Iranian Journal of Numerical Analysis and Optimization
Vol. 14, No. 1, 2024, pp 107–135
https://doi.org/10.22067/ijnao.2023.83112.1288
https://ijnao.um.ac.ir/

Research Article

Approximate proper solutions in vector optimization with variable ordering structure

S. Shahbeyk*,

Abstract

In this paper, we study approximate proper efficient (nondominated and minimal) solutions of vector optimization problems with variable ordering structures (VOSs). In vector optimization with VOS, the partial ordering cone depends on the elements of the image set. Approximate proper efficient/nondominated/ minimal solutions are defined in different senses (Henig, Benson, and Borwein) for problems with VOSs from new standpoints. The relationships among the introduced notions are studied, and some scalarization approaches are developed to characterize these solutions. These scalarization results based on new functionals defined by elements from the dual cones are given. Moreover, some existing results are addressed.

AMS subject classifications (2020): Primary 90C29; Secondary 90C30; 90C59.

*Corresponding author

Received 1 July 2023; revised 5 October 2023; accepted 9 October 2023

Shokouh Shahbeyk
Department of Mathematics, Faculty of Statistics, Mathematics, and Computer Science, Allameh Tabataba'i University , Tehran, Iran. e-mail: sh_shahbeyk@atu.ac.ir

How to cite this article
Shahbeyk, S., Approximate proper solutions in vector optimization with variable ordering structure. *Iran. J. Numer. Anal. Optim.*, 2024; 14(1): 107-135. https://doi.org/10.22067/ijnao.2023.83112.1288

Keywords: Approximate proper solutions; Variable ordering structure; Scalarization; Vector optimization.

1 Introduction

In vector optimization, elements in the objective space are compared by ordering (nontrivial, convex, closed, and pointed) cone. In recent years, the variable ordering structure (VOS) concept has been introduced [5, 10, 15, 44]. In vector optimization with VOS, the partial ordering cone depends on the elements of the image set. A candidate element is said to be nondominated optimal if and only if it is not dominated by other reference elements with respect to their corresponding ordering. Another notion of optimal is called minimality. For that notion, only the ordering of the candidate element itself is considered. Vector optimization with a VOS has recently gained more interest due to several applications in medical image registration, dynamical models, economic theory, behavioral sciences, and so on [2, 14, 18, 31, 42].

One of the most exciting and important notions in vector optimization is proper optimality. Properly optimal elements (solutions) are optimal with additional properties. This notion has been defined and investigated by various scholars; see, for example, Kuhn and Tucker [29], Geoffrion [20], Benson [4], Borwein [6], and Henig [25], among others. See also [22, 19, 35, 36, 45]. The introduction of a properly optimal solution in vector optimization with a VOS was first done by Eichfelder and Kasimbeyli [17]. The relationships between properly optimal solutions with a VOS in different senses, including Borwein, Benson, and Henig, were studied in [12, 13, 17, 16]. After that, Hartley properly optimal solutions, super optimal solutions, and robust solutions in vector optimization with a VOS were investigated [37].

On the other hand, in recent decades, there has been a lot of attention to approximate solutions to optimization problems due to two facts. Firstly, numerical algorithms may generate only approximate solutions, and secondly, the set of exact solutions might be empty in some practical problems, where approximate solutions exist. For Problems with a fixed order structure, the definition of approximate solutions has been extended by various scholars [3, 21, 23, 28, 30, 33, 40, 41, 42, 43]. Based on different ideas of proper efficiency, some researchers proposed concepts of approximate proper efficiency and investigated their characterizations and applications in vector optimization. For example, Maghri and Pareto-Fenchel [32] proposed approximate proper efficiency in the sense of Henig based on the ideas of Henig proper efficiency. Rong proposed the concept of Benson approximate proper efficiency based on the ideas of Benson proper efficiency [34]. Recently, approximate proper efficiency in an infinite-dimensional space was identified [26]. Soleimani and Tammer incorporated VOSs in approximate solutions of vector optimization problems [38, 39].

In the present paper, we introduce approximate properly optimal solutions in different senses (Henig, Benson, and Borwein) for problems with a VOS. Each of these proper solutions has its own advantages. Henig's notion provides easy-to-check criteria for proper optimality, and optimality has a close connection with stability. Benson and Borwein optimality notions deal with efficient solutions, which are efficient, respectively, for a closed cone generated by the transferred image space and the tangent cone of the transferred image space. Besides the well-featured approximate solutions, mentioned in the literature review, these concepts do not need to consider the compactness conditions to show the existence. The relationships between the introduced notions are studied. Furthermore, we give necessary and sufficient conditions for various approximate proper optimal notions.

The rest of the paper is organized as follows. In Section 2, we give some preliminary definitions and results. Section 3 is devoted to introducing proper approximate solutions with a VOS and investigating their relations. Section 4 continues the paper with necessary and sufficient conditions.

2 Preliminaries

Let X and Y be two real normed spaces. A set $C \subseteq Y$ is said to be cone if $\lambda C \subseteq C$ for each $\lambda \in [0, +\infty)$. A cone C is said to be convex if $C + C \subseteq C$, and it is pointed if $C \cap (-C) \subseteq \{0\}$. Furthermore, it is called nontrivial if $C \neq \{0\}$, and $C \neq Y$. A cone C is called an ordering cone if it is nontrivial, convex, closed, and pointed. A nonempty convex subset B of a cone C is called a base of C if each $y \in C$ has a unique representation of the form $y = \lambda d$ for some $\lambda > 0$ and some $d \in B$.

For a set $A \subseteq X$, int$A$, cl$A$, and bd$A$ stand for the interior, the closure, and the boundary of A, respectively. Furthermore, $cone(A) := \bigcup_{\lambda \geq 0} \lambda A$ is the cone generated by A. The set A is called starshaped at $\bar{y} \in A$, if $\lambda y + (1-\lambda)\bar{y} \in A$ for every $y \in A$ and every $\lambda \in [0, 1]$.

Let $A \subseteq X$ and let $\bar{x} \in A$. The contingent cone or Bouligand tangent cone to A at $\bar{x}$, denoted by $T(A, \bar{x})$, is defined as the collection of all $v \in X$ such that there are sequences $\{x_j\} \subseteq A$ and $\{t_j\} \subseteq (0, +\infty)$ satisfying

$$x_j \to \bar{x} \quad \text{and} \quad t_j(x_j - \bar{x}) \to v \quad as \quad j \to \infty.$$

This cone is closed, while it is not convex necessarily. Consider a vector optimization problem,

$$\min\{f(x) : x \in \Omega\}, \tag{1}$$

where $f : X \to Y$ and $\emptyset \neq \Omega \subset X$.

In the following, we present one of the approximate solution concepts, which was first introduced by Kutateladze [30]. It is the most popular notion of approximate efficiency; see [24, 43] for more details. Let C be an ordering cone.

Definition 1. Let $\bar{x} \in \Omega$, let $\varepsilon > 0$, and let $k^0 \in C \setminus \{0\}$.
a) $\bar{x}$ is called an εk^0-efficient solution to (1) with respect to C if

$$\Big(f(\bar{x}) - \varepsilon k^0 - C \setminus \{0\}\Big) \cap f(\Omega) = \emptyset.$$

b) If int$C \neq \emptyset$, then $\bar{x}$ is called a weakly εk^0-efficient solution to (1) with respect to C if

$$\Big(f(\bar{x}) - \varepsilon k^0 - \text{int}C\Big) \cap f(\Omega) = \emptyset.$$

We denote the set of εk^0-efficient and weakly εk^0-efficient solutions of (1) with respect to C by $\varepsilon k^0 - E(\Omega, f, C)$, and $\varepsilon k^0 - wE(\Omega, f, C)$, respectively.

Now, we will study approximate solutions of vector optimization problems with a VOS.

We assume that $C : Y \rightrightarrows 2^Y$ is a given set-valued mapping that satisfies $0 \in bdC(y)$ and that $C(y)$ is an ordering cone for each $y \in Y$. Furthermore, assume that $k^0 \in Y$ is a nonzero vector satisfying $C(y) + [0, \infty)k^0 \subseteq C(y)$ for all $y \in Y$. Moreover, let $\varepsilon > 0$ be given.

The following definitions of minimality and nondomination for vector optimization problems with VOS, defined by a cone-valued mapping $C(\cdot)$, are known in the literature; see, for example, [7, 8, 9, 13, 15, 17, 44].

In order to introduce these notions, we consider the following two domination relations: For $y^1, y^2 \in Y$, we have

$$y^1 \leq_1 y^2 \quad if \quad y^2 \in y^1 + \big(C(y^1) \setminus \{0\}\big),$$
$$y^1 \leq_2 y^2 \quad if \quad y^2 \in y^1 + \big(C(y^2) \setminus \{0\}\big).$$

Definition 2. [38] Let $\bar{x} \in \Omega$.

a) $\bar{x}$ is called an εk^0-minimal solution to (1) with respect to the mapping $C(\cdot)$ if there is no feasible solution $x \in \Omega$ such that $f(x) + \varepsilon k^0 \leq_2 f(\bar{x})$, that is,

$$\Big(f(\bar{x}) - \varepsilon k^0 - \big(C(f(\bar{x})) \setminus \{0\}\big)\Big) \cap f(\Omega) = \emptyset.$$

b) If $\mathrm{int}C(f(\bar{x})) \neq \emptyset$, then $\bar{x}$ is called a weakly εk^0-minimal solution to (1) with respect to the mapping $C(\cdot)$ if

$$\Big(f(\bar{x}) - \varepsilon k^0 - \mathrm{int}C(f(\bar{x}))\Big) \cap f(\Omega) = \emptyset.$$

We denote the set of εk^0-minimal and weakly εk^0-minimal solutions of (1) with respect to the mapping $C(\cdot)$ by $\varepsilon k^0 - M(\Omega, f, C)$ and $\varepsilon k^0 - wM(\Omega, f, C)$, respectively.

Definition 3. [38] Let $\bar{x} \in \Omega$.

a) $\bar{x}$ is called an εk^0-nondominated solution to (1) with respect to the mapping $C(\cdot)$ if

$$\Big(f(\bar{x}) - \varepsilon k^0 - \big(C(f(x)) \setminus \{0\}\big)\Big) \cap \{f(x)\} = \emptyset, \qquad \text{for all} \;\; x \in \Omega.$$

b) Assuming $\mathrm{int}C(f(x)) \neq \emptyset$ for all $x \in \Omega$, the vector $\bar{x}$ is called a weakly εk^0-nondominated solution to (1) with respect to the mapping $C(\cdot)$ if

$$\left(f(\bar{x}) - \varepsilon k^0 - \mathrm{int}C(f(x))\right) \cap \{f(x)\} = \emptyset, \quad \text{for all } \ x \in \Omega.$$

We denote the set of εk^0-nondominated and weakly εk^0-nondominated solutions of (1) with respect to the mapping $C(\cdot)$ by $\varepsilon k^0 - N(\Omega, f, C)$ and $\varepsilon k^0 - wN(\Omega, f, C)$, respectively.

3 Proper approximate solutions

A fundamental solution concept in vector optimization, which plays a vital role from both theoretical and practical points of view, is the proper solution. This concept has been studied in many publications to eliminate the situations in which the trade-off between the criteria is unbounded. In a recent paper, Soleimani [38] investigated the concept of approximate solutions in vector optimization problems with VOSs. To our knowledge, no studies have been done on approximate proper solutions with VOSs in infinite-dimensional spaces.

We start the section by reviewing the definitions of proper approximate solutions with a fixed ordering structure.

Definition 4. [32, 34] Let $\bar{x} \in \Omega$.
a) $\bar{x}$ is called a properly εk^0-efficient solution to (1) with respect to C in the sense of Henig if there is a convex cone C' such that $C \setminus \{0\} \subseteq \mathrm{int}C'$ and $\bar{x} \in \varepsilon k^0 - E(\Omega, f, C')$.
b) $\bar{x}$ is called a properly εk^0-efficient solution to (1) with respect to C in the sense of Benson if $f(\bar{x})$ is an εk^0-efficient element of the set

$$\{f(\bar{x}) - \varepsilon k^0\} + clcone\left(f(\Omega) + C - \{f(\bar{x}) - \varepsilon k^0\}\right)$$

with respect to C.
c) $\bar{x}$ is called a properly εk^0-efficient solution to (1) with respect to C in the sense of Borwein if $f(\bar{x})$ is an εk^0-efficient element of the set

$$\{f(\bar{x}) - \varepsilon k^0\} + T\left(f(\Omega) + C, f(\bar{x}) - \varepsilon k^0\right)$$

with respect to C.

We denote the set of properly εk^0-efficient solutions of (1) with respect to C in the sense of Henig, Benson, and Borwein by $\varepsilon k^0 - He(\Omega, f, C)$, $\varepsilon k^0 - Be(\Omega, f, C)$, and $\varepsilon k^0 - Bo(\Omega, f, C)$, respectively.

In the following, we introduce and study proper approximate solutions of vector optimization problems with a VOS in the sense of Henig, Benson, and Borwein. The notions introduced in Definitions 5 and 6 are extensions to the approximate case of the concepts of exact proper efficiency with VOSs given by Eichfelder and Kasimbeyli in [17, Definition 6].

Definition 5. Let $\bar{x} \in \Omega$.
a) $\bar{x}$ is called a properly εk^0-minimal solution to (1) with respect to the mapping $C(\cdot)$ in the sense of Henig if there is a set-valued mapping $C' : Y \rightrightarrows 2^Y$ with $C'(f(x))$ a convex cone and $C(f(x)) \setminus \{0\} \subseteq \text{int}C'(f(x))$ for all $x \in \Omega$ such that $\bar{x} \in \varepsilon k^0 - M(\Omega, f, C')$.
b) $\bar{x}$ is called a properly εk^0-minimal solution to (1) with respect to the mapping $C(\cdot)$ in the sense of Benson if $f(\bar{x})$ is an εk^0-minimal element of the set

$$A_1 := \{f(\bar{x}) - \varepsilon k^0\} + clcone\Big(f(\Omega) + C(f(\bar{x})) - \{f(\bar{x}) - \varepsilon k^0\}\Big) \qquad (2)$$

with respect to the mapping $C(\cdot)$.
c) $\bar{x}$ is called a properly εk^0-minimal solution to (1) with respect to the mapping $C(\cdot)$ in the sense of Borwein if $f(\bar{x})$ is an εk^0-minimal element of the set

$$A_2 := \{f(\bar{x}) - \varepsilon k^0\} + T\Big(f(\Omega) + C(f(\bar{x})), f(\bar{x}) - \varepsilon k^0\Big) \qquad (3)$$

with respect to the mapping $C(\cdot)$.

We denote the set of properly εk^0-minimal solutions of (1) with respect to the mapping $C(\cdot)$ in the sense of Henig, Benson, and Borwein by $\varepsilon k^0 - MHe(\Omega, f, C)$, $\varepsilon k^0 - MBe(\Omega, f, C)$, and $\varepsilon k^0 - MBo(\Omega, f, C)$, respectively.

Definition 6. Let $\bar{x} \in \Omega$.

a) $\bar{x}$ is called a properly εk^0-nondominated solution to (1) with respect to the mapping $C(\cdot)$ in the sense of Henig if there is a set-valued mapping $C' : Y \rightrightarrows 2^Y$ with $C'(f(x))$ a convex cone and $C(f(x)) \setminus \{0\} \subseteq \text{int}C'(f(x))$ for all $x \in \Omega$ such that $\bar{x} \in \varepsilon k^0 - N(\Omega, f, C')$.

b) $\bar{x}$ is called a properly εk^0-nondominated solution to (1) with respect to the mapping $C(\cdot)$ in the sense of Benson if $f(\bar{x})$ is an εk^0-nondominated element of the set

$$A_3 := \{f(\bar{x}) - \varepsilon k^0\} + clcone\Bigg(\bigcup_{\omega\in\Omega} \big(\{f(\omega)\} + C(f(\omega))\big) - \{f(\bar{x}) - \varepsilon k^0\}\Bigg) \tag{4}$$

with respect to the mapping $C(\cdot)$.

c) The element $\bar{x}$ is a properly εk^0-nondominated solution to (1) with respect to the mapping $C(\cdot)$ in the sense of Borwein if $f(\bar{x})$ is an εk^0-nondominated element of the set

$$A_4 := \{f(\bar{x}) - \varepsilon k^0\} + T\Bigg(\bigcup_{\omega\in\Omega} \big(\{f(\omega)\} + C(f(\omega))\big), f(\bar{x}) - \varepsilon k^0\Bigg) \tag{5}$$

with respect to the mapping $C(\cdot)$.

We denote the set of properly εk^0-nondominated solutions of (1) with respect to the mapping $C(\cdot)$ in the sense of Henig, Benson, and Borwein by $\varepsilon k^0 - NHe(\Omega, f, C)$, $\varepsilon k^0 - NBe(\Omega, f, C)$, and $\varepsilon k^0 - NBo(\Omega, f, C)$, respectively.

Lemma 1. It holds that $\bar{x}$ is a properly εk^0-minimal solution to (1) with respect to mapping $C(\cdot)$ in the sense of Henig if and only if there is a convex cone C with $C(f(\bar{x})) \setminus \{0\} \subseteq \text{int}C$ such that

$$\big(f(\bar{x}) - \varepsilon k^0 - C \setminus \{0\}\big) \cap f(\Omega) = \emptyset.$$

Proof. The proof of this lemma is similar to that of [17, Lemma 4] and is hence omitted. □

Lemma 2. [11] Let $P \subseteq Y$ be a weakly closed cone, and let C be a cone with a weakly compact base such that $P \cap C = \{0\}$. Then there exists an ordering cone C' such that $C \setminus \{0\} \subseteq \mathrm{int} C'$ and $P \cap C' = \emptyset$.

Corollary 1. By the definitions and Lemma 1, $\bar{x}$ is a properly εk^0-minimal solution to (1) with respect to the mapping $C(\cdot)$ in the sense of Henig (resp., Benson/Borwein) if and only if it is a properly εk^0-efficient solution to (1) with respect to the ordering cone $C := C(f(\bar{x}))$ in the sense of Henig (resp., Benson/Borwein).

Corollary 2. It is not difficult to see that if $\bar{x}$ is a properly εk^0-efficient solution to (1) with respect to C in the sense of Henig (Benson/Borwein), then it is an εk^0-efficient solution of (1) with respect to C.

The following theorem shows that every properly εk^0-minimal (resp., εk^0-nondominated) solution is an εk^0-minimal (resp., εk^0-nondominated) solution with respect to the ordering mapping $C(\cdot)$.

Theorem 1. Let $\bar{x} \in \Omega$.

a) If $\bar{x} \in \varepsilon k^0 - MHe(\Omega, f, C)$ or $\bar{x} \in \varepsilon k^0 - MBe(\Omega, f, C)$ or $\bar{x} \in \varepsilon k^0 - MBo(\Omega, f, C)$ then $\bar{x} \in \varepsilon k^0 - M(\Omega, f, C)$

b) If $\bar{x} \in \varepsilon k^0 - NHe(\Omega, f, C)$ or $\bar{x} \in \varepsilon k^0 - NBe(\Omega, f, C)$ or $\bar{x} \in \varepsilon k^0 - NBo(\Omega, f, C)$, then $\bar{x} \in \varepsilon k^0 - N(\Omega, f, C)$

Proof. a) Let $C'(\cdot)$ be the mapping in definition of proper εk^0-minimality in the sense of Henig. We have $C(f(x)) \subseteq C'(f(x))$ for all $x \in \Omega$, $f(\Omega) \subseteq A_1$, where A_1 is as defined in (2) and $f(\Omega) \subseteq A_3$, where A_3 is as defined in (4). Hence, $\bar{x} \in \varepsilon k^0 - MHe(\Omega, f, C)$ or $\bar{x} \in \varepsilon k^0 - MBe(\Omega, f, C)$ implies $\bar{x} \in \varepsilon k^0 - M(\Omega, f, C)$.

To complete the proof of part (a), we have

$$\begin{aligned} & \bar{x} \in \varepsilon k^0 - MBo(\Omega, f, C) \\ \overset{Corollary\,1}{\Longrightarrow} \quad & \bar{x} \in \varepsilon k^0 - Bo(\Omega, f, C(f(\bar{x})), \\ \overset{Corollary\,2}{\Longrightarrow} \quad & \bar{x} \in \varepsilon k^0 - E(\Omega, f, C(f(\bar{x})), \\ \Longrightarrow \quad & \bar{x} \in \varepsilon k^0 - M(\Omega, f, C). \end{aligned}$$

b) The proof for Henig and Benson cases is similar to part (a). Now, let $\bar{x} \in \varepsilon k^0 - NBo(\Omega, f, C)$ and let $\bar{x} \notin \varepsilon k^0 - N(\Omega, f, C)$. Then there exist $\hat{y} \in f(\Omega) \subseteq \bigcup_{\omega\in\Omega}\Big(\{f(\omega)\} + C(f(\omega)\Big)$ and $d \in C(\hat{y}) \setminus \{0\}$ such that $f(\bar{x}) - \varepsilon k^0 = \hat{y} + d$. Now define $d^n := (1 - \frac{1}{n})d \in C(\hat{y})$, $t_n := n$ and $y^n := \hat{y} + d^n \in \{\hat{y}\} + C(\hat{y}) \subseteq \bigcup_{\omega\in\Omega}\Big(\{f(\omega)\} + C(f(\omega)\Big)$. Hence, we have

$$y^n = \hat{y} + d^n = \big(f(\bar{x}) - \varepsilon k^0 - \frac{1}{n}d\big) \to f(\bar{x}) - \varepsilon k^0,$$
$$t_n\Big(y^n - (f(\bar{x}) - \varepsilon k^0)\Big) = n\big(\hat{y} + d^n - f(\bar{x}) + \varepsilon k^0\big) = n(d^n - d) \to -d.$$

Therefore, we conclude

$$-d \in T\Bigg(\bigcup_{\omega\in\Omega}\big(\{f(\omega)\} + C(f(\omega))\big), f(\bar{x}) - \varepsilon k^0\Bigg)$$

and

$$\hat{y} = f(\bar{x}) - \varepsilon k^0 - d \in \{f(\bar{x}) - \varepsilon k^0\} + T\Bigg(\bigcup_{\omega\in\Omega}\big(\{f(\omega)\} + C(f(\omega))\big), f(\bar{x}) - \varepsilon k^0\Bigg).$$

This implies that $\bar{x}$ is not properly εk^0-nondominated solution to (1) with respect to C in the sense of Borwein, and the proof is completed.

□

It is shown in the following propositions that the set of approximate proper solutions does not grow as tolerance (ε) gets smaller.

Proposition 1. Let $0 \le \varepsilon_1 \le \varepsilon_2$. Then
a) $\varepsilon_1 k^0 - MBe(\Omega, f, C) \subseteq \varepsilon_2 k^0 - MBe(\Omega, f, C)$.
b) $\varepsilon_1 k^0 - MBo(\Omega, f, C) \subseteq \varepsilon_2 k^0 - MBo(\Omega, f, C)$.
c) $\varepsilon_1 k^0 - MHe(\Omega, f, C) \subseteq \varepsilon_2 k^0 - MHe(\Omega, f, C)$.

Proof. a)Let $\bar{x} \in \varepsilon_1 k^0 - MBe(\Omega, f, C)$. Then $\bar{x} \in \varepsilon_1 k^0 - M(A_1, C)$ in which A_1 is defined in (2). By [39, Theorem 2], $\bar{x} \in \varepsilon_2 k^0 - M(A_1, C)$, and therefore, $\bar{x} \in \varepsilon_2 k^0 - MBe(\Omega, f, C)$. Parts (b) and (c) can be proved similarly. □

The following proposition can be proved similar to Proposition 1.

Proposition 2. Let $0 \le \varepsilon_1 \le \varepsilon_2$. Then
a) $\varepsilon_1 k^0 - NBe(\Omega, f, C) \subseteq \varepsilon_2 k^0 - NBe(\Omega, f, C)$.

b) $\varepsilon_1 k^0 - NBo(\Omega, f, C) \subseteq \varepsilon_2 k^0 - NBo(\Omega, f, C)$.

c) $\varepsilon_1 k^0 - NHe(\Omega, f, C) \subseteq \varepsilon_2 k^0 - NHe(\Omega, f, C)$.

Proposition 3. a) If $C(f(x)) \subseteq C(f(\bar{x}))$ for all $x \in \Omega$, then every properly εk^0-minimal solution to (1) with respect to the mapping $C(\cdot)$ in the sense of Henig (resp., Benson/Borwein) is a properly εk^0-nondominated solution to (1) with respect to the mapping $C(\cdot)$ in the sense of Henig (resp., Benson/Borwein).

b) If $C(f(\bar{x})) \subseteq C(f(x))$ for all $x \in \Omega$, then every properly εk^0-nondominated solution to (1) with respect to the mapping $C(\cdot)$ in the sense of Henig (resp., Benson/Borwein) is a properly εk^0-minimal solution to (1) with respect to the mapping $C(\cdot)$ in the sense of Henig (resp., Benson/Borwein).

Proof. According to definitions of properly εk^0-minimal and nondominated solution and the inequality $\varepsilon k^0 - M(\Omega, f, C) \subseteq \varepsilon k^0 - N(\Omega, f, C)$ holds [39, Theorem 6], the proof is straightforward $\square$

In the following, the relationships between the properly approximate solutions in different senses have been studied.

Theorem 2. Let $\bar{x} \in \Omega$. Then the following properties hold:

a)

$$\varepsilon k^0 - MBe(\Omega, f, C) \subseteq \varepsilon k^0 - MBo(\Omega, f, C)$$

$$\varepsilon k^0 - NBe(\Omega, f, C) \subseteq \varepsilon k^0 - NBo(\Omega, f, C).$$

b) If $f(\Omega) + C(f(\bar{x}))$ is starshaped with respect to $f(\bar{x})$, then

$$\varepsilon k^0 - MBo(\Omega, f, C) \subseteq \varepsilon k^0 - MBe(\Omega, f, C).$$

c) If $\bigcup_{\omega\in\Omega}(\{f(\omega)\} + C(f(\omega)))$ is starshaped with respect to $f(\bar{x})$, then

$$\varepsilon k^0 - NBo(\Omega, f, C) \subseteq \varepsilon k^0 - NBe(\Omega, f, C).$$

Proof. For a given set Λ and element $\bar{y} \in \Lambda$, we have $T(\Lambda, \bar{y}) \subseteq cl(cone(\Lambda - \{\bar{y}\})$, and the inclusion holds as equality when Λ is starshaped at $\bar{y}$. Therefore, parts (a), (b), and (c) are proved. $\square$

Theorem 3. Let $\bar{x} \in \Omega$. Then the following properties hold:

a)

$$\varepsilon k^0 - MHe(\Omega, f, C) \subseteq \varepsilon k^0 - MBe(\Omega, f, C).$$

b) If $C(f(\bar{x}))$ has a weakly compact base, then

$$\varepsilon k^0 - MBe(\Omega, f, C) \subseteq \varepsilon k^0 - MHe(\Omega, f, C).$$

c) If $C(f(x))$ has a weakly compact base for all $x \in X$, then

$$\varepsilon k^0 - NBe(\Omega, f, C) \subseteq \varepsilon k^0 - NHe(\Omega, f, C).$$

Proof. a) Let $\bar{x} \in \varepsilon k^0 - MHe(\Omega, f, C)$. We have $\bar{x} \in \varepsilon k^0 - M(\Omega, f, C')$, where C' is a set-valued map $C' : Y \rightrightarrows 2^Y$ with $C'(f(x))$ an convex cone and $C(f(x)) \setminus \{0\} \subseteq \mathrm{int}C'(f(x))$ for all $x \in X$. Therefore

$$\Big(f(\Omega) - \{f(\bar{x}) - \varepsilon k^0\}\Big) \cap \Big(-C'(f(\bar{x})) \setminus \{0\}\Big) = \emptyset.$$

Hence, by [25, Lemma 3.7],

$$\Big(f(\Omega) + C(f(\bar{x})) - \{f(\bar{x}) - \varepsilon k^0\}\Big) \cap \Big(-C'(f(\bar{x})) \setminus \{0\}\Big) = \emptyset.$$

Regarding to Definition of $cone(A)$ and since $C'(f(x))$ is an convex cone, we have

$$cone\Big(f(\Omega) + C(f(\bar{x})) - \{f(\bar{x}) - \varepsilon k^0\}\Big) \cap \Big(-C'(f(\bar{x})) \setminus \{0\}\Big) = \emptyset. \quad (6)$$

If $\bar{x} \notin \varepsilon k^0 - MBe(\Omega, f, C)$, then there exists $\hat{y} \in A_1$ (as defined in (2)) such that $\hat{y} = f(\bar{x}) - \varepsilon k^0 + d$ with $d \in -C(f(\bar{x})) \setminus \{0\}$. Therefore

$$d \in clcone\Big(f(\Omega) + C(f(\bar{x})) - \{f(\bar{x}) - \varepsilon k^0\}\Big) \cap -C(f(\bar{x})) \setminus \{0\}.$$

Thus

$$d \in clcone\Big(f(\Omega) + C(f(\bar{x})) - \{f(\bar{x}) - \varepsilon k^0\}\Big) \cap -\mathrm{int}C'(f(\bar{x})).$$

Therefore, there exists a nonzero sequence $\{d_n\}$ such that

$$d_n \in cone\Big(f(\Omega) + C(f(\bar{x})) - \{f(\bar{x}) - \varepsilon k^0\}\Big) \cap -C'(f(\bar{x})),$$

with

$$\lim_{n\to+\infty} d_n = d.$$

This contradicts (6).

b) Let $\bar{x} \in \varepsilon k^0 - MBe(\Omega, f, C)$. By Corollary 1, $\bar{x} \in \varepsilon k^0 - Be(\Omega, f, C(f(\bar{x}))$. Thus

$$clcone\bigg(f(\Omega) + C - \{f(\bar{x}) - \varepsilon k^0\}\bigg) \cap -C(f(\bar{x})) = \{0\}.$$

By Lemma 2, there exists a closed convex pointed cone C' with $C(f(\bar{x})) \setminus \{0\} \subseteq \mathrm{int}C'$ and $clcone\bigg(f(\Omega) - (f(\bar{x}) - \varepsilon k^0)\bigg) \cap -C' = \{0\}$. Hence,

$$\bigg(f(\bar{x}) - \varepsilon k^0 - C' \setminus \{0\}\bigg) \cap f(\Omega) = \emptyset.$$

This implies $\bar{x} \in \varepsilon k^0 - He(\Omega, f, C(f(\bar{x})))$ and by Corollary 1, $\bar{x} \in \varepsilon k^0 - MHe(\Omega, f, C)$.

c) Let $\bar{x} \in \varepsilon k^0 - NBe(\Omega, f, C)$ and let $\hat{x} \in \Omega$. Since $f(\Omega) \subseteq A_3$, by Definition 6, we have

$$f(\hat{x}) - (f(\bar{x}) - \varepsilon k^0) \notin -C(f(\hat{x})) \setminus \{0\}.$$

Thus

$$clcone\bigg(f(\hat{x}) - (f(\bar{x}) - \varepsilon k^0)\bigg) \cap -C(f(\hat{x})) = \{0\}.$$

By Lemma 2, there exists a closed convex pointed cone $C'(f(\hat{x}))$ with $C(f(\hat{x})) \setminus \{0\} \subseteq \mathrm{int}C'(f(\hat{x}))$ and $clcone\big(f(\hat{x}) - (f(\bar{x}) - \varepsilon k^0)\big) \cap -C'(f(\hat{x})) = \{0\}$. Hence,

$$f(\hat{x}) - (f(\bar{x}) - \varepsilon k^0) \notin -C'(f(\hat{x}))$$

and

$$\big(f(\bar{x}) - \varepsilon k^0 - \big(C'(f(\hat{x})) \setminus \{0\}\big)\big) \cap \{f(\hat{x})\} = \emptyset.$$

Therefore, $\bar{x} \in \varepsilon k^0 - NHe(\Omega, f, C)$.

$\square$

The following example shows that the converse of part (c) of Theorem 3 may not hold.

Example 1. Let $\Lambda = f(\Omega) = \{(y_1, y_2) \in \mathbb{R}^2 \mid 0 \leq y_1 \leq 1,\ 0 \leq y_2 \leq 1\}$. Let $\bar{y} = (\varepsilon, \varepsilon)$, $\varepsilon \in (0, 1)$, let $k^0 = (1, 1)$ and let

$$C(y) := \begin{cases} \mathbb{R}^2_+, & y \in \Lambda \setminus \{(1,0)\}, \\ cone(conv(\{(1,1),(1,-1)\})), & y = (1,0), \\ cone(conv(\{(1,1),(-1,1)\})), & y \in Y \setminus \Lambda. \end{cases}$$

If we define

$$C'(y) := \begin{cases} cone(conv(\{(1,-\delta),(-\delta,1)\})), & y \in \Lambda \setminus \{(1,0)\}, \\ cone(conv(\{(1,1+\delta),(1,-1-\delta)\})), & y = (1,0), \\ cone(conv(\{(1+\delta,1),(-1-\delta,1)\})), & y \in Y \setminus \Lambda, \end{cases}$$

for some small $\delta > 0$, then $C(y) \setminus \{0\} \subseteq \text{int}C'(y)$ for all $y \in Y$ and $\bar{y}$ is an εk^0-nondominated solution to (1) with respect to the mapping $C'(\cdot)$. Therefore $\bar{y} \in \varepsilon k^0 - NHe(\Omega, f, C)$.
Furthermore,

$$\hat{y} = (1, -1) \in A_3 = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 + y_2 \geq 0,\ y_1 \geq 0\},$$

and

$$\big(\bar{y} - \varepsilon k^0 - \big(C(\hat{y}) \setminus \{0\}\big)\big) \cap \{\hat{y}\} \neq \emptyset.$$

Hence $\bar{y} \notin \varepsilon k^0 - NBe(\Omega, f, C)$.

4 Scalarization

In this section, some scalarization results are provided to characterize the properly εk^0 minimal, nondominated solution of (1). Let Y^* denote the topological dual space of Y, and it is equipped with the norm $\|\cdot\|_*$.
The dual cone of a set $C \subseteq Y$ is defined as

$$C^* := \{\zeta^* \in Y^* : \langle \zeta^*, c \rangle \geq 0, \ \text{ for all } c \in C\},$$

and the positive dual cone of C is defined as

$$C^{*^0} := \{\zeta^* \in Y^* : \langle \zeta^*, c\rangle > 0, \text{ for all } c \in C \setminus \{0\}\}.$$

Let a map $y^* : Y \to Y^*$ and an element $\bar{x} \in \Omega$ be given. We consider two functionals $\varphi^{\varepsilon}_{\bar{x}}, \psi^{\varepsilon}_{\bar{x}} : Y \to \mathbb{R}$ with

$$\begin{aligned} \varphi^{\varepsilon}_{\bar{x}}(y) &:= \langle y^*(f(\bar{x})), y - (f(\bar{x}) - \varepsilon k^0)\rangle \qquad \textit{for all } \; y \in Y,\\ \psi^{\varepsilon}_{\bar{x}}(y) &:= \langle y^*(y), y - (f(\bar{x}) - \varepsilon k^0)\rangle \qquad \textit{for all } \; y \in Y. \end{aligned}$$

Therefore,

$$\begin{aligned} \varphi^{\varepsilon}_{\bar{x}}(f(\bar{x}) - \varepsilon k^0) &= \langle y^*(f(\bar{x})), (f(\bar{x}) - \varepsilon k^0) - (f(\bar{x}) - \varepsilon k^0)\rangle = 0,\\ \psi^{\varepsilon}_{\bar{x}}(f(\bar{x}) - \varepsilon k^0) &= \langle y^*(f(\bar{x}) - \varepsilon k^0), (f(\bar{x}) - \varepsilon k^0) - (f(\bar{x}) - \varepsilon k^0)\rangle = 0. \end{aligned}$$

We use the following lemmas for some forthcoming proofs.

Lemma 3.

a) [27, Theorem 3.18] Let S be a nonempty closed convex subset of the normed space Y. Then $y \in Y \setminus S$ if and only if there are a continuous linear functional $l \in Y^* \setminus \{0\}$ and a real number α with

$$l(y) < \alpha \leq l(s) \quad \textit{for all } s \in S.$$

b) [27, Theorem 3.22] Let the topology give Y as the topological dual space of Y^*. Moreover, let S and T be closed convex cones in Y with $\text{int}S^* \neq \emptyset$. Then $(-S) \cap (T) = \{0\}$ if and only if there is a continuous linear functional $l \in Y^* \setminus \{0\}$ with

$$l(x) \leq 0 \leq l(y) \quad \textit{for all } x \in -S \;\textit{ and }\; y \in T$$

and

$$l(x) < 0 \quad \textit{for all } x \in -S \setminus \{0\}.$$

Definition 7. Let Λ be a nonempty subset of a subset Y of a partially ordered linear space with an ordering cone C.
a) A functional $y^* : Y \to \mathbb{R}$ is called monotonically increasing on Λ, if for

every $\bar{y} \in \Lambda$

$$y \in (\bar{y} - C) \cap \Lambda \quad \Rightarrow \quad y^*(y) \leq y^*(\bar{y}).$$

b) A functional $y^* : Y \to \mathbb{R}$ is called strongly monotonically increasing on Λ, if for every $\bar{y} \in \Lambda$

$$y \in (\bar{y} - C) \cap \Lambda,\ y \neq \bar{y} \quad \Rightarrow \quad y^*(y) < y^*(\bar{y}).$$

Remark 1. Let Λ be any subset of a partially ordered linear space Y with the ordering cone C. Every linear functional $y^* \in C^*$ is monotonically increasing on Λ. Furthermore, every linear functional $y^* \in {C^*}^0$ is strongly monotonically increasing on Λ.

Lemma 4. Let Λ be a nonempty subset of a partially ordered linear space Y with a pointed cone C.
a) If there is a linear functional $y^* \in C^*$ and an element $\bar{y} \in \Lambda$ with

$$y^*(\bar{y} - \varepsilon k^0) < y^*(y) \quad for\ all\ \ y \in \Lambda \setminus \{\bar{y} - \varepsilon k^0\}, \tag{7}$$

then $\bar{y} \in \varepsilon k^0 - E(\Lambda, C)$.
b) If there are a linear functional $y^* \in {C^*}^0$ and an element $\bar{y} \in \Lambda$ with

$$y^*(\bar{y} - \varepsilon k^0) \leq y^*(y) \quad for\ all\ \ y \in \Lambda, \tag{8}$$

then $\bar{y} \in \varepsilon k^0 - E(\Lambda, C)$.

Proof. For the proof of both parts, we let $\bar{y} \notin \varepsilon k^0 - E(\Lambda, C)$. Then there exists $\hat{y} \neq \bar{y} - \varepsilon k^0$ such that

$$\hat{y} \in (\bar{y} - \varepsilon k^0 - C) \cap \Lambda.$$

In part (a), since $y^* \in C^*$, y^* is monotonically increasing, and hence $y^*(\hat{y}) \leq y^*(\bar{y} - \varepsilon k^0)$, which contradicts (7).
In the part (b), since $y^* \in {C^*}^0$, y^* is strongly monotonically increasing, and hence $y^*(\hat{y}) < y^*(\bar{y} - \varepsilon k^0)$, which contradicts (8). □

Example 2. Let $\Lambda = f(\Omega) = \{(y_1, y_2) \in \mathbb{R}^2 \mid 0 \leq y_1 \leq 1,\ 0 \leq y_2 \leq 1\}$. Let $\bar{y} = (\varepsilon, \varepsilon)$, let $\varepsilon \in (0, 1)$, let $k^0 = (1, 1)$, and let $C = \mathbb{R}^2_+$. It is easy to see that $\bar{y} \in \varepsilon k^0 - E(\Lambda, C)$. We define a map $y^* : \mathbb{R}^2 \to \mathbb{R}$ by $y^*(y) := (1, 1)^T y$.

Then for any $y \in \Lambda \setminus \{\bar{y} - \varepsilon k^0\} = \Lambda \setminus \{(0,0)\}$, we obtain $y^*(\bar{y} - \varepsilon k^0) = 0 < y^*(y) = y_1 + y_2$.

4.1 Characterizing approximate minimal elements

In this subsection, we provide some characterizations of approximate minimal elements. Theorem 4, given below, provides two characterizations of εk^0-minimal elements.

Theorem 4. Let $\bar{x} \in \Omega$.
a) Let $y^* : Y \to Y^*$ be a map such that $y^*(f(\bar{x})) \in C(f(\bar{x}))^*$. If

$$\varphi^{\varepsilon}_{\bar{x}}(y) > \varphi^{\varepsilon}_{\bar{x}}(f(\bar{x}) - \varepsilon k^0) = 0 \qquad \textit{for all} \;\; y \in f(\Omega) \setminus \{f(\bar{x})\}, \tag{9}$$

then $\bar{x} \in \varepsilon k^0 - M(\Omega, f, C)$.
b)If $\bar{x} \in \varepsilon k^0 - M(\Omega, f, C)$, then there is a map $y^* : Y \to Y^*$ with $y^*(f(\bar{x})) \in C(f(\bar{x}))^*$ such that

$$\psi^{\varepsilon}_{\bar{x}}(y) > \psi^{\varepsilon}_{\bar{x}}(f(\bar{x}) - \varepsilon k^0) = 0 \qquad \textit{for all} \;\; y \in f(\Omega) \setminus \{f(\bar{x})\}.$$

c) If $\mathrm{int} C(f(\bar{x}))^* \neq \emptyset$, then $\bar{x} \in \varepsilon k^0 - M(\Omega, f, C)$ if and only if there is a map $y^* : Y \to Y^*$ with $y^*(f(\bar{x})) \in C(f(\bar{x}))^{*^0} \setminus \{0_{Y^*}\}$ such that

$$\varphi^{\varepsilon}_{\bar{x}}(y) \geq \varphi^{\varepsilon}_{\bar{x}}(f(\bar{x}) - \varepsilon k^0) = 0 \qquad \textit{for all} \;\; y \in f(\Omega) \setminus \{f(\bar{x})\}. \tag{10}$$

Proof. a) Let $y^* : Y \to Y^*$ be a map such that $y^*(f(\bar{x})) \in C(f(\bar{x}))^*$ and $\varphi^{\varepsilon}_{\bar{x}}(y) > 0$ for all $y \in f(\Omega) \setminus \{f(\bar{x})\}$. If $\bar{x} \notin \varepsilon k^0 - M(\Omega, f, C)$, then

$$\big(f(\bar{x}) - \varepsilon k^0 - \big(C(f(\bar{x})) \setminus \{0\}\big)\big) \cap f(\Omega) \neq \emptyset.$$

Therefore, there exists $\hat{y} \in f(\Omega)$ such that $f(\bar{x}) - \varepsilon k^0 - \hat{y} \in C(f(\bar{x})) \setminus \{0\}$. Since $y^*(f(\bar{x})) \in C(f(\bar{x}))^*$, then $\langle y^*(f(\bar{x})), f(\bar{x}) - \varepsilon k^0 - \hat{y} \rangle \geq 0$, and hence

$$\varphi^{\varepsilon}_{\bar{x}}(\hat{y}) = \langle y^*(f(\bar{x})), \hat{y} - (f(\bar{x}) - \varepsilon k^0) \rangle \leq 0,$$

and this is in contradiction with (9).
b) To prove (b), let $\bar{x} \in \varepsilon k^0 - M(\Omega, f, C)$ and $y \in f(\Omega) \setminus \{f(\bar{x})\}$ be arbitrarily

chosen. Then $f(\bar{x}) - \varepsilon k^0 - y \notin C(f(\bar{x}))$. Therefore by Lemma 3(a), there exist $y_1^*(y) \in Y^* \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that

$$\langle y_1^*(y), \big((f(\bar{x}) - \varepsilon k^0) - y\big)\rangle < \alpha \leq \langle y_1^*(y), d\rangle \qquad \text{for all } d \in C(f(\bar{x})).$$

We can show $\alpha = 0$. Hence by setting $y^*(y) = y_1^*(y)$ for all $y \in f(\Omega)\setminus\{f(\bar{x})\}$, we obtain a map $y^* : Y \to Y^*$ with $y^*(f(\bar{x})) \in C(f(\bar{x}))^*$ and $\psi_{\bar{x}}^{\varepsilon}(y) > \psi_{\bar{x}}^{\varepsilon}(f(\bar{x}) - \varepsilon k^0) = 0$ for all $y \in f(\Omega) \setminus \{f(\bar{x})\}$.
c) The argument of this part is similar to the first part.

$\square$

Remark 2. By the definitions, $\bar{x} \in \varepsilon k^0 - M(\Omega, f, C)$ if and only if $\bar{x} \in \varepsilon k^0 - E(\Omega, f, C(f(\bar{x})))$.

Theorem 5 presents necessary and sufficient conditions for a given feasible solution to be Benson and Borwein properly εk^0 minimal solutions.

Theorem 5. Let $\bar{x} \in \Omega$.

a) If there exists a map $y^* : Y \to Y^*$ such that $y^*(f(\bar{x})) \in {C^*}^0 = C(f(\bar{x}))^{*^0}$ and

$$\varphi_{\bar{x}}^{\varepsilon}(y) \geq 0 \qquad \textit{for all } \; y \in f(\Omega) \setminus \{f(\bar{x})\}, \tag{11}$$

then $\bar{x} \in \varepsilon k^0 - MBe(\Omega, f, C)$ and $\bar{x} \in \varepsilon k^0 - MBo(\Omega, f, C)$.

b) If the topology gives Y as the topological dual space of Y^*, $\mathrm{int}C(f(\bar{x}))^* \neq \emptyset$ and $f(\Omega) + C(f(\bar{x}))$ is convex, then $\bar{x} \in \varepsilon k^0 - MBe(\Omega, f, C) = \varepsilon k^0 - MBo(\Omega, f, C)$ if and only if there exists a map $y^* : Y \to Y^*$ with $y^*(f(\bar{x})) \in C(f(\bar{x}))^{*^0}$ such that (11) holds.

c) If the topology gives Y as the topological dual space of Y^*, $\mathrm{int}C(f(\bar{x}))^* \neq \emptyset$ and $\bar{x} \in \varepsilon k^0 - MBe(\Omega, f, C)$, then there exists a map $y^* : Y \to Y^*$ with $y^*(y) \in C(f(\bar{x}))^{*^0}$ for all $y \in f(\Omega) \setminus \{f(\bar{x})\}$ such that

$$\psi_{\bar{x}}^{\varepsilon}(y) \geq 0 \qquad \textit{for all } \; y \in f(\Omega). \tag{12}$$

Proof. a) From remark 2, it is sufficient to prove that $\bar{x} \in \varepsilon k^0 - Be(\Omega, f, C)$ and $\bar{x} \in \varepsilon k^0 - Bo(\Omega, f, C)$.
Let $y \in A_1$. Then $y = f(\bar{x}) - \varepsilon k^0 + d$ such that $d \in \mathit{clcone}\Big(f(\Omega) + C(f(\bar{x})) -$

$\{f(\bar{x})-\varepsilon k^0\}\Big)$. Therefore there are sequences $\{y_n\}\subseteq f(\Omega)+C(f(\bar{x}))$ and $\{t_n\}\subseteq[0,+\infty)$ such that

$$d=\lim_{n\to\infty}t_n(y_n-f(\bar{x})-\varepsilon k^0).$$

Then

$$\langle y^*(f(\bar{x})),d\rangle=\langle y^*(f(\bar{x})),\lim_{n\to\infty}t_n(y_n-f(\bar{x})-\varepsilon k^0)\rangle.$$

Since the linear functional y^* is continuous and strongly monotonically increasing on $f(\Omega)$, it is strongly monotonically increasing on $f(\Omega)+C(f(\bar{x}))$, and therefore,

$$\langle y^*(f(\bar{x})),d\rangle)=\lim_{n\to\infty}t_n\big(\langle y^*(f(\bar{x})),y_n\rangle-\langle y^*(f(\bar{x})),(f(\bar{x})-\varepsilon k^0)\rangle\big)\geq 0$$

and

$$\begin{aligned}\langle y^*(f(\bar{x})),y\rangle&=\langle y^*(f(\bar{x})),d\rangle+\langle y^*(f(\bar{x})),(f(\bar{x})-\varepsilon k^0)\rangle\\&\geq\langle y^*(f(\bar{x})),(f(\bar{x})-\varepsilon k^0)\rangle.\end{aligned}$$

Hence, we obtain $\langle y^*(f(\bar{x})),y\rangle\geq\langle y^*(f(\bar{x})),(f(\bar{x})-\varepsilon k^0)\rangle$ for all $y\in A_1$. Consequently, by Lemma 4, (b) $f(\bar{x})\in\varepsilon k^0-E(A_1,C)$. This completes the proof.

b) Sufficient condition follows immediately from (a) and Theorem 2. To prove the necessary condition, we assume that $\bar{x}\in\varepsilon k^0-MBe(\Omega,f,C)$. Then $\bar{x}$ is εk^0-minimal element of set A_1 with respect to $C(\cdot)$. Therefore

$$\big(A_1-\{f(\bar{x})-\varepsilon k^0\}\big)\cap\big(-C(f(\bar{x}))\big)=\{0\}.\tag{13}$$

Since the set $f(\Omega)+C(f(\bar{x}))$ is convex, A_1 is convex and closed. Then, by Lemma 3(b), the set (13) is equivalent to a continuous linear functional $y_1^*\in Y^*\setminus\{0\}$ with

$$\langle y_1^*,-d\rangle\leq 0\quad\text{ for all } d\in C(f(\bar{x})),$$

$$\langle y_1^*,\hat{y}\rangle\geq 0\quad\text{ for all } \hat{y}\in A_1-\{f(\bar{x})-\varepsilon k^0\},$$

and

$$\langle y_1^*, d\rangle > 0 \quad \text{for all } d \in C(f(\bar{x})) \setminus \{0\}.$$

With these inequalities, we conclude

$$y_1^* \in C(f(\bar{x}))^{*^0},$$
$$\langle y_1^*, y - (f(\bar{x}) - \varepsilon k^0)\rangle \geq 0 \quad \text{for all } y \in A_1.$$

By setting $y^*(f(\bar{x})) := y_1^*$, we are done.

c) Let $\bar{x} \in \varepsilon k^0 - MBe(\Omega, f, C)$ and let $y = f(\bar{x}) - \varepsilon k^0 + d \in A_1$. Then

$$\big(f(\bar{x}) - \varepsilon k^0 - \big(C(f(\bar{x})) \setminus \{0\}\big)\big) \cap \{y\} = \emptyset,$$

and therefore,

$$cone\big(y - (f(\bar{x} - \varepsilon k^0)\big) \cap \big(- C(f(\bar{x}))\big) = \{0\}.$$

By Lemma 3(b), there exists $y_1^* \in Y^* \setminus \{0\}$ such that

$$\langle y_1^*, -d\rangle \leq 0 \quad \text{for all } d \in C(f(\bar{x})),$$

$$\langle y_1^*, \hat{y}\rangle \geq 0 \quad \text{for all } \hat{y} \in cone\big(y - (f(\bar{x} - \varepsilon k^0)\big)$$

and

$$\langle y_1^*, d\rangle > 0 \quad \text{for all } d \in C(f(\bar{x})) \setminus \{0\},$$

and then

$$y_1^* \in C(f(\bar{x}))^{*^0},$$
$$\langle y_1^*, y - (f(\bar{x}) - \varepsilon k^0)\rangle \geq 0.$$

By setting $y^*(y) := y_1^*$ for all $0 \neq y \in A_1$, we are done. □

Theorem 6 presents necessary and sufficient conditions for a given feasible solution to be Henig properly εk^0 minimal solutions with different assumptions.

Theorem 6. Let $\bar{x} \in \Omega$.

a) If $C(f(\bar{x}))$ has a weakly compact base, $y^*: Y \to Y^*$ is a map such that $y^*(f(\bar{x})) \in C(f(\bar{x}))^{*^0}$, and (11) holds, then $\bar{x} \in \varepsilon k^0 - MHe(\Omega, f, C)$.

b) Suppose that Y is a locally convex Hausdorff topological vector space and that there is a locally convex topology compatible with the dual pairing in Y^*. If there exists a map such that $y^*(f(\bar{x})) \in C(f(\bar{x}))^{*^0}$ and (11) holds, then $\bar{x} \in \varepsilon k^0 - MHe(\Omega, f, C)$.

c) If the topology gives Y as the topological dual space of Y^*, $\mathrm{int}C(f(\bar{x}))^* \neq \emptyset$, $f(\Omega) + C(f(\bar{x}))$ is convex and $\bar{x} \in \varepsilon k^0 - MHe(\Omega, f, C)$, then there exists a map $y^*: Y \to Y^*$ with $y^*(f(\bar{x})) \in C(f(\bar{x}))^{*^0}$ such that (11) holds.

d) If $\bar{x} \in \varepsilon k^0 - MHe(\Omega, f, C)$, then there exists a map $y^*: Y \to Y^*$ with $y^*(y) \in C(f(\bar{x}))^{*^0}$ for all $y \in f(\Omega) \setminus \{f(\bar{x})\}$ such that

$$\psi^{\varepsilon}_{\bar{x}}(y) > 0 \qquad for\ all\ \ y \in f(\Omega) \setminus \{f(\bar{x})\}. \tag{14}$$

Proof. a) This part follows immediately from Theorems 5(a) and 3(b).

b) We consider the set-valued mapping C' from Y to 2^Y defined as follows: $C'(y) = Y$, if $y \neq f(\bar{x})$, and

$$C'(f(\bar{x})) := \{y \in Y \ \mid \ \langle y^*(f\bar{x})), y\rangle > 0\} \cup \{0\}.$$

Since $y^*(f(\bar{x})) \in C(f(\bar{x}))^{*^0}$, it is easy to see that $C'(f(x))$ is solid and convex and $C(f(x)) \setminus \{0\} \subset \mathrm{int}C'(f(x))$ for all $x \in \Omega$. It follows that $\bar{x} \in \varepsilon k^0 - M(f, \Omega, C')$. Otherwise, there would exist $\hat{x} \in \Omega$ such that $f(\hat{x}) - f(\bar{x}) + \varepsilon k^0 \in -C'(f\bar{x})) \setminus \{0\}$ (note that from the definition of k^0 and C' it is easy to see that $f(\bar{x}) \neq f(\hat{x})$), which by the definition of C' means that

$$\varphi^{\varepsilon}_{\hat{x}}(y) = \langle y^*(f(\bar{x})), f(\hat{x}) - f(\bar{x}) + \varepsilon k^0\rangle < 0,$$

which contradicts (11). Then $\bar{x} \in \varepsilon k^0 - M(f, \Omega, C')$, and so $\bar{x} \in \varepsilon k^0 - MHe(f, \Omega, C)$.

c) Let $\bar{x} \in \varepsilon k^0 - MHe(\Omega, f, C)$. By Theorem 3(a), $\bar{x} \in \varepsilon k^0 - MBe(\Omega, f, C)$, and then by Theorem 5(b), there exists a map $y^*: Y \to Y^*$ with $y^*(f(\bar{x})) \in C(f(\bar{x}))^{*^0}$ such that (11) holds.

d) If $\bar{x} \in \varepsilon k^0 - MHe(\Omega, f, C)$, then by Lemma 1, there is a convex cone C' with $C(f(\bar{x})) \setminus \{0\} \subseteq \text{int}C'$ such that

$$\big(f(\bar{x}) - \varepsilon k^0 - C' \setminus \{0\}\big) \cap f(\Omega) = \emptyset.$$

We can assume the coneC' to be closed. Let $\hat{y} \in f(\Omega) \setminus \{f(\bar{x})\}$. Then

$$(f(\bar{x}) - \varepsilon k^0) - \hat{y} \notin C'.$$

Therefore by Lemma 3(a), there are a continuous linear functional $y_1^* \in Y^* \setminus \{0\}$ and a real number α with

$$\langle y_1^*, \big((f(\bar{x}) - \varepsilon k^0) - \hat{y}\big)\rangle < \alpha \leq \langle y_1^*, d\rangle \qquad for\ all\ d \in C'.$$

We can show $\alpha = 0$ and hence $y_1^* \in C'^* \setminus \{0\}$ and $\langle y_1^*, \hat{y} - (f(\bar{x}) - \varepsilon k^0)\rangle > 0$. By [27, Lemma 3.21], it holds

$$C(f(\bar{x})) \setminus \{0\} \subseteq \text{int}C' = \{y \in Y : y^*(y) > 0 \text{ for all } y^* \in C'^* \setminus \{0\}\},$$

and then $y_1^* \in C(f(\bar{x}))^{*^0}$. By setting $y^*(y) := y_1^*$ for all $y \in f(\Omega) \setminus \{f(\bar{x})\}$, we are done. □

4.2 Characterizing approximate nondominated elements

The characterization results of approximate nondominated elements are presented in the current subsection work. Theorem 7, given below, provides two characterizations of εk^0-nondominated elements.

Theorem 7. Let $\bar{x} \in \Omega$.
a) $\bar{x} \in \varepsilon k^0 - N(\Omega, f, C)$ if and only if there is a map $y^* : Y \to Y^*$ with $y^*(y) \in C(y)^* \setminus \{0\}$ for all $y \in f(\Omega) \setminus \{f(\bar{x}\})$ such that

$$\psi_{\bar{x}}^{\varepsilon}(y) > \psi_{\bar{x}}^{\varepsilon}(f(\bar{x}) - \varepsilon k^0) = 0 \qquad for\ all\ \ y \in f(\Omega) \setminus \{f(\bar{x})\}.$$

b) Suppose that $\text{int}C(y)^* \neq \emptyset$ for all $y \in f(\Omega)$. Then $\bar{x} \in \varepsilon k^0 - N(\Omega, f, C)$ if and only if there is a map $y^* : Y \to Y^*$ with $y^*(y) \in C(y)^{*^0} \setminus \{0\}$ for all $y \in f(\Omega) \setminus \{f(\bar{x}\})$ such that

$$\psi^{\varepsilon}_{\bar{x}}(y) \geq \psi^{\varepsilon}_{\bar{x}}(f(\bar{x}) - \varepsilon k^0) = 0 \qquad \textit{for all} \;\; y \in f(\Omega) \setminus \{f(\bar{x})\}.$$

We omit proofs as the using same ideas in the proof of Theorem 4. Theorems 8 and 9 provide necessary and sufficient conditions for a given feasible solution to be Benson, Borwein, and Henig properly εk^0 nondominated solutions.

Theorem 8. Let $\bar{x} \in \Omega$, and let

$$\bar{C} := \bigcup_{y \in A_4} C(y).$$

a) If there exists a map $y^* : Y \to Y^*$ such that $y^*(y) \in \bar{C}^{*^0}$ for all $y \in f(\Omega) \setminus \{f(\bar{x}))$ and (11) holds, then $\bar{x} \in \varepsilon k^0 - NBe(\Omega, f, C)$ and $\bar{x} \in \varepsilon k^0 - NBo(\Omega, f, C)$.

b) If $\bar{x} \in \varepsilon k^0 - NBe(\Omega, f, C)$, then there exists a map $y^* : Y \to Y^*$ with $y^*(y) \in C(y)^*$ for all $y \in f(\Omega) \setminus \{f(\bar{x})\}$ such that (14) holds.

c) If the topology gives Y as the topological dual space of Y^*, $\text{int}C(y)^* \neq \emptyset$ for all $y \in f(\Omega)$ and $\bar{x} \in \varepsilon k^0 - NBe(\Omega, f, C)$, then there exists a map $y^* : Y \to Y^*$ with $y^*(y) \in C(y)^{*^0}$ for all $y \in f(\Omega) \setminus \{f(\bar{x})\}$ such that (12) holds.

Proof. a) Let $y^*(y) \in \bar{C}^{*^0}$ and $\langle y^*(y), y - (f(\bar{x}) - \varepsilon k^0))\rangle \geq 0$ for all $y \in f(\Omega)$. Then $y^*(y) \in C(y)^{*^0}$ for all $y \in A_4$. Now, let $y \in A_4 \setminus \{f(\bar{x}) - \varepsilon k^0\}$, therefore there exist sequences $\{y_n\} \subseteq f(\Omega)$, $\{d_n\} \subseteq C(y_n)$, and $\{t_n\} \subseteq [0, \infty)$ such that $y = f(\bar{x}) - \varepsilon k^0 + d$, where

$$d = \lim_{n \to \infty} t_n(y_n + d_n - (f(\bar{x}) - \varepsilon k^0)).$$

Since the linear functional $y^*$is continuous, we have

$$\begin{aligned}
\langle y^*(y), y - (f(\bar{x}) - \varepsilon k^0)\rangle &= y^*(d) \\
&= \lim_{n \to \infty} t_n\big(\langle y^*(y_n - (f(\bar{x}) - \varepsilon k^0))\rangle + \langle y^*(y), d_n\rangle\big) \\
&\geq 0.
\end{aligned}$$

Hence, by Theorem 7(b) $\bar{x} \in \varepsilon k^0 - N(A_4, C)$. Thus $\bar{x} \in \varepsilon k^0 - NBe(\Omega, f, C)$, and then by Theorem 2, $\bar{x} \in \varepsilon k^0 - NBo(\Omega, f, C)$.

b) Let $\bar{x} \in \varepsilon k^0 - NBe(\Omega, f, C)$ and let $y = f(\bar{x}) - \varepsilon k^0 + d \in A_4$. Then

$$\big(f(\bar{x}) - \varepsilon k^0 - \big(C(y)) \setminus \{0\}\big)\big) \cap \{y\} = \emptyset,$$

and therefore,

$$(f(\bar{x}) - \varepsilon k^0) - y \notin C(y).$$

By Lemma 3(a), there are a continuous linear functional $y_1^* \in Y^* \setminus \{0\}$ and a real number α with

$$\langle y_1^*, \big((f(\bar{x}) - \varepsilon k^0) - y\big)\rangle < \alpha \leq \langle y_1^*, d\rangle \qquad \textit{for all } d \in C(y).$$

We can show $\alpha = 0$, and hence $y_1^* \in C(y)^* \setminus \{0\}$ and $\langle y_1^*, y - (f(\bar{x}) - \varepsilon k^0)\rangle > 0$. By setting $y^*(y) = y_1^*$ for all $0 \neq y \in A_4$, we are done.

c) Let $\bar{x} \in \varepsilon k^0 - NBe(\Omega, f, C)$ and let $y = f(\bar{x}) - \varepsilon k^0 + d \in A_4$. Then

$$\big(f(\bar{x}) - \varepsilon k^0 - \big(C(y)) \setminus \{0\}\big)\big) \cap \{y\} = \emptyset,$$

and therefore,

$$cone\big(y - (f(\bar{x} - \varepsilon k^0)\big) \cap \big(-C(f(y))\big) = \{0\}.$$

By Lemma 3(b), there exists $y_1^* \in Y^* \setminus \{0\}$ such that

$$\langle y_1^*, -d\rangle \leq 0 \quad \text{ for all } d \in C(y),$$

$$\langle y_1^*, \hat{y}\rangle \geq 0 \quad \text{ for all } \hat{y} \in cone\big(y - (f(\bar{x} - \varepsilon k^0)\big),$$

and

$$\langle y_1^*, d\rangle > 0 \quad \text{ for all } d \in C(y) \setminus \{0\},$$

and then

$$y_1^* \in C(y)^{*^0},$$
$$\langle y_1^*, y - (f(\bar{x}) - \varepsilon k^0)\rangle \geq 0.$$

By setting $y^*(y) := y_1^*$ for all $0 \neq y \in A_4$, we are done. □

Theorem 9 provides necessary and sufficient conditions for a given feasible solution to be Henig properly εk^0 nondominated solutions.

Theorem 9. Let $\bar{x} \in \Omega$.

a) If there exists a map $y^* : Y \to Y^*$ such that $y^*(y) \in C(y)^{*^0}$ for all $y \in f(\Omega) \setminus \{f(\bar{x})\}$ such that (14) holds, then $\bar{x} \in \varepsilon k^0 - NHe(\Omega, f, C)$.

b) Suppose that $C(f(\bar{x}))$ has a weakly compact base for all $y \in f(\Omega) \setminus \{f(\bar{x})\}$. Then $\bar{x} \in \varepsilon k^0 - NHe(\Omega, f, C)$ if and only if there exists a map $y^* : Y \to Y^*$ with $y^*(y) \in C(y)^{*^0}$ such that (14) holds.

Proof. The proof is similar in spirit to Theorem 6. □

5 Conclusion

In this paper, some new approximate properly efficient solutions such as Henig, Benson, and Borwein minimal/nondominated solutions, in vector optimization with a VOS, were introduced, and theorems establishing the relationship between these concepts were proved. Necessary and sufficient conditions for these solutions were presented. We will consider these solutions in finite- dimensional spaces and provide an algorithm to generate these concepts in future work. It is recommended to study approximate Hartley properly and super solutions with a VOS.

References

[1] Adan, M. and Novo, V. *Proper efficiency in vector optimization on real linear spaces*, J. Optim. Theory Appl. 121 (2004), 515–540.

[2] Bao, T.Q., Mordukhovich, B.S. and Soubeyran, A. *Variational analysis in psychological modeling*, J. Optim. Theory Appl. 164 (2015), 290–315.

[3] Bazgan, C., Ruzika, S., Thielen, C. and Vanderpooten, D. *The power of the weighted sum scalarization for approximating multiobjective optimization problems* , Theory Comput. Syst. 66 (2022), 395–415.

[4] Benson, H.P. *An improved definition of proper efficiency for vector maximization with respect to cone*, J. Math. Appl. 71 (1979), 232–241.

[5] Bergstresser, K., Charnes, A. and Yu, PL. *Generalization of domination structures and nondominated solutions in multicriteria decision making*, J. Optim. Theory Appl., 18 (1976), 3–13.

[6] Borwein, J.M. *Proper efficient points for maximizations with respect to cones*, SIAM J. Control Optim. 15 (1977), 57–63.

[7] Chen, G.Y. *Existence of solutions for a vector variational inequality: an extension of the Hartmann Stampacchia theorem*, J. Optim. Theory Appl. 74 (1992), 445–456.

[8] Chen, G.Y. and Craven, B.D. *Existence and continuity of solutions for vector optimization*, J. Optim. Theory Appl. 81(3) (1994), 459–468.

[9] Chen, G.Y., Huang, X. and Yang, X. *Vector Optimization, Set-Valued and Variational Analysis*, Springer, Berlin, 2005.

[10] Chen, G.Y. and Yang, X.Q. *Characterizations of variable domination structures via nonlinear scalarization*, J. Optim. Theory Appl. 112 (2002), 97–110.

[11] Dauer, J.P. and Saleh, O.A. *A characterization of proper minimal points as solutions of sublinear optimizations problems*, J. Math. Anal. Appl. 178 (1993), 227–246.

[12] Durea, M., Florea E.A. and Strugriu, R. *Henig proper efficiency in vector optimization with variable ordering structure*, J. Ind. Manag. Optim. 15(2) (2019), 791–815.

[13] Eichfelder, G. *Optimal elements in vector optimization with a variable ordering structure*, J. Optim. Theory Appl. 151 (2011), 217–240.

[14] Eichfelder, G. *Vector optimization in medical engineering*, in Pardalos PM, Rassias TM (eds.) Mathematics Without Boundaries Springer, New York, (2014), 181–215.

[15] Eichfelder, G. *Variable Ordering Structures in Vector Optimization*, Springer, Berlin, 2014.

[16] Eichfelder, G. and Gerlach, T. *Characterization of properly optimal elements with variable ordering structures*, Optimization, 65 (2016), 571–588.

[17] Eichfelder, G., Kasimbeyli, R. *Properly optimal elements in vector optimization with variable ordering structure*, J. Global Optim. 60 (2014), 689–712.

[18] Engau, A. *Variable preference modeling with ideal-symmetric convex cones*, J. Global Optim. 42 (2008), 295–311.

[19] Foroutannia, D. and Mahmodinejad, A. *The concept of B-efficient solution in fair multiobjective optimization problems*, Iran. J. Numer. Anal. Optim. 7(1) (2017), 47–63.

[20] Geoffrion, A. *Proper efficiency and the theory of vector maximization*, J. Math. Anal. Appl. 22 (1968), 618–630.

[21] Ghaznavi-Ghosoni, B.A., Khorram E. and Soleimani-Damaneh, M. *Scalarization for characterization of approximate strong/weak/proper efficiency in multi-objective optimization*, Optimization, 62(6) (2013), 703–720.

[22] Guerraggio, A., Molho, E. and Zaffaroni, A. *On the notion of proper efficiency in vector optimization*, J. Optim. Theory Appl. 82 (1994), 1–21.

[23] Helbig, S. *On a new concept for ε-efficiency*, talk at Optimization Days 1992, Montreal, (1992).

[24] Helbig, S. and Pateva, D. *On several concepts for ε-efficiency*, OR Spectrum, 16 (1994), 179–186.

[25] Henig, M.I. *Proper efficiency with respect to cones*, J. Optim. Theory Appl. 36 (1982), 387–407.

[26] Hoseinpour, N. and Ghaznavi, M. *Identifying approximate proper efficiency in an infinite dimensional space*, RARO Operations Research, 57 (2023) 697–714.

[27] Jahn, J. *Vector optimization*, Springer, Berlin, 2012.

[28] Kiyani, E. and Soleimani-damaneh, M. *Approximate proper efficiency on real linear vector spaces*, Pacific Journal of Optimization, 10 (2014), 715–734.

[29] Kuhn, H. and Tucker, A. *Nonlinear programming*, in Neyman, J. (ed.) Proceeding of the Second Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, Berkeley, California, (1951) 481–492.

[30] Kutateladze, S.S. *convex ε-programming*, Soviet. Math. Dokl, 20 (1979), 391–393.

[31] Luc, D.T. and Soubeyran, A. *Variable preference relations: existence of maximal elements*, J. Math. Econ. 49 (2013), 251–262.

[32] Maghri, E.l. and Pareto-Fenchel, M. *ε-subdifferential sum rule and ε-efficiency*, Optim. Lett. 6 (2012), 763–781.

[33] Németh, A.B. *A nonconvex vector minimization problem*, Nonlinear Anal. 10 (1986), 669–678.

[34] Rong, W. *Proper ε-efficiency in vector optimization problems with cone-subconvexlikeness*, Acta Sci. Natur. Univ. NeiMongol, 28 (1997), 609–613.

[35] Shahbeyk, S. and Soleimani-damaneh, M. *Proper minimal points of nonconvex sets in Banach spaces in terms of the limiting normal cone*, Optimization, 66 (2017), 473–489.

[36] Shahbeyk, S. and Soleimani-damaneh, M. *Limiting proper minimal points of nonconvex sets in finite-dimensional spaces*, Carpathian J. Math. 35(3) (2019), 370–384.

[37] Shahbeyk, S., Soleimani-damaneh, M. and Kasimbeyli, R. *Hartley properly and super nondominated solutions in vector optimization with a variable ordering structure*, J. Global Optim. 71 (2018), 383–405.

[38] Soleimani, B. *Characterization of Approximate Solutions of Vector Optimization Problems with a Variable Order Structure*, J. Optim. Theory Appl. 162 (2014), 605–632.

[39] Soleimani, B. and Tammer, C. *Concepts for approximate solutions of vector optimization problems with variable order structure*, Vietnam J. Math. 42 (2014), 43–566.

[40] Tammer, C. *Stability results for approximately efficient solutions*, OR Spectrum, 16 (1994), 47–52.

[41] Tanaka, T. *A new approach to approximation of solutions in vector optimization problems*, in Proceeding of APORS, 1994, M. Fushimi and K. Tone, eds., World Scientific Publishing, Singapore, (1995) 497–504.

[42] Wacker, M. and Deinzer, F. *Automatic robust medical image registration using a new democratic vector optimization approach with multiple measures*, in Yang GZ, Hawkes D, Rueckert D, Noble A, Taylor C, editors. Vol. 5761, Lecture notes in computer science, 12th International Conference, MICCAI 2009, 2009 Sept 20–24. Heidelberg, Springer, London, UK, (2009) 590–597.

[43] White, D.J. *Epsilon efficiency*, J. Optim. Theory Appl. 49 (1986), 319–337.

[44] Yu, P.l. *Cone convexity, cone extreme points, and nondominated solutions in decision problems with multiobjectives*, J. Optim. Theory Appl. 14 (1974), 319–377.

[45] Zamani, M. and Soleimani-damaneh, M. *Proper efficiency, scalarization and transformation in multi-objective optimization: unified approaches*, Optimization, (2020), 1–22.