,0) = 0, \quad \mathrm{v}(-20,0) = \mathrm{v}(20,0) = 0, \end{cases} \tag{24}$$

as the initial conditions. In this case, we choose the parameters $D_0 = 1$, $\mathrm{v}_0 = 2$, $\xi = 1.4$, and $\eta = 1.2$.

The discretization of the coupled system of (23) leads to the solution of the linear systems of equations of the form (5). We assume that these coefficient matrices are A and B. These matrices have the same structure. Tables 5 and 7 show the optimal values of the relaxation parameter of A and B in the BGSOR method for different values of α and m.

Table 3: The optimal parameters ω^* for the BGSOR method with $\alpha = 1.6$ and $n = 6m$ at $t = 2$ for Example 1.

ℓ	800	1600	3200	6400
ω^*	1.010	1.022	1.050	1.108

Table 4: Numerical results with $\alpha = 1.6$ and $n = 6m$ at $t = 2$ for Example 1.

Method	ℓ	800	1600	3200	6400
BGSOR	IT	6	7	8	10
	CPU	0.018	0.068	0.311	2.015
BGS	IT	7	9	14	28
	CPU	0.022	0.093	0.571	4.462
GMRES(10)	IT	8	9	10	13
	CPU	0.112	0.185	0.235	6.941

In Tables 6 and 8, we report the results for the BGSOR, BGS, and GMRES(10) iterative methods at $t = 2$. These results clearly show that the BGSOR method leads to a faster overall convergence time than the other examined methods. Besides, the BGSOR method gets less iteration numbers.

Table 5: The optimal parameters ω^* of A and B for the BGSOR method with $\alpha = 1.3$ and $n = 4m$ at $t = 2$ for Example 2.

ℓ	800	1600	3200	6400
$\omega^*(A)$	1.002	1.004	1.006	1.008
$\omega^*(B)$	1.002	1.004	1.006	1.008

Table 6: Numerical results with $\alpha = 1.3$ and $n = 4m$ at $t = 2$ for Example 2.

Method	ℓ	800		1600		3200		6400	
		A	B	A	B	A	B	A	B
BGSOR	IT	5	5	5	5	5	5	5	5
	CPU	0.013	0.010	0.052	0.023	0.173	0.145	0.938	0.841
BGS	IT	5	5	6	6	6	6	7	7
	CPU	0.020	0.014	0.069	0.064	0.213	0.228	1.641	1.145
GMRES(10)	IT	6	6	7	7	7	7	8	8
	CPU	0.064	0.017	0.093	0.049	0.155	0.139	2.812	1.377

5 Conclusion

In this paper, the BGSOR scheme has been presented to solve the complex symmetric linear systems deriving from the discretization of the space

Table 7: The optimal parameters ω^* of A and B for the BGSOR method with $\alpha = 1.6$ and $n = 6m$ at $t = 2$ for Example 2.

ℓ	800	1600	3200	6400
$\omega^*(A)$	1.010	1.022	1.050	1.122
$\omega^*(B)$	1.010	1.022	1.050	1.122

Table 8: Numerical results with $\alpha = 1.6$ and $n = 6m$ at $t = 2$ for Example 2.

Method	ℓ	800		1600		3200		6400	
		A	B	A	B	A	B	A	B
BGSOR	IT	6	6	7	7	9	9	10	10
	CPU	0.021	0.017	0.071	0.069	0.346	0.248	1.941	2.003
BGS	IT	7	7	10	10	15	15	35	35
	CPU	0.025	0.020	0.106	0.112	0.607	0.592	6.832	5.483
GMRES(10)	IT	8	8	9	9	11	11	13	13
	CPU	0.088	0.061	0.093	0.082	0.448	0.412	3.376	3.251

fractional CNLS equation. We have analyzed the convergence theory of the method, and we have shown that the method is convergent under a suitable condition. The optimal value of the relaxation parameter and the rate of convergence factor for the BGSOR method were also provided. Our results have verified that the BGSOR method performs better than some existing methods.

Acknowledgements

We would like to thank the referees for their helpful comments and suggestions.

This paper is dedicated to Prof. Faezeh Toutounian and Prof. Ali Vahidan Kamyad for their many contributions to numerical linear algebra and optimization theory.

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How to cite this article
H. Aslani, D. Khojasteh Salkuyeh and M. Taghipour A new iteration method for solving space fractional coupled nonlinear Schrödinger equations. *Iranian Journal of Numerical Analysis and Optimization*, 2022; 12(3 (Special Issue), 2022): 704-718. doi: 10.22067/ijnao.2022.77745.1163.