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Research Article

Solving quantum optimal control problems by wavelets method

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Abstract

We present the quantum equation and synthesize an optimal control procedure for this equation. We develop a theoretical method for the analysis of quantum optimal control system given by the time depending Schrödinger equation. The Legendre wavelet method is proposed for solving this problem. This can be used as an efficient and accurate computational method for obtaining numerical solutions of different quantum optimal control problems. The distinguishing feature of this paper is that it makes the method, previously used to solve non-quantum control equations based on Legendre wavelets, usable by using a change of variables for quantum control equations.

AMS subject classifications (2020): 49M25; 35J10.

Keywords: Quantum Equations; Optimal Control Problems; Legendre Wavelets Methods.

1 Introduction

With the advent of the twentieth century, the inability of classical physics in the fields of relativity and microscopy led to the outburst of quantum physics. After the development of quantum physics, the issue of quantum control was inspired by experimental advances and issues raised in sciences such as

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quantum chemistry, quantum optics, quantum information, and atomic and molecular physics [7, 10, 24, 27, 34]. Recently, classical control methods such as optimal control, robust control, Lyapunov control, and feedback control for quantum systems have been studied and expanded [5, 16, 19, 20, 22, 25]. In quantum control, the main goal is to effectively control the system from an initial state to a desired final state using external control fields. However, control in quantum systems is at the beginning of the road, and further research is needed. In particular, optimal control in quantum systems is of particular importance as one of the most widely used issues [3, 9, 12, 15, 17, 23, 26, 31].

Along with the development of analytical control methods in quantum systems, numerical methods have also been developed. Numerical methods that have been considered in solving classical equations and problems are also studied [3, 11, 18, 28]. One of the most useful numerical methods for solving differential problems is wavelets-based numerical method [2, 14, 21, 29, 33]. This article tries to provide a useful way to solve optimal quantum control based on wavelets method.

Let $\mathcal{H}$ be a finite- or infinite-dimensional Hilbert space of a quantum system and let Ψ denote the state of this system. Then the Schrödinger equation can be found as follows [13, 32]:

$$ih\frac{\partial \Psi}{\partial t} = H\Psi, \tag{1}$$

where $\Psi \in H$ is the state variable, h is the Planck constant, and H is a self-adjoint Hamiltonian operator in $\mathcal{H}$.

In every physical system, energy is an important quantity. In quantum systems, Hamiltonian H is corresponding to energy, then we can write $He_i = E_i e_i$, $i = 1, \ldots, N$, where E_i are eigenvalues, e_i are eigenvectors of the physical system concerned, and N is the dimension of $\mathcal{H}$. In this paper, we suppose that e_i is an orthogonal basis; then we expand a state vector Ψ as follows:

$$\Psi = \sum_{i=1}^{N} \psi_i e_i.$$

The population of energy states of level i are the quantities $|\psi_i|^2$.

When a quantum system is operated by an external field, the Schrödinger equation in (1) is modified as

$$ih\frac{\partial \Psi}{\partial t} = (H_0 + \sum_k H_k u_k(t))\Psi,$$

where $H_0 : \mathcal{H} \mapsto \mathcal{H}$, which is the internal Hamiltonian, and the Hamiltonian linear operator $H_k : \mathcal{H} \mapsto \mathcal{H}$ describes the coupling of the system to external fields $u_k(t)$. In this paper, for simplicity, we consider a quantum system with one control, and set $k = 1$. Then we can write

$$ih\frac{\partial \Psi}{\partial t} = (H_0 + H_1 u(t))\Psi.$$

The design of controls in quantum systems is considered for energy-efficient population transfer. These controls perform the desired transmission and in addition optimize a specific performance index. The specific energy performance index considered in this section is shown below:

$$\min J[u] = \int_0^T u^2(t)dt.$$

This cost function for control has been widely used in the literature on optimal control of quantum systems as a part of various objective functions. It is a measure of the energy expended to create a control field.

The main object of this paper is to present an efficient numerical algorithm based on the Legendre wavelets methods to solve the following optimal control problems of the form:

$$\min J[u] = \int_0^T u^2(t)dt,$$

subject to the dynamical quantum systems

$$i\dot{\Psi} = (H_0 + Vu(t))\Psi.$$

The initial condition for the above equation is

$$\Psi(0) = \psi_0$$

It is worthy to note that $V = H_1$.

2 The hat function

In this section, we introduce a family of basic functions, namely, the hat functions. An n-set of the hat functions is defined on the interval $[0, T]$ as follows:

$$h_0(t) = \begin{cases} \frac{k-t}{k}, & 0 \le t \le k, \\ 0, & o.w. \end{cases}$$

$$h_j(t) = \begin{cases} \frac{t-(j-1)k}{k}, & (j-1)k \le t \le jk, \\ \frac{(j+1)k-t}{k}, & (jk) \le t \le (j+1)k, \\ 0, & o.w. \end{cases}$$

$$h_{n-1}(t) = \begin{cases} \frac{t-(1-k)}{k}, & T-k \le t \le T, \\ 0, & o.w. \end{cases}$$

where $k = \frac{T}{n-1}$. For hat functions, we can write $h_i(jk) = \delta_{ij}$, where δ is the Kronecker delta. By the definition of the hat functions, we can expand any function like $g(t) \in L^2[0,T]$ as follows:

$$g(t) \simeq \sum_{j=0}^{n-1} g_j h_j(t) = G^T H(t) = H^T(t) G, \tag{2}$$

where

$$G \triangleq [g_0, g_1, \ldots, g_{n-1}]$$

and

$$H(t) \triangleq [h_0(t), h_1(t), \ldots, h_{n-1}(t)]^T.$$

When we use the hat functions for the $g(t)$, it can be observed that

$$g_j = g(jk), \quad j = 0, 1, \ldots, n-1. \tag{3}$$

Now, we introduce another family of basic functions, namely, Legendre wavelet functions. The set of these functions is an orthogonal set on the interval $[0,1]$ with the weight function $w(t) = 1$. If $P_k(t)$ is the Legendre polynomials of degree k that are orthogonal on the interval $[-1,1]$ with respect to the weight function $w(t) = 1$, then we can write Legendre wavelets as follows [2, 14]:

$$w_{lk}(t) = \begin{cases} \sqrt{2k+1}\, 2^{\frac{n}{2}} P_k(2^{n+1}t - 2l + 1), & t \in [\frac{l-1}{2^n}, \frac{l}{2^n}], \\ 0, & o.w. \end{cases} \tag{4}$$

In fact, $w_{lk}(t) = w(l,k,n,t)$, where $l = 1, 2, \ldots, 2^n$ and n is an arbitrary positive integer.
For any arbitrary function, like $g(t)$ defined over $[0,1]$ and square-integrable over $[0,1]$, we can expand $g(t)$ by the Legendre wavelets as follows:

$$g(t) = \sum_{l=1}^{\infty} \sum_{k=0}^{\infty} a_{lk} w_{lk}(t).$$

By approximating the above infinite series, we can write

$$g(t) \simeq \sum_{l=1}^{2^n} \sum_{k=0}^{K-1} a_{lk} w_{lk}(t) = A^T W(t), \tag{5}$$

where A and $W(t)$ are $\hat{k} = 2^n K$ column vectors.
For the index lk, we can write $j = k(l-1) + k + 1$. Then $a_{lk} = a_j$ and

$w_{lk} = w_j$. Thus (5) can be written as

$$g(t) \simeq \sum_{l=1}^{\hat{k}} a_j w_j(t) = A^T W(t),$$

where

$$A \triangleq [a_1, a_2, \ldots, a_{\hat{k}}]^T$$

and

$$W(t) \triangleq [w_1(t), w_2(t), \ldots, w_{\hat{k}}(t)]. \tag{6}$$

Now by taking $t_j = \frac{j}{\hat{k}-1}$ as the collocation points into (6), we can write

$$P_{\hat{k}\times\hat{k}} \triangleq [W(0), W(\frac{1}{\hat{k}-1}), \ldots, W(1)].$$

It can be simplify verified that the Legendre wavelets can be expanded in $\hat{k}$ in terms of the hat function by using (2) and (3) as follows:

$$W(t) \simeq P_{\hat{k}\times\hat{k}} H(t). \tag{7}$$

Theorem 1. Suppose that $f(t) \simeq F^T H(t)$ and that $g(t) \simeq G^T H(t)$. Then

$$f(t)g(t) \simeq S^T H(t), \tag{8}$$

where $S_{ij} = (F.G)_{ij} = F_{ij}G_{ij}$ denotes pointwise product of F and G.

Proof. By applying (2) and (3) for $f(t)$ and $g(t)$, we can write

$$g(t) \simeq G^T H(t) = \sum_{j=0}^{n-1} g_j h_j(t) = \sum_{j=0}^{\hat{k}-1} g(jk) h_j(t),$$

$$f(t) \simeq F^T H(t) = \sum_{j=0}^{n-1} f_j h_j(t) = \sum_{j=0}^{\hat{k}-1} f(jk) h_j(t).$$

Then by using the point wise product, we have

$$f(t)g(t) \simeq \sum_{j=0}^{\hat{k}-1} f(jk)g(jk)h_j(t) = D^T H(t),$$

which completes the proof. □

Corollary 1. Suppose that $g(t) \simeq G^T H(t)$ by hat functions. Then, for any integer number $n \geq 2$, we have

$$(g(t))^n \simeq [g_0^n, g_1^n, \ldots, g_{\hat{k}-1}^n] H(t). \tag{9}$$

Proof. For $n = 2$ by Theorem 1, we have

$$(g(t))^2 \simeq [g_0^2, g_1^2, \ldots, g_{\hat{k}-1}^2]H(t);$$

then by induction for $n > 2$, we have

$$(g(t))^n \simeq [g_0^n, g_1^n, \ldots, g_{\hat{k}-1}^n]H(t),$$

which completes the proof.

□

Theorem 2. Suppose that $g(t) \simeq A^T W(t)$ and $f(t) \simeq B^T W(t)$ by Legendre wavelets. Then we can write

$$f(t)g(t) \simeq Q^T P_{\hat{k}\times\hat{k}}^{-1} W(t), \tag{10}$$

where $A_1^T = A^T P_{\hat{k}\times\hat{k}}$, $B_1^T = B^T P_{\hat{k}\times\hat{k}}$, and $Q = A_1.B_1$.

Proof. By Theorem 1 and equation (7), we have

$$g(t) \simeq A^T W(t) \simeq A^T P_{\hat{k}\times\hat{k}} H(t) = A_1^T H(t),$$

$$f(t) \simeq B^T W(t) \simeq B^T P_{\hat{k}\times\hat{k}} H(t) = B_1^T H(t),$$

and then

$$f(t)g(t) \simeq (A_1.B_1)^T H(t) = Q^T H(t) \simeq Q^T P_{\hat{k}\times\hat{k}}^{-1} W(t).$$

□

In Theorem 1, the multiplication of two functions is obtained according to the hat functions, and in Theorem 2 by using Theorem 1, the multiplication of two functions is obtained according to the Legendre wavelets. If $g(t) \simeq A^T W(t)$ by the Legendre wavelets, then by Theorem 2 and Corollary 1, we can write

$$(g(t))^n \simeq [\tilde{a}_1^n, \tilde{a}_2^n, \ldots, \tilde{a}_{\hat{k}}^n] P_{\hat{k}\times\hat{k}}^{-1} W(t). \tag{11}$$

3 Analysis of the proposed method

In previous works, using Legendre wavelet, numerical solutions for non-quantum control equations have been obtained. In this section, we try to implement one of these methods, which is based on the Legendre wavelet, for quantum control equations. To overcome this problem, first change of variables to make the equation usable is performed.

In this section, we consider the quantum control systems of the form

$$\min J[u] = \int_0^T u^2(t)dt,$$

subject to the dynamical system

$$i\dot{\Psi} = (H_0 + Vu(t))\Psi, \tag{12}$$

with the initial condition $\Psi(0) = \Psi_0$.
First, we introduce a change of variables that generalizes

$$\Psi = e^{-iH_0t}x,$$

such that for (12), we obtain

$$\begin{aligned} i(-iH_0e^{-iH_0t}x + e^{-iH_0t}\dot{x}) = (H_0 + Vu(t))e^{-iH_0t}x &\to H_0e^{-iH_0t}x + e^{-iH_0t}\dot{x} \\ &= H_0e^{-iH_0t}x + Vu(t)e^{-iH_0t}x. \end{aligned}$$

Thus we can write

$$\dot{x} = e^{iH_0t}Vu(t)e^{-iH_0t}x.$$

By considering $E(t) = e^{iH_0t}Ve^{-iH_0t}$, equation (12) can be written as

$$\dot{x} = E(t)u(t)x(t), \tag{13}$$

and the new initial condition is $x(0) = \Psi_0$.
By the Legendre wavelets for the derivative of the state variable $\dot{x}$ and the control variable $u(t)$, we can write

$$\dot{x} \simeq X^TW(t) \tag{14}$$

and

$$u(t) \simeq U^TW(t), \tag{15}$$

where

$$U^T = [u_1, u_2, \ldots, u_{\hat{k}}]$$

and

$$X^T = [x_1, x_2, \ldots, x_{\hat{k}}].$$

We can apply integration on both sides of (14) and considering the initial condition

$$x(t) \simeq X^TQW(t) + \Psi_0.$$

Let η be the coefficients vector of the unit function. Then

$$x(t) \simeq (X^TQ + \Psi_0\eta^T)W(t). \tag{16}$$

Now (16) can be written as

$$x(t) \simeq A^T W(t) = [a_1, a_2, \ldots, a_{\hat{k}}]W(t). \tag{17}$$

In this part, by applying (11) and (9) for the above approximation, we have

$$(x(t))^n \simeq [\tilde{a}_1^n, \tilde{a}_2^n, \ldots, \tilde{a}_{\hat{k}}^n]P^{-1}_{\hat{k}\times\hat{k}}W(t) \simeq [\tilde{a}_1^n, \tilde{a}_2^n, \ldots, \tilde{a}_{\hat{k}}^n]H(t),$$

and also for (15), we have

$$(u(t))^n \simeq [\tilde{u}_1^n, \tilde{u}_2^n, \ldots, \tilde{u}_{\hat{k}}^n]P^{-1}_{\hat{k}\times\hat{k}}W(t) \simeq [\tilde{u}_1^n, \tilde{u}_2^n, \ldots, \tilde{u}_{\hat{k}}^n]H(t). \tag{18}$$

We can also approximate $E(t)$ by Legendre wavelets as

$$E(t) \simeq E^T W(t), \tag{19}$$

where

$$E^T = [e_1, e_2, \ldots, e_{\hat{k}}].$$

In this part, by using (8) and (10) for $E(t)u(t)x(t)$, we have

$$\begin{aligned} E(t)u(t)x(t) &\simeq [\tilde{e_1}\tilde{u_1}\tilde{x_1}, \tilde{e_2}\tilde{u_2}\tilde{x_2}, \ldots, \tilde{e}_{\hat{k}}\tilde{u}_{\hat{k}}\tilde{x}_{\hat{k}}]P^{-1}_{\hat{k}\times\hat{k}}W(t) \\ &\simeq [\tilde{e_1}\tilde{u_1}\tilde{x_1}, \tilde{e_2}\tilde{u_2}\tilde{x_2}, \ldots, \tilde{e}_{\hat{k}}\tilde{u}_{\hat{k}}\tilde{x}_{\hat{k}}]H(t) \\ &= \Delta_1^T H(t), \end{aligned} \tag{20}$$

also by using (18) for $n = 2$, we have

$$(u(t))^2 \simeq [\tilde{u}_1^2, \tilde{u}_2^2, \ldots, \tilde{u}_{\hat{k}}^2]P^{-1}_{\hat{k}\times\hat{k}}W(t) \simeq [\tilde{u}_1^2, \tilde{u}_2^2, \ldots, \tilde{u}_{\hat{k}}^2]H(t) = \Delta_2^T H(t). \tag{21}$$

Now by using (21), the index J can be written as

$$J \simeq J[U] = \Delta_2^T \Omega, \tag{22}$$

where

$$\Omega = [\int_0^T h_o(t)dt, \int_0^T h_1(t)dt, \ldots, \int_0^T h_{\hat{k}-1}(t)dt]. \tag{23}$$

By applying (20) for (13) and (12), we can write

$$X^T - \Delta_1 P^{-1}_{\hat{k}\times\hat{k}} \simeq 0. \tag{24}$$

In this part, by Lagrange multiplier method for minimization index J in (22) subject to systems of algebraic equation (24), we can write

$$\tilde{J}[X, U, L] = J[U] + X^T - \Delta_1 P^{-1}_{\hat{k}\times\hat{k}}L = \Delta_2^T\Omega + X^T - \Delta_1 P^{-1}_{\hat{k}\times\hat{k}}L, \tag{25}$$

where

$$L = [L_1, L_2, \ldots, L_{\hat{k}}]^T.$$

Hence L is the vector of Lagrange multiplier.
For minimizing by the Lagrange method, the necessary conditions are

$$\begin{cases} \dfrac{\partial \tilde{J}}{\partial X} = 0, \\ \dfrac{\partial \tilde{J}}{\partial U} = 0, \\ \dfrac{\partial \tilde{J}}{\partial L} = 0. \end{cases} \tag{26}$$

By the Newton iteration method, we can solve equations of (26) for X, U, and L. Then the approximation of $x(t)$ and $u(t)$ can be determined by (17) and (15).

4 Proposed algorithm

The object of this algorithm is designed to solve the Schrödinger equation:
Input: T (final time), N(dimension of $\mathcal{H}$), H_0, and H_1 .
Step1: Make a change of variable $\Psi = e^{-iH_0 t}x$ in dynamical system (12) to obtain (13).
Step2: Define $x(t), u(t)$, and $E(t)$ by (17),(15), and (19), respectively.
Step3: Write the index J as (22) and (23).
Step4: Compute dynamical system (13) by applying (20) to the form (24).
Step5: Compose new index $\tilde{J}[X, U, L]$ as (25) by (24) and (23).
Step6: Solve equation systems in (26) and obtain X and U.
Step7: Compute $x(t)$ and $u(t)$ by (17) and (15).
Step8: Compute $\Psi = e^{-iH_0 t}x$.
Output: The approximate solution $u(t)$ and Ψ.

5 Numerical experiments

Example 1. In this example, we consider the two-level system $i\dot{\Psi} = (H_0 + Vu(t))\Psi$, where $\Psi \in \mathbb{C}^2$ as follows:

$$\mathbf{H_0} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}. \qquad \mathbf{V} = \begin{pmatrix} 0 & v_{12} \\ v_{12}^* & 0 \end{pmatrix}.$$

The concept of optimal control of two-level quantum systems was presented in [1, 3, 4, 6, 8, 11, 13, 30]. In the most of these researches, optimal control is constructed on the basis of geometric arguments. If we suppose $E_1 = 2$,

$E_2 = -2$, and $v_{12} = 2+3i$, then we can obtain the numerical results as follows:

Table 1: Approximate and exact value of $u(t)$ in Example 1 for different values of t.

t	approximate of $u(t)$	exact of $u(t)$	error
1	-0.0359	-0.0459	0.0120
5	-0.0496	-0.0578	0.0102
10	0.1010	0.0999	0.0051
15	-0.0922	-0.0878	0.0016
20	0.0663	0.0594	0.0119
25	0.0286	0.0066	0.0220
30	-0.0630	-0.0635	0.0020

Table 2: Approximate and exact value of $x(t)$ in Example 1 for different values of t.

t	approximate of $\psi_1(t)$	exact of $\psi_1(t)$	approximate of $\psi_2(t)$	exact of $\psi_2(t)$	error of $\psi_1(t)$	error of $\psi_2(t)$
1	1.0097	1.0078	0.0690	0.0590	0.0109	0.0190
5	0.9869	0.9769	0.2664	0.2564	0.0108	0.0190
10	0.8666	0.8766	0.4681	0.4764	0.0110	0.0113
15	0.7197	0.7297	0.6656	0.6756	0.0120	0.0160
20	0.5513	0.5413	0.8645	0.8445	0.0110	0.0230
25	0.3033	0.3133	0.9270	0.9470	0.0120	0.0220
30	0.0830	0.0737	1.0105	1.0005	0.0123	0.0130

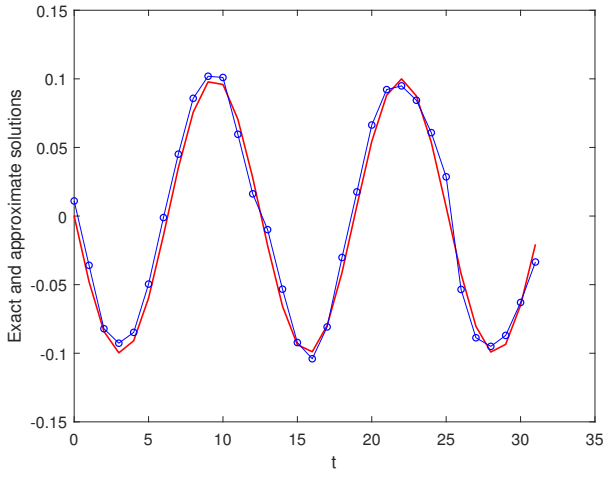

Figure 1: Plots of approximate and exact results of control variable for Example 1

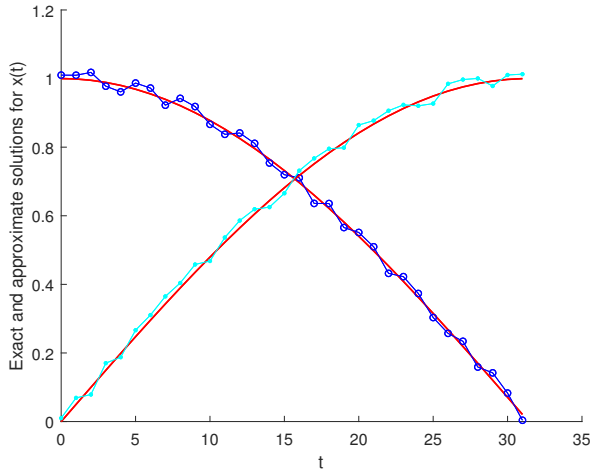

Figure 2: Plots of the approximate and exact results of state variable for Example 1

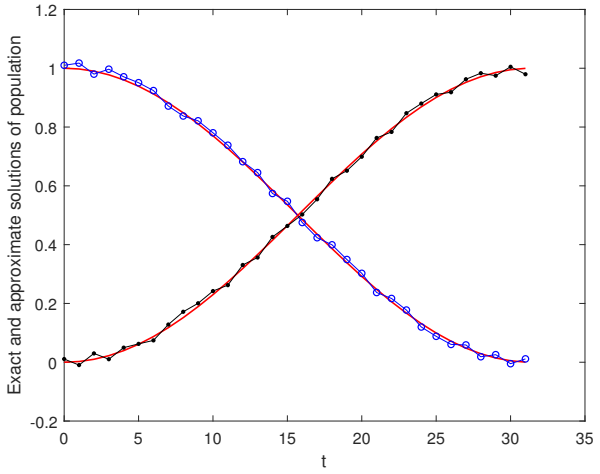

Figure 3: Plots of the approximate and exact results of population for Example 1

We solved the above problem by our proposed algorithm with $n = 2$ and $K = 10$ or $\hat{k} = 40$. The stopping condition is $|u^*(t) - u(t)| < 5 \times 10^{-2}$, where $u^*(t)$ and $u(t)$ are approximate and exact results of control variable, respectively, and by the stopping condition, we have 73 iterations. The plots of the approximate and exact values of $u(t)$ is shown in Figure 1. The plots

of the approximate and exact values of $\psi_1(t)$ and $\psi_2(t)$ are shown in Figure 2. The plots of the approximate and exact values of $|\psi_1(t)|^2$ and $|\psi_2(t)|^2$ are shown in Figure 3.

Approximation and exact results of control variable $u(t)$ and their absolute errors for different t are presented in Table 1. In fact, because the index $J(t)$ in each step depends on the control variable, so by approximating the control variable in each step, the index $J(t)$ is also approximated. Approximation and exact results of state variables $\psi_1(t)$ and $\psi_2(t)$ and their absolute errors for different t are presented in Table 2. The above results show that using the algorithm and method mentioned numerically is very useful and efficient. The main advantage of this method is to provide a simple solution based on classical numerical methods in the field of optimal quantum control. In [32], an index similar to the index used in this article has been used and numerical and graphical results have been obtained. Carefully in these results, it is observed that the solutions obtained in this article have been obtained with less repetition and more accuracy.

Example 2. In this example, we consider the three-level system $i\dot{\Psi} = (H_0 + Vu(t))\Psi$, where $\Psi \in \mathbb{C}^3$ as follows [32]:

$$\mathbf{H_0} = \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} 0 & v_{12} & v_{13} \\ v_{12}^* & 0 & v_{23} \\ v_{13}^* & v_{23}^* & 0 \end{pmatrix}.$$

The concept of optimal control of three-level quantum systems were presented in [3, 4, 13, 32]. In the most of these works, the optimal control is constructed on the basis of geometric arguments. If we suppose $E_1 = 2$, $E_2 = 0$, $E_3 = 6$, $v_{12} = i + 1$, $v_{13} = 4$, and $v_{23} = 2 + 3i$, then we can obtain the numerical results as follows:

Table 3: Approximate and exact values of $u(t)$ in Example 2 for different values of t.

t	approximate of $u(t)$	exact values of $u(t)$	error
2	0.0074	0.0074	3.3×10^{-4}
10	0.0029	0.0029	5.2×10^{-4}
20	-0.0107	-0.0105	2.0×10^{-4}
30	-0.0002	-0.0001	1.1×10^{-4}
40	0.0062	0.0062	3.3×10^{-4}
50	-0.0032	-0.0032	2.3×10^{-4}
60	0.0060	0.0063	4.1×10^{-4}

Table 4: Approximate and exact values of Ψ in Example 2 for different values of t.

t	approximate of $\psi_1(t)$	approximate of $\psi_2(t)$	approximate of $\psi_3(t)$	error of $\psi_1(t)$	error of $\psi_2(t)$	error of $\psi_3(t)$
2	1.0188	-0.1011	0.0731	0.0200	0.0220	0.0220
10	0.9837	-0.3944	0.2748	0.0140	0.0154	0.0220
20	0.9016	-0.6478	0.5012	0.0210	0.0141	0.0123
30	0.7168	-0.7977	0.6782	0.0220	0.0215	0.0144
40	0.5713	-0.6806	0.8288	0.0180	0.0125	0.0214
50	0.3356	-0.4449	0.9638	0.0000	0.0112	0.0124
60	0.0799	-0.0525	0.9767	0.0225	0.0114	0.0126

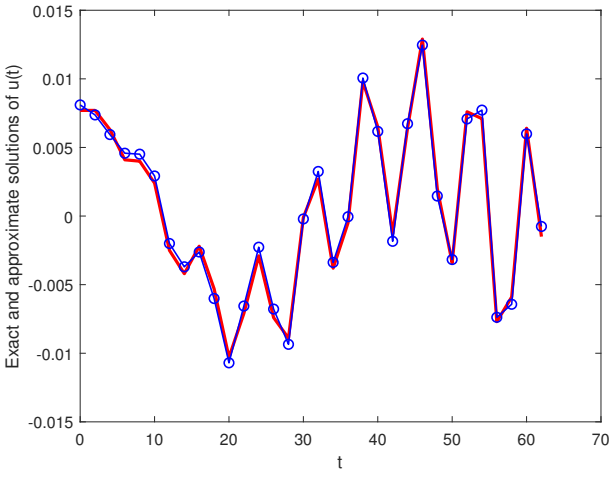

Figure 4: Plots of approximate and exact results of control variable for Example 2

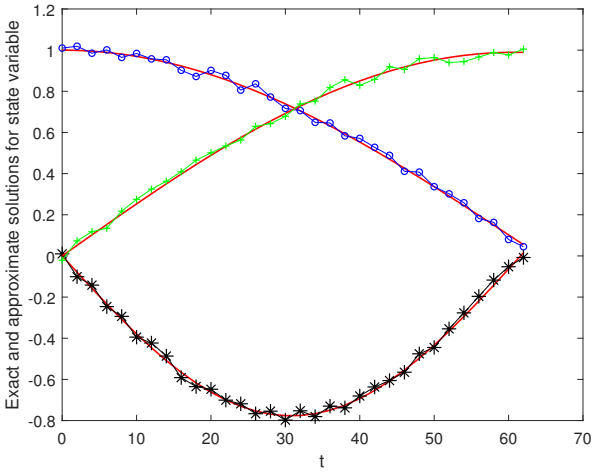

Figure 5: Plots of the approximate and exact results of state variable for Example 2

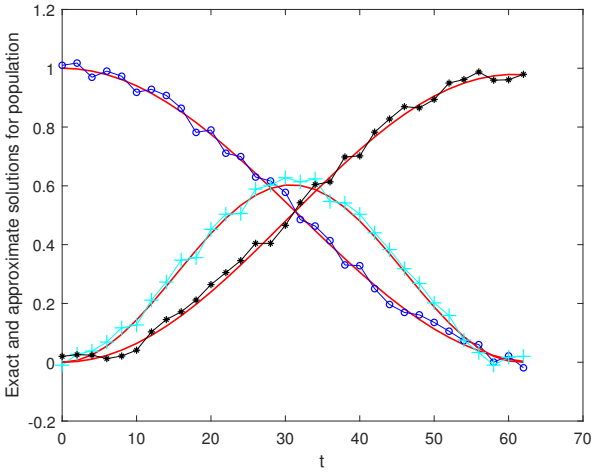

Figure 6: Plots of the approximate and exact results of population for Example 2

We solved the above problem by our proposed algorithm with $n = 2$ and $K = 10$ or $\hat{k} = 40$. The stopping condition is $|u^*(t) - u(t)| < 5 \times 10^{-4}$, where $u^*(t)$ and $u(t)$ are approximate and exact results of control variable, respectively, and by the stopping condition, we have 97 iterations. The plot of the approximate and exact values of $u(t)$ is shown in Figure 4. The plots

of the approximate and exact values of $\psi_1(t)$, $\psi_2(t)$, and $\psi_3(t)$ are shown in Figure 5. The plots of the approximate and exact values of $|\psi_1(t)|^2$, $|\psi_2(t)|^2$ and $|\psi_3(t)|^2$ are shown in Figure 6.
Approximation and exact results of control variable $u(t)$ and their absolute errors for different t are presented in Table 3. Approximation and exact results of state variables $\psi_1(t)$, $\psi_2(t)$, and $\psi_3(t)$ and their absolute errors for different t are presented in Table 4. Figures and tables obtained above show that the numerical method used in this paper is simpler and more useful than other methods. In [32], an index similar to the index used in this article has been used, and numerical and graphical results have been obtained. Investigating these results carefully, it can be observed that the solutions obtained in this article endure less repetition while having more accuracy.

6 Conclusion

Today, the issue of quantum optimal control is one of the most widely used issues in many basic sciences and engineering. At the same time, numerical methods are one of the most useful solutions to these problems. In this paper, a numerical method based on wavelets was proposed to solve the problem of optimal quantum control. This method was benefited by using topics related to applied mathematics in the field of classical equations and presented usable in the field of quantum systems. The above results showed that using the algorithm and method mentioned numerically is very useful and efficient. The merit of this method is to provide a simple solution based on classical numerical methods in the field of optimal quantum control.

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