Iranian Journal of Numerical Analysis and Optimization
Vol. 10, No. 2, (2020), pp 197–221
DOI:10.22067/ijnao.v10i2.86884

Research Article

On the numerical solution of optimal control problems via Bell polynomials basis

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Abstract

We present a new numerical approach to solve the optimal control problems (OCPs) with a quadratic performance index. Our method is based on the Bell polynomials basis. The properties of Bell polynomials are explained. We also introduce the operational matrix of derivative for Bell polynomials. The chief feature of this matrix is reducing the OCPs to an optimization problem. Finally, we discuss the convergence of the new technique and present some illustrative examples to show the effectiveness and applicability of the proposed scheme. Comparison of the proposed method with other previous methods shows that this method is accurate.

AMS(2010): Primary 49J15; Secondary 65M70.

Keywords: Optimal control problems; Bell polynomial; Best approximation; Operational matrix of derivative.

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Received 14 May 2020 ; revised 10 August 2020 ; accepted 31 August 2020

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1 Introduction

Researchers believe that the analytical solution of optimal control problems (OCPs) is usually complicated, or, most of the time, impossible [14]. Hence we use numerical methods to solve them. The first attempt to solve OCPs was applying numerical methods to related Bellman and Ponteryagin in 1950; see [16]. The main problem in numerical methods is the accuracy of computations and the amount of manipulations.

In recent years, some studies have been done in this regard, which we will refer only [1, 7, 8, 9, 11, 18, 20, 21, 25]. The investigation of OCPs and solving them by using the Bezier polynomials were proposed by Yari, Mirnia, and Lakestani [25]. To be more specific, they chose Bezier polynomials as basic functions and considered the state vector $x(t)$ and control vector $u(t)$ as Bezier polynomials with unknown coefficients. Grigoryev, Mustafina, and Larin [9] proposed the numerical solution of OCPs, using a successive approximations method. Ramezani [18] presented the numerical solution of OCPs, using the second kind Chebyshev wavelet. Rose [20] described numerical methods to resolve control issues. Kafash and his colleagues [11] proposed a numerical approach for solving OCPs, using the Boubaker polynomials expansion scheme.

In the present study, we offer a numerical solution for the linear constrained quadratic OCPs via Bell polynomials basis. The Bell polynomials were studied extensively by Bell [4] in 1934. These polynomials naturally occur from differentiating a composite function several times. Bell polynomials have many applications in number theory and classical analysis, and there is a vast literature about their applications. They are frequently applied in combinatorial analysis and statistics. Also, these polynomials have been used in many other contexts such as the Blissard problem; see [15].
This method relies on approximating the state and control variables with Bell polynomials. It includes reducing the OCPs to an optimization one by first expanding the state rate $\dot{x}(t)$ and the control $u(t)$ as a Bell polynomial with unknown coefficients. The operational matrix of differentiation D_ϕ is given to approximate the differential part of the problem and the performance index.

This paper is arranged as follows. In Section 2, we present some important preliminaries needed for our subsequent development and approximation of a function by using the Bell polynomials basis. Section 3 is devoted to the numerical method of solving the OCPs. In Section 4, we discuss the proposed method for solving the OCPs. In Section 5, we present the convergence of the proposed method. In Section 6, we consider the applicability as well as the accuracy of the proposed technique with several examples. Finally, the main results are summarized in Section 7.

2 Preliminaries

In this section, we present some preliminaries needed in some other parts of the paper.

2.1 Bell polynomials and their properties

The incomplete exponential Bell polynomials are given as

$$B_{m,n}(t_1,\dots,t_{m-n+1}) = \sum_{\pi(m,n)} \frac{m!}{d_1!\dots d_{m-n+1}!}(\frac{t_1}{1!})^{d_1}\dots(\frac{t_{m-n+1}}{(m-n+1)!})^{d_{m-n+1}},$$

where $m,n \in Z^+,\ n \le m$, and the summation takes place over all integers $d_1, d_2, \dots, d_{m-n+1} \ge 0$, such that

$$\begin{aligned} &d_1 + d_2 + \dots + d_{m-n+1} = n, \\ &d_1 + 2d_2 + \dots + (m+n-1)d_{m-n+1} = m. \end{aligned}$$

The sum

$$B_m(t_1,t_2,\dots,t_m) = \sum_{n=1}^{m} B_{m,n}(t_1,t_2,\dots,t_{m-n+1}),$$

is called the mth complete exponential Bell polynomial [4].

Suppose that $t_1 = t_2 = \dots = t_m = t$, and let $B_m(t,t,\dots,t) = B_m(t)$, then

$$B_m(t) = \sum_{n=0}^{m} \{m,n\} t^n, \tag{1}$$

is called the complete Bell polynomial (Touchard polynomial), where

$$\{m,n\} = \frac{1}{n!}\sum_{j=0}^{n}(-1)^j \binom{n}{j}(n-j)^m,$$

are called Sterling numbers of the second kind and some properties of them are as follows (see [4, 23]):

(1) $\{m,0\} = \delta_{m,0}$,

(2) $\{m,1\} = 1$,

(3) $\{m,2\} = 2^{m-1} - 1$,

(4) $\{m,n\} = n\{m-1,n\} + \{m-1,n-1\}$,

(5) $\{m,m\} = 1$,

(6) $\{m, n\} = 0, n > m$.

Some sterling numbers of the second kind are given in Table 1:

Table 1: Sterling numbers of the second kind

m	$\{m,0\}$	$\{m,1\}$	$\{m,2\}$	$\{m,3\}$	$\{m,4\}$	$\{m,5\}$	$\{m,6\}$	$\{m,7\}$	$\{m,8\}$	$\{m,9\}$
0	1									
1	0	1								
2	0	1	1							
3	0	1	3	1						
4	0	1	7	6	1					
5	0	1	15	25	10	1				
6	0	1	31	90	65	15	1			
7	0	1	63	301	350	140	21	1		
8	0	1	127	966	1701	1050	266	28	1	
9	0	1	255	3025	7770	6951	2646	462	36	1

The first six Bell polynomials are as

$$
\begin{aligned}
B_0(t) &= 1, \\
B_1(t) &= t, \\
B_2(t) &= t^2 + t, \\
B_3(t) &= t^3 + 3t^2 + t, \\
B_4(t) &= t^4 + 6t^3 + 7t^2 + t, \\
B_5(t) &= t^5 + 10t^4 + 25t^3 + 15t^2 + t, \\
B_6(t) &= t^6 + 15t^5 + 65t^4 + 90t^3 + 31t^2 + t.
\end{aligned}
$$

Some properties of Bell polynomials are as follows (see [4, 5, 6]):

(1) $B_m(1) = B_m$ (Bell number),

(2) $B_m(-1) = \widetilde{B}_m$ (complement of Bell number),

(3) for all $m, n \in Z^+$ and $n \le m$,

1.
$$B_m(t) = e^{-t} \sum_{n=0}^{\infty} \frac{n^m}{n!} t^n,$$

2.
$$(t\frac{d}{dt})^j e^{at} = \sum_{i=0}^{j} \{j, i\} t^j (\frac{d}{dt})^j e^{at} = B_j(at) e^{at},$$

3.
$$B_{m+1}(t) = t(B_m(t) + B'_m(t)), \tag{2}$$

4.
$$t\frac{d}{dt}B_m(t)e^t = B_{m+1}(t)e^t,$$

5.
$$t\frac{d}{dt}B_m(t)e^t = t(B_m(t) + B'_m(t))e^t,$$

6.
$$B_{m+1}(t) = t\sum_{n=0}^{m}\binom{m}{n}B_n(t), \tag{3}$$

7.
$$B_{m+1}(p+q) = \sum_{n=0}^{m}\binom{m}{n}B_n(p)B_{m-n}(q).$$

Theorem 1.[semi-orthogonality of $B_m(t)$] For all $m, n \in N$, we have

$$\int_0^\infty B_m(-t)B_n(-t)\frac{e^{-2t}}{t}dt = (-1)^{m-1}\frac{2^{m+n}-1}{m+n}\mathcal{B}_{m+n}, \tag{4}$$

where $\mathcal{B}_{m+n}$ are the Bernoulli numbers. Note that the right-hand side of (4) is zero, when $m+n$ is odd. Also all Bernoulli numbers $\mathcal{B}_k$ with indices odd and larger than one, are zero.

Proof. For the proof, see [5]. □

We can express the standard basis $\{1, t, t^2, \ldots, t^m\}$ in terms of the Bell polynomials as follows:

$$\begin{aligned}
1 &= B_0(t),\\
t &= B_0(t) + B_1(t),\\
t^2 &= B_0(t) - B_1(t) + B_2(t),\\
t^3 &= B_0(t) + 2B_1(t) - 3B_2(t) + B_3(t),\\
t^4 &= B_0(t) - 6B_1(t) + 11B_2(t) - 6B_3(t) + B_4(t).
\end{aligned}$$

We expand t^m in terms of Bell polynomials as follows:

$$t^m = \sum_{n=0}^{m}(-1)^{m-n}\,[m,n]\,B_n(t), \tag{5}$$

where $[m, n]$ are called Sterling numbers of the first kind and given as follows (see [22]):

$$[m,n] = \sum_{0\le i_1<\cdots<i_{m-n}<m} i_1 i_2 \dots i_{m-n} = \sum_{i_1=0}^{m-1}\sum_{i_2=0}^{i_1-1}\cdots\sum_{i_{m-n}=0}^{i_{m-n-1}-1} i_1 i_2 \dots i_{m-n}.$$

We can write (5) as in the following way:

$$t^j = d_j B_j(t), \qquad j = 0,1,2,\dots,m-1,$$

where $d_j,\ j = 0,1,2,\dots,m-1$, are constant vectors of order $1\times(m+1)$ and are given as

$$d_j = \sum_{i=0}^{j}(-1)^{j-i}\,[j,i]\,.$$

Some properties of Sterling numbers of the first kind are as follows (see [4, 23]):

(1) $[m,0] = \delta_{m,0}$,

(2) $[m,1] = (n-1)!$,

(3) $[m,n] = (m-1)\,[m-1,n] + [m-1,n-1]\,$,

(4) $[m,m] = 1$,

(5) $[m,n] = 0, n > m$.

Some sterling numbers of the first kind are given in Table 2:

Table 2: Sterling numbers of the first kind

m	$[m,1]$	$[m,1]$	$[m,2]$	$[m,3]$	$[m,4]$	$[m,5]$	$[m,6]$	$[m,7]$	$[m,8]$	$[m,9]$
0	1									
1	0	1								
2	0	1	1							
3	0	2	3	1						
4	0	6	11	6	1					
5	0	24	50	35	10	1				
6	0	120	274	225	85	15	1			
7	0	720	1764	1624	735	175	21	1		
8	0	5040	13068	13132	6769	1960	322	28	1	
9	0	40320	109584	118124	67284	22249	4536	546	36	1

2.2 The operational matrix of the Bell polynomials

In this section, we define the operational matrix of derivative for the Bell polynomials. Suppose that $\Phi_m(t_0)$ on $[\alpha,\beta]$ is defined as

$$\Phi_m(t_0) = [B_0(t_0), B_1(t_0), \dots, B_m(t_0)]^T. \tag{6}$$

Then we have

$$\Phi'_m(t_0) = D_\phi \Phi_m(t_0), \tag{7}$$

where D_ϕ is the $(m+1) \times (m+1)$ operational matrix of derivative for the Bell polynomials and it is given as follows:

$$D_\phi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots & \dots & 0 \\ 1 & 0 & 0 & 0 & 0 & \dots & \dots & 0 \\ 1 & 2 & 0 & 0 & 0 & \dots & \dots & 0 \\ 1 & 3 & 3 & 0 & 0 & \dots & \dots & 0 \\ 1 & 4 & 6 & 4 & 0 & \dots & \dots & 0 \\ 1 & 5 & 10 & 10 & 5 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ \binom{m}{0} & \binom{m}{1} & \binom{m}{2} & \binom{m}{3} & \binom{m}{4} & \dots & \binom{m}{m-1} & 0 \end{bmatrix}.$$

Theorem 2. For all $i, j \in Z^+$ and $i \le j$, we have

$$B'_j(t_0) = \sum_{i=0}^{j-1} \binom{j}{i} B_i(t_0).$$

Proof. Using (2) and (3), we have

$$t_0(B_j(t_0) + B'_j(t_0)) = t_0 \sum_{i=0}^{j} \binom{j}{i} B_i(t_0);$$

then

$$B_j(t_0) + B'_j(t_0) = \sum_{i=0}^{j} \binom{j}{i} B_i(t_0).$$

Therefore

$$B'_j(t_0) = \sum_{i=0}^{j} \binom{j}{i} B_i(t_0) - B_j(t_0) = \sum_{i=0}^{j-1} \binom{j}{i} B_i(t_0) + \binom{j}{j} B_j(t_0) - B_j(t_0),$$

so that

$$B'_j(t_0) = \sum_{i=0}^{j-1} \binom{j}{i} B_i(t_0).$$

□

2.3 Function approximation

Suppose that $H = L^2([\alpha, \beta])$ and that $Y = span\{B_0(t), B_1(t), \ldots, B_m(t)\}$. Since Y is a finite-dimensional and closed subspace, so Y is a complete subspace of H; see [12]. Thus for an arbitrary element $h(t)$ in H, there exists a unique best approximation out of Y such as $h_0(t)$, such that

$$\text{for all } g(t) \in Y, \quad \parallel h(t) - h_0(t) \parallel \leq \parallel h(t) - g(t) \parallel .$$

Since $h_0(t) \in Y$, there exist the unique coefficients $c_0, \ldots, c_m$, such that

$$h(t) \simeq h_0(t) = \sum_{j=0}^{m} c_j B_j(t) = C^T \Phi_m(t), \tag{8}$$

where C and $\Phi_m(t)$ are given by

$$\Phi_m(t_0) = [B_0(t_0), B_1(t_0), \ldots, B_m(t_0)]^T, \quad C = [c_0, c_0, \ldots, c_m]^T,$$

and T denotes the transposition. For computing the coefficients c_j, we let

$$k_i = \langle h(t), B_i(t) \rangle = \int_{\alpha}^{\beta} h(t) B_i(t) dt, \quad i = 0,\, 1,\, 2,\, \ldots,\, m.$$

Hence, using (8), we have

$$\begin{aligned} k_i &= \int_{\alpha}^{\beta} h(t) B_i(t) dt = \int_{\alpha}^{\beta} \sum_{j=0}^{m} c_j B_j(t) B_i(t) = \sum_{j=0}^{m} c_j \int_{\alpha}^{\beta} B_j(t) B_i(t) \\ &= \sum_{j=0}^{m} c_j \langle B_j(t), B_i(t) \rangle dt = \sum_{j=0}^{m} c_j r_{ji} dt, \quad i = 0,\, 1,\, 2,\, \ldots,\, m, \end{aligned} \tag{9}$$

where

$$r_{ji} = \langle B_j(t), B_i(t) \rangle = \int_{\alpha}^{\beta} B_j(t) B_i(t) dt,$$

and $\langle \cdot, \cdot \rangle$ denotes the inner product. Let

$$R = [r_{ji}]_{(m+1)\times(m+1)}, \quad K = [k_1, k_2, \ldots, k_m]^T.$$

The matrix R is called the dual matrix of Φ_m, which can be obtained as follows:

$$R_{i+1,j+1} =< B_i, B_j >= \int_{\alpha}^{\beta} B_i(t) B_j(t) dt = \int_{\alpha}^{\beta} \sum_{p=0}^{i} \{i,p\} t^p \sum_{q=0}^{j} \{j,q\} t^q dt$$

$$= \sum_{p=0}^{i} \sum_{q=0}^{j} \{i,p\}\{j,q\} \frac{\beta^{p+q+1} - \alpha^{p+q+1}}{p+q+1}.$$

From (9), we have

$$K^T = C^T R \ \Rightarrow C^T = K^T R^{-1}.$$

Therefore, any function $h(t) \in L^2([\alpha, \beta])$ can be expanded in terms the Bell polynomials as

$$h(t) \simeq C^T \Phi_m(t), \tag{10}$$

where

$$C^T = \langle h(t), \Phi_m(t) \rangle R^{-1}.$$

Also, the Bell polynomials form a complete basis over the interval $[\alpha, \beta]$. Because using (1), we write

$$\Phi_m(t) = SX(t), \tag{11}$$

where

$$\Phi_m(t) = [B_0(t), B_1(t), \ldots, B_m(t)]^T, \quad X(t) = [1, t, t^2, \ldots, t^m]^T,$$

and S is a lower triangular matrix with nonzero diagonal elements. Since from [15] the matrix S is nonsingular, hence S^{-1} exists. Let

$$\sum_{j=0}^{m} c_j B_j(t) = C^T \Phi_m(t) = 0, \tag{12}$$

where $C = [c_0, c_0, \ldots, c_m]^T$. using (11) and (12), we have

$$C^T \Phi_m(t) = C^T SX(t) = 0. \tag{13}$$

According to the fundamental theorem of algebra, $X(t) = \{1, t, t^2, \ldots, t^m\}$ forms a basis; see [12]. Hence they are a linearly independent set. Therefore in (13), we have

$$C^T S = 0 \ \ \to C^T = [c_0, c_0, \ldots, c_m] = 0;$$

hence $\Phi_m(t)$ is a linearly independent set. Therefore using (10), the Bell polynomials form a complete basis in $[\alpha, \beta]$.

3 Problem statement

In this section, we discuss the numerical solution of the OCPs. Consider the nonlinear system [17]

$$\dot{x}(t) = \mathcal{A}(t)x(t) + \mathcal{B}(t)u(t), \tag{14}$$

$$x(\alpha) = x^0,\ x(\beta) = x^1, \quad t \in [\alpha, \beta], \tag{15}$$

where $[\mathcal{A}(t)]_{m\times m}$ and $[\mathcal{B}(t)]_{n\times n}$ are matrix functions and $[x(t)]_{m\times 1}$ and $[u(t)]_{n\times 1}$ are state and control vectors, respectively. The purpose is finding the optimal control $u(t)$ and the corresponding state trajectory $x(t)$, which satisfy (14) and (15) while maximize (or minimize) the quadratic performance index

$$\mathcal{Z} = \frac{1}{2}x^T(\beta)\mathcal{G}(t)x(\beta) + \frac{1}{2}\int_\alpha^\beta (x^T(t)\mathcal{Q}(t)x(t) + u^T(t)\mathcal{R}(t)u(t))dt, \tag{16}$$

where $[\mathcal{G}(t)]_{m\times m}$ and $[\mathcal{Q}(t)]_{m\times m}$ are symmetric positive semi-definite matrices and $[\mathcal{R}(t)]_{n\times n}$ is a symmetric positive definite matrix.

4 The proposed method

In this section, we discuss a new approach to solve the OCPs by using Bell polynomials. Let

$$x_i(t_0) \simeq \Phi_m^T(t_0)X^i, \quad i = 1, \ldots, m, \tag{17}$$

$$u_j(t_0) \simeq \Phi_m^T(t_0)U^j, \quad j = 1, \ldots, n, \tag{18}$$

where $\Phi_m(t_0)$ is defined in (6) and $\left[X^i\right]_{(m+1)\times 1}$ and $\left[U^j\right]_{(m+1)\times 1}$ are state and control coefficient vectors, respectively. Then by using (7), we get

$$\dot{x}_i(t_0) \simeq [D_\phi \Phi_m(t_0)]^T X^i,$$
$$\dot{u}_j(t_0) \simeq [D_\phi \Phi_m(t_0)]^T U^j.$$

Using (17) and (18), we have

$$x(t_0) \simeq [\Phi_n^T(t_0)X]^T = [\sum_{i=0}^m B_i(t_0)X_i^1, \ldots, \sum_{i=0}^m B_i(t_0)X_i^m]^T, \tag{19}$$

$$u(t_0) \simeq [\Phi_m^T(t_0)U]^T = [\sum_{j=0}^m B_j(t_0)U_j^1, \ldots, \sum_{j=0}^m B_j(t_0)U_j^n]^T, \tag{20}$$

where $X = (X_i^r)_{(m+1)\times m}$ and $U = (U_j^s)_{(m+1)\times n}$ are state and control coefficient matrices, respectively. The boundary conditions in (15) can be rewritten as

$$\begin{aligned} x(\alpha) &= x^0 = d^0 \otimes E\Phi_m(t_0), \\ x(\beta) &= x^1 = d^1 \otimes E\Phi_m(t_0), \end{aligned} \tag{21}$$

where $\left[d^0\right]_{m\times 1}$, $\left[d^1\right]_{m\times 1}$ and $E = [1, \ldots, 1]_{1\times(m+1)}$ are constant vectors, and the symbol $\otimes$ denotes the Kronecker product [13]. If $x(\alpha)$ or $x(\beta)$ is unknown in (15), then we put

$$\begin{aligned} x(\alpha) &\simeq [\Phi_m^T(\alpha)X]^T = [\sum_{i=0}^{m} B_i(\alpha)X_i^1, \ldots, \sum_{i=0}^{m} B_i(\alpha)X_i^m]^T, \\ x(\beta) &\simeq [\Phi_m^T(\beta)X]^T = [\sum_{j=0}^{m} B_j(\beta)X_j^1, \ldots, \sum_{j=0}^{m} B_j(\beta)X_j^n]^T. \end{aligned} \tag{22}$$

4.1 Performance index approximation

By substituting (19), (20), and (21) in (16), we obtain

$$\begin{aligned} \max(\min)\mathcal{Z} =& \frac{1}{2}{x^\beta}^T \mathcal{G}(\beta)x^\beta + \frac{1}{2}X^T[\int_\alpha^\beta \Phi_m^T(t)\mathcal{Q}(t)\Phi_m(t)dt]X \\ &+ \frac{1}{2}U^T[\int_\alpha^\beta \Phi_m^T(t)\mathcal{R}(t)\Phi_m(t)dt]U. \end{aligned} \tag{23}$$

Let

$$P_x^* = \int_\alpha^\beta \Phi_m^T(t)\mathcal{Q}(t)\Phi_m(t)dt \quad \text{and} \quad P_u^* = \int_a^\beta \Phi_m^T(t)\mathcal{R}(t)\Phi_m(t)dt. \tag{24}$$

By replacing (22) and (24) in (23), we obtain

$$\mathcal{Z}[X,U] = \frac{1}{2}X^T(P^* + P_x^*)X + \frac{1}{2}U^T P_u^* U, \tag{25}$$

where

$$P^* = \Phi_m^T(\beta)\mathcal{G}(\beta)\Phi_m(\beta).$$

The boundary conditions in (15) can be expressed as

$$q_k^0 = x_k(a) - x_k^0 \quad , \quad k = 1, \ldots, m, \tag{26}$$

$$q_k^1 = x_k(b) - x_k^1 \quad , \quad k = 1, \ldots, m. \tag{27}$$

We now find the extremum of (25) subject to (26) and (27) by using the Lagrange multiplier technique. Let

$$\mathcal{Z}[X, U, \lambda^0, \lambda^1] = \mathcal{Z}[X, U] + \lambda^0 \mathcal{Q}^0 + \lambda^1 \mathcal{Q}^1, \tag{28}$$

where $\mathcal{Q}^0 = \left[q_k^0\right]_{m\times 1}$ and $\mathcal{Q}^1 = \left[q_k^1\right]_{m\times 1}$ are constant vectors. The necessary condition for the extremum of (28), is

$$\nabla \mathcal{Z}[X, U, \lambda^0, \lambda^1] = 0.$$

5 Convergence analysis

In this section, we present the convergence of the proposed method.

Theorem 3. Suppose that the function $h : [\alpha, \beta] \to R$ is $m+1$ times continuously differentiable (i.e., $f \in C^{m+1}[\alpha, \beta]$) and that S_m=span$\{\Phi_m(t)\}$. If $C^T B$ is the best approximation of h out of S_m, then

$$\|h - C^T B\|_{L^2[\alpha,\beta]} \leq \frac{\hat{U}}{(m+1)!}\sqrt{\frac{\beta^{2m+3} - \alpha^{2m+3}}{2m+3}},$$

where $\hat{U}$=max $|h^{(m+1)}(t)|$, $t \in [\alpha, \beta]$.

Proof. We know that, the set $\{1, t, t^2, \dots, t^m\}$ is a basis for polynomials space of degree m. Considering

$$y_1(t) = h(\alpha) + t h'(\alpha) + \cdots + \frac{t^m}{m!} h^m(\alpha),$$

and applying Taylor expansion, we have

$$|h(t) - y_1(t)| = |h^{(m+1)}(\xi_t)\frac{t^{(m+1)}}{(m+1)!}|, \quad \xi_t \in (\alpha, \beta). \tag{29}$$

Since $C^T B$ is the best approximation of h out of S_m and $y_1(t) \in S_m$, using (29), we obtain

$$\begin{aligned}
\|h - C^T B\|^2_{L^2[\alpha,\beta]} \leq \|h - y_1\|^2_{L^2[\alpha,\beta]} &= \int_\alpha^\beta |h(t) - y_1(t)|^2 dx \\
&= \int_\alpha^\beta |h^{(m+1)}(\xi_t)|^2 (\frac{t^{(m+1)}}{(m+1)!})^2 dt \\
&\leq (\frac{\hat{U}}{(m+1)!})^2 \int_\alpha^\beta t^{2m+2} dt = (\frac{\hat{U}}{(m+1)!})^2 \frac{\beta^{2m+3} - \alpha^{2m+3}}{2m+3},
\end{aligned}$$

so

$$\|h - C^T B\|_{L^2[\alpha,\beta]} \le \frac{\hat{U}}{(m+1)!}\sqrt{\frac{\beta^{2m+3}-\alpha^{2m+3}}{2m+3}}.$$

Also, we can write

$$\frac{\hat{U}}{(m+1)!}\sqrt{\frac{\beta^{2m+3}-\alpha^{2m+3}}{2m+3}} = \frac{\hat{U}}{(m+1)!}\sqrt{\frac{\beta^{2m+3}(1-(\frac{\alpha}{\beta})^{2m+3})}{2m+3}}. \tag{30}$$

Therefore, using (30), we have

$$\|h - C^T B\|_{L^2[\alpha,\beta]} \le \frac{\hat{U}}{(m+1)!}\sqrt{\frac{\beta^{2m+3}(1-(\frac{\alpha}{\beta})^{2m+3})}{2m+3}} \approxeq \frac{\hat{U}\beta^m}{(m+1)!}\sqrt{\frac{b^3}{2m+3}},$$

which shows that the error vanishes as $m \to \infty$. □

6 Illustrative examples

To illustrate the proposed technique, we solve some numerical examples and make a comparison with some of the results in the literature in this section. We implemented our method with MATLAB.

Example 1. This example has been taken from [25]. Consider

$$\min \mathcal{Z} = \int_0^1 (3x^2(t) + u^2(t))dt, \tag{31}$$

subject to

$$\dot{x}(t) = x(t) + u(t),\ x(0) = 1.$$

We solve the problem (31), for $m = 3$. Let

$$x_{app}(t) = X\Phi_3(t), \quad u_{app}(t) = U\Phi_3(t),$$

we obtain

$$X = [1, -\frac{905}{236}, \frac{285}{118}, -\frac{35}{118}],\ U = [-\frac{641}{236}, \frac{1835}{236}, -\frac{195}{59}, \frac{35}{118}].$$

Then, the approximate solutions are as follows:

$$\begin{aligned} x_{app}(t) &= B_0 - \frac{905}{236}B_1 + \frac{285}{118}B_2 - \frac{35}{118}B_3 \\ &= -\frac{35}{118}t^3 + \frac{90}{59}t^2 - \frac{405}{236}t + 1, \end{aligned} \tag{32}$$

$$\begin{aligned} u_{app}(t) &= -\frac{641}{236}B_0 + \frac{1835}{236}B_1 - \frac{195}{59}B_2 + \frac{35}{118}B_3 \\ &= \frac{35}{118}t^3 - \frac{285}{118}t^2 + \frac{1125}{236}t - \frac{641}{236}. \end{aligned} \tag{33}$$

The analytical solutions are as follows:

$$x_{exa}(t) = \frac{3e^{-4}}{3e^{-4}+1}e^{2t} + \frac{1}{3e^{-4}+1}e^{-2t}, \tag{34}$$

$$u_{exa}(t) = \frac{3e^{-4}}{3e^{-4}+1}e^{2t} - \frac{3}{3e^{-4}+1}e^{-2t}, \tag{35}$$

and the optimal value of objective function is $\mathcal{Z}_{exa} = 2.791659975310063$.

In Table 3, we compare the absolute errors of objective function $\mathcal{Z}$ of the proposed method with methods of [25, 24] for different values of m. Obviously, the estimated results of our proposed scheme are coincided with the results in [25, 24]. Figures 1 and 2 show, respectively, plots of errors for state and control functions for Example 1, by using the presented method.

Table 3: Comparing absolute errors of $\mathcal{Z}$ of the proposed method with the methods of [24, 25] for different values of m for Example 1.

m	proposed method	model of [25]	model of [24]
5	1.9512×10^{-15}	1.9512×10^{-15}	1.9512×10^{-15}
6	1.3590×10^{-10}	1.3590×10^{-10}	1.3590×10^{-10}
7	3.7582×10^{-12}	3.7582×10^{-12}	3.7582×10^{-12}
8	1.9512×10^{-15}	2.0000×10^{-15}	2.0228×10^{-15}

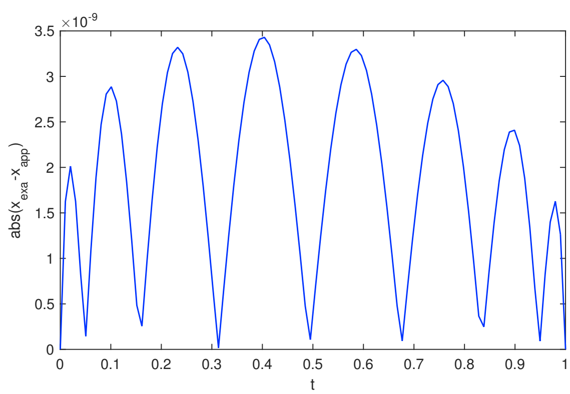

Figure 1: Absolute error for the state function $x(t)$ for Example 1, $m = 8$.

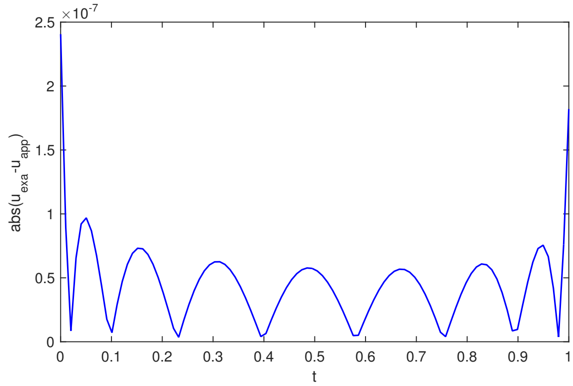

Figure 2: Absolute error for the control function $u(t)$ for Example 1, $m = 8$.

Example 2. This example has been adapted from [19]. Consider

$$\min \mathcal{Z} = \int_0^1 (x_2(t) + u^2(t))dt, \tag{36}$$

subject to

$$\dot{x}_1(t) = x_2(t), \;\; x_1(0) = 0, \;\; x_1(1) = 1, \;\; \dot{x}_2(t) = u(t), \;\; x_2(0) = 0.$$

We solve the problem (36), for $m = 3$. Let

$$x^1_{app}(t) = \Phi_3^T(t)X_1, \; x^2_{app}(t) = \Phi_3^T(t)X_2, \; u_{app}(t) = \Phi_3^T(t)U.$$

we obtain

$$X_1 = [0, -\frac{5}{2}, 3, -\frac{1}{2}], \;\; X_2 = [0, \frac{9}{2}, -\frac{3}{2}, 0], \;\; U = [3, -3, 0, 0].$$

Then, the approximate solutions are as follows:

$$\begin{aligned} x^1_{app}(t) &= -\frac{5}{2}B_1 + 3B_2 - \frac{1}{2}B_3 = \frac{3}{2}t^2 - \frac{1}{2}t^3 \\ x^2_{app}(t) &= \frac{9}{2}B_1 - \frac{3}{2}B_2 = 3t - \frac{3}{2}t^2, \\ u_{app}(t) &= 3B_0 - 3B_1 = 3 - 3t. \end{aligned}$$

The analytical solutions are as follows:

$$x^1_{exa}(t) = \frac{3}{2}t^2 - \frac{1}{2}t^3, \quad x^2_{exa}(t) = 3t - \frac{3}{2}t^2, \quad u_{exa}(t) = 3 - 3t,$$

and the optimal value of objective function is $\mathcal{Z}_{exa} = 4$.

Example 3. In this example, extracted from [10], the vibration of a spring-mass-damper system subjected to an external force is considered. In particular, the response to harmonic excitations, impulses, and step forcing functions is examined. In many environments, rotation machinery, motors, and so on cause periodic motions of structures to induce vibrations into other mechanical devices and structures nearby. On summing the forces, the equation for the forced vibration of the system in Figure 3 becomes. It is common to approximate the driving forces, $F(t)$,

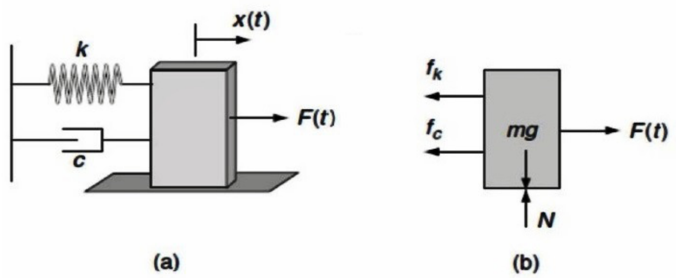

Figure 3: (a) Schematic of the forced-mass-damper system assuming no friction the surface and (b) free body diagram of the system of part (a) for Example 2.

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t),$$

where m, c, and k are fixed numbers and $F(t)$ represents the control force derived from the action of one actuator force represented by $F(t) = bu(t)$, where b is a fixed number. The linear regulator problem specific application in vibration suppression, the performance index for this problem is

$$\mathcal{Z} = \frac{1}{2}\int_{t_0}^{t_f} (x^2(t) + au^2(t))dt, \tag{37}$$

where t_0 and t_f are initial and final times, respectively. We introduce the usual state variable notation

$$x_1(t) = x(t), \quad x_2(t) = \dot{x}(t).$$

Then

$$\dot{x_1}(t) = x_2(t), \quad \dot{x_2}(t) = \ddot{x}(t) = -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{b}{m}.$$

The boundary conditions for this example are considered as

$x(0) = \dot{x}(0) = 1,\ \ x(1) = \dot{x}(1) = 0\ \ or\ \ x_1(0) = x_2(0) = 1,\ \ x_1(1) = x_2(1) = 0.$

We solve the problem (37), using Bell polynomials for $m = 3$, $t_0 = 0, t_f = 1, c = 2$, and $a = b = m = k = 1$. Let

$$x^1_{app}(t) = \Phi^T_3(t)X_1,\ x^2_{app}(t) = \Phi^T_3(t)X_2,\ u_{app}(t) = \Phi^T_3(t)U.$$

We obtain

$$X_1 = [1, 12, -14, 3],\ \ X_2 = [1, -19, 9, 0],\ \ U = [-7, -8, 4, 3].$$

Then, the approximate solutions are as follows:

$$\begin{aligned}
x^1_{app}(t) &= B_0 + 12B_1 - 14B_2 + 3B_3 = 3t^3 - 5t^2 + t + 1,\\
x^2_{app}(t) &= B_0 - 19B_1 + 9B_2 = 9t^2 - 10t + 1,\\
u_{app}(t) &= -7B_0 - 8B_1 + 4B_2 + 3B_3 = 3t^3 + 13t^2 - t - 7.
\end{aligned}$$

The analytical solutions are as follows:

$$\begin{aligned}
x^1_{exa}(t) =&[\frac{1393}{95}e^{-\frac{167}{152}t} + \frac{679}{219}e^{\frac{167}{152}t}]\sin(\frac{67}{167}t)\\
&+[\frac{1574}{423}e^{-\frac{167}{152}t} - \frac{1151}{423}e^{\frac{167}{152}t}]\cos(\frac{67}{167}t),\\
x^2_{exa}(t) =&[\frac{2838}{613}e^{-\frac{167}{152}t} - \frac{12872}{723}e^{\frac{167}{152}t}]\sin(\frac{67}{167}t)\\
&+[\frac{1021}{395}e^{-\frac{167}{152}t} - \frac{1151}{423}e^{\frac{626}{423}t}]\cos(\frac{67}{167}t),\\
u_{exa}(t) =&[-\frac{2931}{1145}e^{-\frac{167}{152}t} + \frac{15449}{851}e^{\frac{167}{152}t}]\sin(\frac{67}{167}t)\\
&+[\frac{2591}{1263}e^{-\frac{167}{152}t} + \frac{2326}{421}e^{\frac{167}{152}t}]\cos(\frac{67}{167}t),
\end{aligned}$$

and the optimal value of objective function is $\mathcal{Z}_{exa} = 13.00484054100315$.

Table 4 shows the errors of x_1, x_2, u and approximate values of objective function $\mathcal{Z}$, and Figures 4, 5, and 6 show the absolute errors for states of x_1, x_2 and control of u functions for Example 3, by using the presented method.

Table 4: Approximations of $\mathcal{Z}$ and errors of x_1, x_2, u, for Example 3.

m	$\mathcal{Z}_{app}$	errors of x_1	errors of x_2	errors of u
4	13.01222484276730	1.9070×10^{-3}	1.2649×10^{-2}	1.2145×10^{-1}
5	13.01900000000000	7.6740×10^{-6}	7.2401×10^{-5}	9.4197×10^{-4}
6	13.00484749962340	2.9364×10^{-6}	2.6746×10^{-5}	4.1151×10^{-4}
7	13.00484741498875	1.9166×10^{-6}	2.2206×10^{-6}	8.6153×10^{-6}

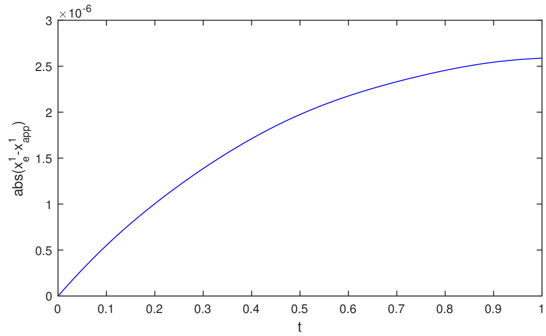

Figure 4: Absolute errors of x_1 for Example 3, $m = 7$.

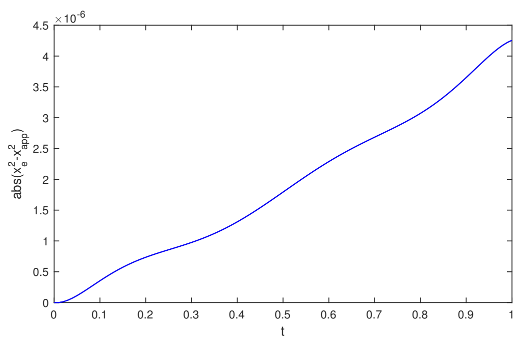

Figure 5: Absolute errors of x_2 for Example 3, $m = 7$.

Example 4. This example has been taken from [2]. Consider

$$\min \mathcal{Z} = \frac{1}{2}\int_0^1 (x^2(t) + u^2(t))dt,$$

subject to

$$\dot{x}(t) = tx(t) + u(t),\ \ x(0) = 1.$$

In Table 5, we compare the results of the optimal values of objective function $\mathcal{Z}$ of the proposed method with methods of [2, 3] for $m = 8$. Obviously, the estimated results for the minimum values of $\mathcal{Z}$ of the proposed method are more accurate than the results in models of [2, 3]. Figures 7 and 8 illustrate the behavior of state variable $x(t)$ and control variable $u(t)$ respectively, by using the presented method for $m = 8$.

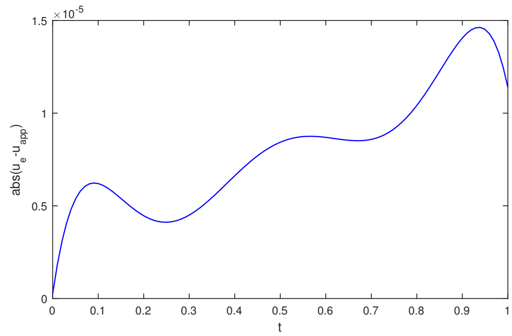

Figure 6: Absolute errors of u for Example 3, $m = 7$.

Table 5: Comparing approximate values of $\mathcal{Z}$ of the proposed method with the methods of [2, 3] for Example 4, $m = 8$.

m	proposed method	model of [2]	model of [3]
8	0.48426769622877	0.484268	0.48427

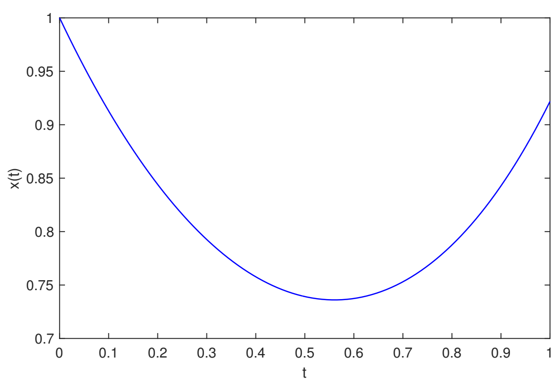

Figure 7: Behavior of $x(t)$ for Example 4, $m = 8$.

Example 5. This example has been adapted from [26]. Consider

$$\min \mathcal{Z} = \frac{1}{2}\int_0^1 (x_1^2(t) + x_2^2(t) + u^2(t))dt,$$

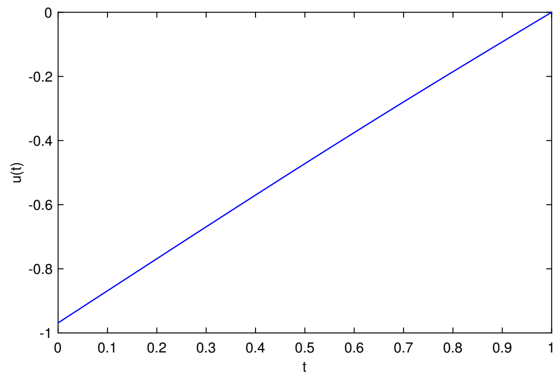

Figure 8: Behavior of $u(t)$ for Example 4, $m = 8$.

subject to

$$\begin{aligned}
&\dot{x}_1(t) = -x_1(t) + x_2(t) + u(t),\ x(0) = 1,\\
&\dot{x}_2(t) = -2x_2(t),\\
&x_1(0) = x_2(0) = 1.
\end{aligned}$$

The analytical solutions of this problem are as follows:

$$\begin{aligned}
x_1(t) &= \frac{-3}{2}e^{-2t} + 2.48165e^{-\sqrt{2}t} + 0.018352e^{\sqrt{2}t},\\
x_2(t) &= e^{-2t},\\
u_(t) &= \frac{1}{2}e^{-2t} - 1.027922e^{-\sqrt{2}t} + 0.044305e^{\sqrt{2}t},
\end{aligned}$$

and $\mathcal{Z} = 0.431988328237349 28111$. In Tables 6, 7, and 8, we compare the results of the absolute errors of $x_1(t)$, $x_2(t)$, and $u(t)$ of the proposed method with the method of [26] for different points of time and values of m, respectively. Obviously, the estimated results of the proposed method are more accurate than the results in [26]. Figures 9, 10, and 11 illustrate the behavior of state variables $x_1(t)$, $x_2(t)$ and control variable $u(t)$ by using the presented method for $m = 8$, respectively.

Table 6: Comparing absolute errors of $x_1(t)$ of the proposed method with the method of [26] for Example 5, different values of m.

	proposed method		model of [26]	
t	$m=4$	$m=6$	$m=4$	$m=6$
0.1	1.5215×10^{-4}	2.4791×10^{-6}	6.2840×10^{-5}	6.9470×10^{-6}
0.2	2.9911×10^{-5}	4.1683×10^{-7}	3.8617×10^{-4}	5.6627×10^{-6}
0.3	1.2214×10^{-4}	1.5184×10^{-6}	4.7516×10^{-4}	1.4208×10^{-6}
0.4	1.0726×10^{-4}	2.1591×10^{-6}	2.5623×10^{-4}	5.4834×10^{-7}
0.5	2.2672×10^{-5}	8.9684×10^{-7}	1.1356×10^{-4}	3.5375×10^{-6}
0.6	1.2331×10^{-4}	8.4596×10^{-8}	4.0262×10^{-4}	5.8476×10^{-6}
0.7	8.1944×10^{-5}	9.0533×10^{-7}	4.2904×10^{-4}	3.9087×10^{-6}
0.8	8.3818×10^{-5}	1.8245×10^{-6}	1.5915×10^{-4}	1.1543×10^{-7}
0.9	1.4779×10^{-4}	2.4645×10^{-7}	2.1417×10^{-4}	1.8431×10^{-6}

Table 7: Comparing absolute errors of $x_2(t)$ of the proposed method with the models of [26] for Example 5, different values of m.

	proposed method		model of [26]	
t	$m=4$	$m=6$	$m=4$	$m=6$
0.1	1.6805×10^{-4}	6.0534×10^{-7}	5.9651×10^{-5}	1.8022×10^{-6}
0.2	4.1122×10^{-5}	9.0967×10^{-7}	4.0570×10^{-4}	1.2818×10^{-6}
0.3	1.2262×10^{-4}	1.4305×10^{-7}	5.1060×10^{-4}	1.7540×10^{-6}
0.4	1.1136×10^{-4}	8.0849×10^{-7}	2.8780×10^{-4}	2.1099×10^{-6}
0.5	2.4713×10^{-5}	1.0834×10^{-7}	1.0392×10^{-4}	6.6340×10^{-7}
0.6	1.3364×10^{-4}	8.2323×10^{-7}	4.2076×10^{-4}	2.8671×10^{-6}
0.7	9.2876×10^{-5}	6.4668×10^{-8}	6.6499×10^{-4}	1.5552×10^{-6}
0.8	8.3272×10^{-5}	8.9367×10^{-7}	1.9246×10^{-4}	1.5283×10^{-6}
0.9	1.5574×10^{-4}	7.4550×10^{-7}	1.9951×10^{-4}	1.2855×10^{-6}

Table 8: Comparing absolute errors of $u(t)$ of the proposed method with the models of [26] for Example 5, different values of m.

	proposed method		model of [26]	
t	$m=4$	$m=6$	$m=4$	$m=6$
0.1	2.9430×10^{-5}	1.1055×10^{-6}	1.0466×10^{-5}	2.5978×10^{-6}
0.2	9.1300×10^{-7}	1.2850×10^{-6}	8.0579×10^{-5}	2.1213×10^{-6}
0.3	3.4541×10^{-5}	5.5935×10^{-7}	8.9374×10^{-5}	8.6406×10^{-7}
0.4	2.7359×10^{-5}	9.289×10^{-10}	2.5833×10^{-5}	5.0892×10^{-7}
0.5	4.9785×10^{-6}	1.4324×10^{-8}	6.7342×10^{-5}	1.3463×10^{-6}
0.6	2.7466×10^{-5}	5.9741×10^{-8}	1.3411×10^{-5}	1.9603×10^{-6}
0.7	1.5686×10^{-5}	7.1574×10^{-7}	1.3507×10^{-4}	1.3192×10^{-6}
0.8	2.2792×10^{-5}	1.3485×10^{-6}	6.8786×10^{-5}	1.1611×10^{-6}
0.9	3.4034×10^{-5}	1.1203×10^{-6}	1.3754×10^{-5}	1.0032×10^{-7}

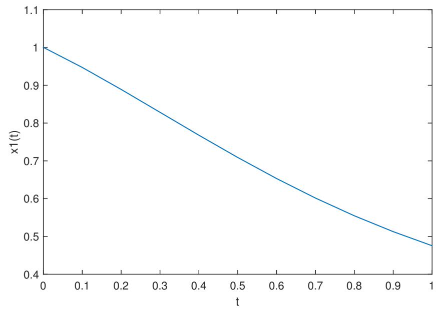

Figure 9: Behavior of $x_1(t)$ for Example 5, $m = 8$.

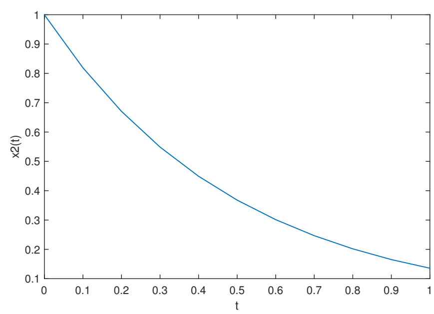

Figure 10: Behavior of $x_2(t)$for Example 5, $m = 8$.

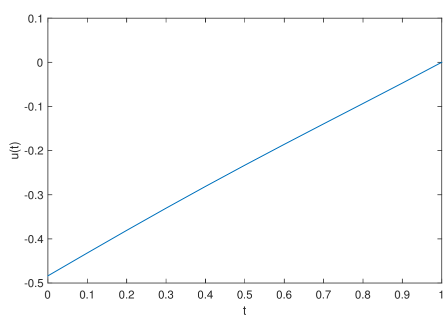

Figure 11: Behavior of $u(t)$ for Example 5, $m = 8$.

7 Conclusion

In the presented study, we have provided a computational approach to solve linear constrained quadratic OCPs by employing the Bell polynomials. Several test problems have been studied to illustrate the applicability and efficiency of the proposed method. Results have revealed that the new technique can solve the OCPs effectively. Comparison of the proposed method with other previous methods showed that this method is accurate.

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