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Research Article

Chebyshev wavelet-based method for solving various stochastic optimal control problems and its application in finance†

M. Yarahmadi*, and S. Yaghobipour

Abstract

In this paper, a computational method based on parameterizing state and control variables is presented for solving Stochastic Optimal Control (SOC) problems. By using Chebyshev wavelets with unknown coefficients, state and control variables are parameterized, and then a stochastic optimal control problem is converted to a stochastic optimization problem. The expected cost functional of the resulting stochastic optimization problem is approximated by sample average approximation thereby the problem can

*Corresponding author

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Majid Yarahmadi
Department of Mathematics and Computer Science, Lorestan University. e-mail: yarahmadi.m@lu.ac.ir

Saba Yaghobipour
Department of Mathematics and Computer Sciences, Lorestan University, Lorestan, Iran. e-mail: sabaphd88@gmail.com.

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be solved by optimization methods more easily. For facilitating and guaranteeing convergence of the presented method, a new theorem is proved. Finally, the proposed method is implemented based on a newly designed algorithm for solving one of the well-known problems in mathematical finance, the Merton portfolio allocation problem in finite horizon. The simulation results illustrate the improvement of the constructed portfolio return.

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1 Introduction

Most practical problems in various fields, such as economics and finance, lead us to solve continuous finite-horizon Stochastic Optimal Control (SOC) problems. These problems are driven by Stochastic Differential Equations (SDEs) in mathematical finance and especially in portfolio management, which is started with classical equations such as Merton [10].

The solution of an SDE requires the evaluation of stochastic It$\hat{o}$ integral $\int_0^T S(u)dB(u)$, where $\{B(t)\}_t \geq 0$ is a Brownian motion and S is a stochastic process. The common methods for evaluating of the integral, are direct numerical ideas [13]. For example, the approaches of [20, 3] use an optimal wavelet approximation of the Browning motion $\{B(t)\}_t \geq 0$. Also, the approaches of [30, 23, 22] apply the Legendre wavelets to compute numerical solutions of integral and differential equations. For instance, in [22], coefficient functions of the SDE are approximated in terms of Legendre wavelets. Then according to the relation between the block pulse functions and the It$\hat{o}$ integral of this function, the integrals of the coefficient function are approximated.

The most common approach for solve SOC problems is the indirect dynamic programming approach in which the corresponding Hamiltonian–Jacobi–Bellman (HJB) equation has to be considered [14, 27, 33]. However, the second-order partial differential HJB equation is usually solved under some prior assumptions, and it is not solvable analytically in most cases, which is why applying appropriate numerical ideas can be applicable to design optimal decision (control) rules. Among the numerical ideas, the direct ones are more common methods. In these methods, the dynamical system under consideration is discretized in space and time domain, based on the application of the Markov chain. For example, the presented methods of [17, 31] obtain the transition probabilities of the Markov chain from the HJB equation by using finite difference and finite-element ideas. Also, there are some approaches that apply approximation schemes to SOC problems. For instance, the approaches of [4, 16] are Markov decision chain methods from the original SOC problem by approximating the solution of the corresponding SDE and then solving the resulting Bellman equation by using value iteration. Additionally, Huschto and Sager [9] utilized the Wiener chaos expansion developed by Ledoux [19] to reformulate the original SOC problem as a deterministic optimal control problem. Indeed, the considered stochastic process is expressed in terms of deterministic coefficient and orthonormal basis polynomials spanning the underlying Wiener chaos space. Therefore, the approach of [9] is a direct numerical method based on using orthonormal polynomials to solve SOC problems. For this purpose, wavelet polynomials can be used, because they have the property of local approximating discontinuous functions [29]. For example, Haar wavelet orthogonal functions [26], Legendre wavelets [30], and Chebyshev wavelets [35] can be applied to SOC problems, and then the resulting parameterized problems can be solved. Indeed, the application of wavelets as a basis function in a numerical solution of integral equations and optimal control have been attended recently [29, 1, 32]. In comparison with other algorithms, the wavelets techniques provide very fast algorithms. This is because of specific properties when it is used as a basis function. In fact, the wavelets techniques convert SOC problems to stochastic optimization (SO) ones [21, 2, 8] that can be solved by using common methods of SO problems. These methods are commonly divided into two categories, solution methods for single-stage problems and multistage problems [8]. These methods minimize or maximize an expected objective function due to random variables in the formulation of stochastic problems. There are four well-known solution methods, sample average approximation (SAA) [8, 5], stochastic approximation [18], response surfaces [34], and metamodels [6].

In this paper, a new direct numerical method based on using Chebyshev wavelets is applied to solve SOC problems. For this purpose, stochastic state and control processes are approximated by the finite series of Chebyshev wavelet polynomials. The resulting Chebyshev approximation is the best approximation for the continuous state and control variables. Chebyshev wavelets are well-behaved basic functions that are orthonormal on the interval $[0, 1]$. The advantages of Chebyshev wavelets are that the values of the degree of polynomial M and the number of subintervals 2^k are adjustable. Therefore, it can yield a more accurate approximation than piecewise constant orthogonal functions. Thereby, stochastic state and control processes and the expected cost functional of the SOC problem are parameterized. Therefore, the SOC problem is converted to an SO problem. The resulting SO problem is a single-stage problem. Thus, the parameterized expected cost functional is approximated by sample average approximation thereby the problem is formulated as a deterministic optimal control problem, which can be solved by optimization methods more easily. Also, the proposed method does not require operational matrix that is an obstacle for works which use Chebychev wavelets to convert a differential equation into an algebraic one. Additionally, to solve an SOC problem by the proposed method does not require to solve a partial differential HJB equation, which is a limitation for solving SOC problems, whereas the optimal trajectory and optimal control are approximated properly. The convergence of the proposed method is proved through a new theorem. Finally, the proposed method is implemented for solving one of the well-known problems in mathematical finance, the Merton portfolio allocation problem in finite horizon. The simulation results illustrate the improvement of the constructed portfolio return.

This paper is organized as follows: The mathematical preliminaries of the proposed method are presented in section 2. The Chebyshev wavelet-based method is presented in section 3. The Merton portfolio allocation problem as an application of the proposed method is simulated in section 4. Finally, the conclusion is presented in section 5.

2 Mathematical preliminaries

2.1 Formulation of a stochastic optimal control problem

Assume that $\{X_t\}_{t\in[0,T]}$ is an n-dimensional stochastic process within the probability space $(\Omega, \mathcal{F}, \mathbb{R})$ and that $\{B_t\}_{t\in[0,T]}$ is a d-dimensional Brownian motion. The finite-horizon SOC problem is defined as follows [28]:

$$\min_u E\Big[\int_0^T L(X_t, u_t)dt + G(T, X_T)\Big], \tag{1a}$$

$$s.t. \qquad dX_t = b(X_t, u_t)dt + \sigma(X_t, u_t)dB_t, \tag{1b}$$

$$X(0) = X_0. \tag{1c}$$

The process $\{X_t\}_{t\in[0,T]}$ in $L^2(\Omega \times [0,T])$ is driven by controlled Ito stochastic differential equation (1b) with initial condition (1c). Also, b and σ are describing the drift and diffusion parts of the random state process $X_t \in \mathbb{R}^n$ for all $t \in [0,T]$, respectively. The stochastic control process $\{u_t\} = \{u(\omega,t)\}_{u_t\in\mathbb{R}^n}$ is chosen over a set $\mathcal{A}$ of admissible controls such that the cost function (1a) be minimized.
Additionally, the taken decision at time $t \in [0,T]$ depends on what already has happened up to t. Therefore, the stochastic control process must be $\mathcal{F}_t$-adapted [12]. The admissible control functions can be chosen as "deterministic" controls $u(t,\omega) = u(t)$, "open-loop" $u(t,\omega)$ controls, which are nonanticipative with respect to the Brownian motion $\{B_t\}$ and "Markov" controls $u(t,\omega) = u_M(t, X(\omega))$, where $u_M(\cdot)$ is a nonrandom and Lebesgue-measurable function. The process$\{X_t\}$ becomes an Ito diffusion under Markov control [25]. Also, the function u_M must be smooth.

2.2 Chebyshev wavelets

In this subsection, Chebyshev wavelet polynomials are described, briefly. A family of wavelets can be constituted by dilation and translation of a function Ψ called mother wavelet.

Consider the following family of continuous wavelet:

$$\Psi_{a,b}(t) = |a|^{-\frac{1}{2}}\Psi(\frac{t-a}{b}), \quad a, b \in \mathbb{R}, a \neq 0, \tag{2}$$

where the dilation parameter a and translation parameter b are varying continuously. The family of Chebyshev wavelets $\Psi_{nm}(t) = \Psi(k, n, m, t)$ is one of the applicable wavelets that is defined on interval $[0, 1]$ as follows [29]:

$$\Psi_{nm}(t) = \begin{cases} \frac{\alpha_m 2^{\frac{k}{2}}}{\sqrt{\pi}} T_m(2^{k+1}t - 2n + 1), & \frac{n-1}{2^k} \leqslant t \leqslant \frac{n}{2^k}, \\ 0, & otherwise, \end{cases} \tag{3}$$

where $k \in \mathbb{N}$ and $n = 1, 2, 3, \ldots, 2^k$. Also, $m = 0, 1, 2, \ldots, M-1$ for M-order Chebyshev polynomials and

$$\alpha_m = \begin{cases} \sqrt{2}, & m = 0, \\ 2, & m = 1, 2, \ldots, \end{cases} \tag{4}$$

Additionally, $T_m(t)$'s in (3) are Chebyshev polynomials that can be constructed by the following recursive relations:

$$\begin{cases} T_0(t) = 1, \\ T_1(t) = t, \\ \vdots \\ T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t). \end{cases} \tag{5}$$

3 Solving stochastic optimal control problem via Chebyshev wavelet-based method

In this section, a proposed Chebyshev wavelet-based method for solving general SOC problems is presented. This procedure is presented in three subsections. In subsection 3.1, stochastic state and control process are expanded in Chebyshev Wavelet polynomials. In subsection 3.2, the expected cost function approximation is described. Finally, the convergence analysis of the method is proved in subsection 3.3

3.1 Chebyshev wavelet expansion of stochastic state and control process

Let $Q \subseteq C^1[0,T]$ be the set of all piecewise-continuous functions with initial condition (1c) and let $Q_{2^k,M-1} \subseteq Q$ be the class of combinations of Chebyshev wavelet polynomials of degree up to $(M-1)$.
Consider the Chebyshev wavelet approximations of state process $\{X_t\}_{t\in[0,T]}$ and control process $\{u_t\}$ as follows:

$$\hat{X}(t) = \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} a_{nm} \Psi_{nm}(t), \tag{6}$$

and

$$\hat{U}(t) = \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} c_{nm} \Psi_{nm}(t). \tag{7}$$

Therefore, the SOC problem (1) can be interpreted as a stochastic minimization problem on Q. For this purpose, the interval $[0,T]$ is divided to 2^k subintervals; that is,

$$[0,T] = [0, \frac{1}{2^k}T] \bigcup [\frac{1}{2^k}T, \frac{2}{2^k}T] \bigcup \cdots \bigcup [\frac{2^k-1}{2^k}T, T]. \tag{8}$$

Thus, the state variable (6) and the control variable (7) is rewritten as follows:

$$\hat{X}(t) = \begin{cases} \hat{x}_1(t) = \sum_{m=0}^{M-1} a_{1m}\Psi_{1m}(t), & 0 \leqslant t \leqslant \frac{1}{2^k}T, \\ \hat{x}_2(t) = \sum_{m=0}^{M-1} a_{2m}\Psi_{2m}(t), & \frac{1}{2^k}T \leqslant t \leqslant \frac{2}{2^k}T, \\ \vdots \\ \hat{x}_{2^k}(t) = \sum_{m=0}^{M-1} a_{2^k m}\Psi_{2^k m}(t), & \frac{2^k-1}{2^k}T \leqslant t \leqslant T. \end{cases} \tag{9}$$

Also,

$$\hat{U}(t) = \begin{cases} \hat{u}_1(t) = \sum_{m=0}^{M-1} c_{1m}\Psi_{1m}(t), & 0 \leqslant t \leqslant \frac{1}{2^k}T, \\ \hat{u}_2(t) = \sum_{m=0}^{M-1} c_{2m}\Psi_{2m}(t), & \frac{1}{2^k}T \leqslant t \leqslant \frac{2}{2^k}T, \\ \vdots \\ \hat{u}_{2^k}(t) = \sum_{m=0}^{M-1} c_{2^k m}\Psi_{2^k m}(t), & \frac{2^k-1}{2^k}T \leqslant t \leqslant T. \end{cases} \tag{10}$$

Now, from (9) and (10), the cost functional (1a) becomes

$$E[\hat{J}(a_{10}, \ldots, a_{2^k M-1}, c_{10}, \ldots, c_{2^k M-1})] = E\Big[\int_0^T L(\hat{X}, \hat{U})dt + G(T, \hat{X}_T)\Big].$$

In other words, we have

$$E[\hat{J}] = E\Big[\int_0^{\frac{1}{2^k}T} L(\hat{x}_1(t), \hat{u}_1(t))dt + \int_{\frac{1}{2^k}T}^{\frac{2}{2^k}T} L(\hat{x}_2(t), \hat{u}_2(t))dt \\ + \cdots + \int_{\frac{2^k-1}{2^k}T}^{T} L(\hat{x}_{2^k}(t), \hat{u}_{2^k}(t))dt + G(T, \hat{x}_{2^k}(T))\Big]. \tag{11}$$

Additionally, from (9) and (10), the controlled Ito SDE (1b) becomes

$$\dot{\hat{X}}_t = b(\hat{X}_t, \hat{U}_t) + \sigma(\hat{X}_t, \hat{U}_t)\dot{B}_t. \tag{12}$$

More precisely, the following constraints are yielded

$$\begin{cases} \dot{\hat{x}}_1(t) = b(\hat{x}_1, \hat{u}_1) + \sigma(\hat{x}_1, \hat{u}_1)\dot{B}_1, \\ \dot{\hat{x}}_2(t) = b(\hat{x}_2, \hat{u}_2) + \sigma(\hat{x}_2, \hat{u}_2)\dot{B}_2, \\ \vdots \\ \dot{\hat{x}}_{2^k}(t) = b(\hat{x}_{2^k}, \hat{u}_{2^k}) + \sigma(\hat{x}_{2^k}, \hat{u}_{2^k})\dot{B}_{2^k}. \end{cases} \tag{13}$$

Also, according to the initial condition (1c), we have the following constraint:

$$\sum_{m=0}^{M-1} a_{1m}\Psi_{1m}(0) - X_0 = 0. \tag{14}$$

According to the continuity property of the state variable and (8) for

$$t_i = \frac{i}{2^k}, \qquad i = 1, 2, \ldots, 2^k - 1, \tag{15}$$

one can yield

$$\begin{array}{l} \sum_{m=0}^{M-1} a_{1m}\Psi_{1m}(t_1) = \sum_{m=0}^{M-1} a_{2m}\Psi_{2m}(t_1), \\ \sum_{m=0}^{M-1} a_{2m}\Psi_{1m}(t_2) = \sum_{m=0}^{M-1} a_{3m}\Psi_{3m}(t_2), \\ \vdots \\ \sum_{m=0}^{M-1} a_{2^k-1m}\Psi_{2^k-1m}(t_{2^k-1}) = \sum_{m=0}^{M-1} a_{t_{2^k m}}\Psi_{2^k m}(t_{2^k-1}). \end{array} \tag{16}$$

According to the SOC problem (1) and the parameterization process, which was mentioned above, the following SO problem with objective function (11) subject to constraints (13)–(14) and (16) is designed, which can be solved by SO methods [8, 18], more easily.

$$\min E[\hat{J}(\alpha_t, \gamma_t, B_t)], \tag{17a}$$

$$subject\,to: \qquad P[\alpha_t, \gamma_t, B_t]^T = C, \tag{17b}$$

where $\alpha = [a_{10}, \dots, a_{1M-2}, a_{20}, \dots, a_{2M-1}, \dots, a_{2^k0}, \dots, a_{2^kM-1}]$,
$\gamma = [c_{10}, \dots, c_{1M-2}, c_{20}, \dots, c_{2M-1}, \dots, c_{2^k0}, \dots, c_{2^kM-1}]$ and $\{B_t\}$ is the Brownian motion.

3.2 Expected cost function approximation

Consider the SO problem (17). Let $\mathbb{R}^{2k+1M}$ be the domain of all the feasible decisions and let $(\alpha_t, \gamma_t) = \mathcal{X}$ be a specific decision. The goal is to find a decision that minimizes the cost function, $\hat{J}$. Let the Brownian motion $\{\xi_t\}$ denote random information that is available only after a taken decision. Since we cannot directly optimize $\hat{J}(\mathcal{X}, \xi)$, we compute minimize the expected value, $E[\hat{J}(\mathcal{X}, \xi)]$. SAA is the most common approach of a choice for the approximation of $E[\hat{J}(\mathcal{X}, \xi)]$. The first step in SAA is sampling. let $\{\xi_1, \dots, \xi_n\}$ be a set of independent identically distributed realizations of random process $\{\xi_t\}$ and let $\hat{J}(\alpha_t, \gamma_t, \xi_i)$ be the cost function realization for ξ_i. According to the SAA approach, the expected cost function is approximated by the average of the realizations:

$$E[\hat{J}(\mathcal{X}, \xi)] \approx \frac{1}{n} \sum_{i=1}^{n} \hat{J}(\mathcal{X}, \xi_i). \tag{18}$$

The second step in SAA is the solving an optimal problem. The right-hand side of (18) is deterministic, so deterministic optimization methods can be used for solving the following approximated problem for all $n \in \mathbb{N}$:

$$\beta_n^* = \min\nolimits_{(\alpha_t, \gamma_t) \in \mathbb{R}^{2k+1M}} E[\hat{J}(\alpha_t, \gamma_t, \xi)] = \min\nolimits_{\mathcal{X} \in \mathbb{R}^{2k+1M}} \frac{1}{n} \sum_{i=1}^{n} \hat{J}(\mathcal{X}, \xi_i). \tag{19}$$

Deterministic search is the main benefit of SAA [8].

3.3 Convergence analysis

In this section, the convergence of the proposed method is investigated.

Theorem 1. Suppose that $f(x) \in L^2[0,1]$ with bounded second derivative $|\ddot{f}| \leq L$, and let $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{nm}\Psi_{nm}(x)$ be its infinite Chebyshev wavelet expansion. Then

$$|c_{nm}| \leq \frac{\sqrt{(2\pi)}L}{(2n)^{\frac{5}{2}}(m^2-1)}. \tag{20}$$

This means that the Chebyshev wavelets series converges uniformly to $f(x)$; that is,

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\Psi_{nm}(x). \tag{21}$$

Proof. See [7]. □

Theorem 2. Let $\beta = \inf_{(X,u)\in Q} E[J]$. If $\beta_{2^kM-1} = \inf_{(X,u)\in Q_{2^kM-1}} E[J(X,u)]$ for $k, M = 1, 2, \ldots$, where Q_{2^kM-1} is a subset of Q, then $\lim_{k,M\to\infty} \beta_{2^kM-1} = \beta$.

Proof. Since $(X_t, u_t) \in Q_{2^kM-1}$, by considering the Chebyshev expansions (6) and (7) for stochastic processes $\{X_t\}$ and $\{u_t\}$, respectively, we have

$$\beta_{2^kM-1} = \min_{(\alpha_t,\gamma_t)\in\mathbb{R}^{2k+1M}} E[\hat{J}(\alpha_t,\gamma_t,\xi)]. \tag{22}$$

Now, let $\{\xi_1,\ldots,\xi_n\}$ be a set of independent identically distributed realizations of random process $\{\xi_t\}$ and let $\hat{J}(\alpha_t,\gamma_t,\xi_i)$ be the cost function realization for ξ_i. According to the SAA approach, the expected cost function is approximated by the average of the realizations:

$$E[\hat{J}(\alpha,\gamma,\xi)] \approx \frac{1}{n}\sum_{i=1}^{n}\hat{J}(\alpha,\gamma,\xi_i). \tag{23}$$

If we define

$$\beta^*_{n,2^kM-1} = \min_{(\alpha,\gamma)\in\mathbb{R}^{2k+1M}} \frac{1}{n}\sum_{i=1}^{n}\hat{J}(\alpha,\gamma,\xi_i), \tag{24}$$

then there exists an optimal value $(\alpha^*,\gamma^*,\xi_i)$ for

$$\arg\min\{\frac{1}{n}\sum_{i=1}^{n}\hat{J}(\alpha,\gamma,\xi_i) : (\alpha,\gamma,\xi_i) \in (\mathbb{R}^{2k+1M}\times\mathbb{R})\}, \tag{25}$$

such that

$$\beta^*_{n,2^kM-1} = \frac{1}{n}\sum_{i=1}^{n} \hat{J}(\alpha^*, \gamma^*, \xi_i). \tag{26}$$

Also, one can conclude that there exists a corresponding optimal value (X^*, u^*) from $\arg\min\{\frac{1}{n}\sum_{i=1}^{n} \hat{J}(X, u) : (X, u) \in Q_{2^kM-1}\}$ such that

$$\beta^*_{n,2^kM-1} = \frac{1}{n}\sum_{i=1}^{n} \hat{J}(X^*, u^*). \tag{27}$$

Furthermore, since $Q_{2^kM-1} \subseteq Q_{2^kM}$, we have

$$\beta^*_{n,2^kM} = \frac{1}{n}\sum_{i=1}^{n} \hat{J}(X^*, u^*) = \min_{(X,u)\in Q_{2^kM}} \frac{1}{n}\sum_{i=1}^{n} \hat{J}(X, u) \tag{28}$$

$$\leq \min_{(X,u)\in Q_{2^kM-1}} \frac{1}{n}\sum_{i=1}^{n} \hat{J}(X, u) \tag{29}$$

$$= \beta^*_{n,2^kM-1}. \tag{30}$$

Therefore, for all k, M and n, we have $\beta^*_{n,2^kM} \leq \beta^*_{n,2^kM-1}$. Since, the sequence $\{\beta^*_{n,2^kM}\}$ is a nonincreasing sequence. Additionally, this sequence is upper bounded and therefore is convergent and this completes the proof. □

For implementing of the proposed method, Chebyshev Wavelet-based Method algorithm for solving SOC problems, is presented as follows:

Chebyshev wavelet-based method algorithm for stochastic optimal control problems

Input : SOC problem (1).

Output : The approximated Stochastic optimal trajectory and approximated optimal control.

Step 1 : Choose N, k, and M, and set $i = 1$

Step 2 : Approximate the state and control variable by Chebyshev wavelet series from equations (9) and (10).

Step 3 : Find an expression of $E[\hat{J}]$ from equation (11).

Step 4 : Determine the set of equality constraints, by (13), (14), and (16).

Step 5 : Consider a set of independent identically distributed realizations of Brownian motion $\{B_t\}$ and approximate the expected cost function $E[\hat{J}(\alpha, \gamma, B)]$ from (18).

Step 6 : Determine the optimal parameters $[\alpha^*, \gamma^*]$ by solving the optimization problem with the resulting cost function of **Step 5** subject to (17b) and substitute these parameters into (9) and (10) to find the approximated optimal trajectory, approximated optimal control.

Step 7 : If $i = N$ go to **Step 8** else set $i = i + 1$ and go to **Step 2**.

Step 8 : End.

4 Simulation results

In this section, one of the well-known problems in continuous-time finance, the Merton portfolio allocation problem, is considered for illustrating the advantages and efficiency of the proposed method.

Example 1 (Merton portfolio allocation problem in finite hoziron[15])**.** Consider an investing process on time interval $[0, T]$, and denote wealth at time $t \in [0, T]$ by W_t. Also, suppose that this investing is started with a known initial wealth W_0. At time t, we must choose a fraction of wealth to invest in a (set of risky assets) stock portfolio $U_1(t)$ whereas the remaining fraction,

$1-U_1(t)$ is invested in a (the risk-free asset) bond . Additionally, the variable $U_2(t)$ is considered as wealth consumption value at time t. Therefore, the control vector has two component, $0 \leq U_1(t) \leq 1$ and $U_2(t) \geq 0$.

Let the interest rate for risky and risk-free investment be R and r, respectively, such that $0 < r < R$. Thus, the bond price b_t and the stock price S_t evolve according to the following Black-Scholes models:

$$\frac{db(t)}{b} = rdt, \qquad \frac{dS_t}{S} = Rdt + \sigma dB_t, \tag{31}$$

where σ is a real positive constant that denotes the volatility of S, and B_t is a Brownian motion described. The objective is to find an optimal two-dimensional control vector $(U_1(t), U_2(t))$, which maximizes the following expected cost functional:

$$E[J] = E\Big[\int_0^T e^{-\beta t} F(U_2(t))dt\Big] \tag{32}$$

subject to

$$\begin{aligned} dW_t &= \tfrac{db(t)}{b} W_t + \tfrac{dS(t)}{S} W_t - U_2(t)dt \\ &= [(1-U_1(t))r + U_1 R]W_t dt + U_1(t)\sigma W_t dB_t - U_2(t)dt, \end{aligned} \tag{33}$$

$$W(0) = W_0, \tag{34}$$

where β and F are discount rate and utility function, respectively.

Consider the utility function $F(U_2(t)) = U_2^{\alpha}(t), 0 < \alpha < 1$. This problem is solved by the proposed method for $k = 1, M = 3, r = 0.05, R = 0.11, \alpha = 0.5, \beta = 0.11, W_0 = 10^5$ and $\sigma = 0.4$ on the interval $[0, 1]$. Therefore, the state and control variables are approximated as follows:

$$\hat{W}(t) = \begin{cases} \hat{w}_1(t) = \sum_{m=0}^{2} a_{1m}\Psi_{1m}(t), & 0 \leqslant t \leqslant \frac{1}{2}, \\ \hat{w}_2(t) = \sum_{m=0}^{2} a_{2m}\Psi_{2m}(t), & \frac{1}{2} \leqslant t \leqslant 1, \end{cases} \tag{35}$$

$$\hat{U}_1(t) = \begin{cases} \hat{u}_1(t) = \sum_{m=0}^{2} c_{1m}\Psi_{1m}(t), & 0 \leqslant t \leqslant \frac{1}{2}, \\ \hat{u}_2(t) = \sum_{m=0}^{2} c_{2m}\Psi_{2m}(t), & \frac{1}{2} \leqslant t \leqslant 1, \end{cases} \tag{36}$$

$$\hat{U}_2(t) = \begin{cases} \hat{v}_1(t) = \sum_{m=0}^{2} d_{1m}\Psi_{1m}(t), & 0 \leqslant t \leqslant \frac{1}{2}, \\ \hat{v}_2(t) = \sum_{m=0}^{2} d_{2m}\Psi_{2m}(t), & \frac{1}{2} \leqslant t \leqslant 1, \end{cases} \tag{37}$$

where

$$\Psi(t) = [\Psi_{10}(t), \Psi_{11}(t), \Psi_{12}(t), \Psi_{20}(t), \Psi_{21}(t), \Psi_{22}(t)],$$
$$a(t) = [a_{10}, a_{11}, a_{12}, a_{20}, a_{21}, a_{22}],$$
$$\gamma(t) = [c_{10}, c_{11}, c_{12}, c_{20}, c_{21}, c_{22}, d_{10}, d_{11}, d_{12}, d_{20}, d_{21}, d_{22}].$$

From (35) and (36), the expected cost functional (32) is parameterized as follows:

$$E[\hat{J}] = E\Big[\int_0^{\frac{1}{2}} e^{-\beta t}\hat{v}_1^{\alpha}(t)dt + \int_{\frac{1}{2}}^{1} e^{-\beta t}\hat{v}_2^{\alpha}(t)dt\Big]. \tag{38}$$

By substituting (35) and (36) into the SED equation (33) and applying the differential transformation $\frac{dB}{dt} = t_k B(t_k)$ [24], the following equality constraints can be obtained:

$$\begin{cases} [\dot{\hat{w}}_1 - [(1-\hat{u}_1)r + \hat{u}_1 R]\hat{w}_1]_{t=t_k} + [\hat{u}_1\sigma\hat{w}_1]_{t=t_k} t_k B_k - \hat{v}_1(t_k) = 0, t_k \in \{0, \frac{1}{2}\}, \\ (\dot{\hat{w}}_2 - [(1-\hat{u}_2)r + \hat{u}_2 R]\hat{w}_2)_{t=t_k} + [\hat{u}_2\sigma\hat{w}_2]_{t=t_k} t_k B_k - \hat{v}_2(t_k) = 0, t_k \in \{\frac{1}{2}, 1\}. \end{cases} \tag{39}$$

The continuity of state variable must be satisfied at point $t_1 = \frac{1}{2}$. Therefore, the following equality constraint is considered as well:

$$\hat{w}_1(t_1) - \hat{w}_2(t_1) = 0. \tag{40}$$

Also, from initial condition (34) another constraint is produced as follows:

$$\hat{w}_1(0) - 10^5 = 0. \tag{41}$$

Now, consider a Brownian motion, $B_t \sim N(0,1)$. Let $\{B_t(t_1), B_t(t_2), B_t(t_3)\}$ be a sample of random process B_t. According to the SAA method, the expected cost functional (38) is approximated as

$$E[\hat{J}(a,\gamma,B_t)] \approx \frac{1}{3}\sum_{i=1}^{3}\hat{J}(a,\gamma,B_i). \tag{42}$$

By maximizing (42) subject to constraints (39)–(41), the optimal vector $[a^*, \gamma^*]$ is obtained. Therefore, the optimal approximated trajectory of wealth $\{W_t\}$ and the optimal approximated control $\{U_2(t)\}$ can be obtained. The simulation results are shown in Figures 1 and 2.

In the following, the simulation results based on presented method in this paper are comprised with the simulation result of method proposed in [11].

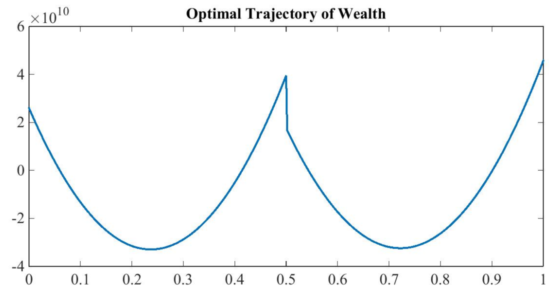

Figure 1: The wealth trajectory by Chebyshev wavelet-based method.

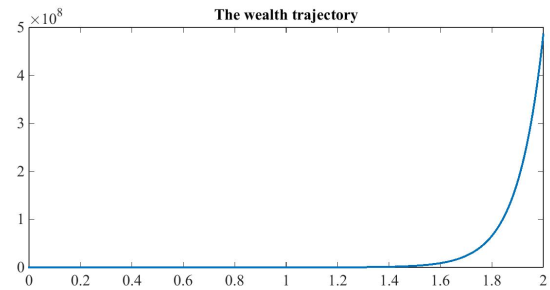

Figure 2: The wealth trajectory by method proposed in [11]

The optimal trajectory of the wealth $\{W_t\}$ by the Chebyshev wavelet-based method and method proposed in [11] is given in Figures 1 and 2, respectively. Figure 1 shows that the Chebyshev wavelet-based portfolio increases the initial wealth value from 10^5\$ to $5{*}10^{10}$\$ while according to Figure 2, the proposed method in [11] has increased it to $5 * 10^8$\$. Therefore, from comparison point of view, the resulting portfolio of Chebyshev wavelet-based method has a higher return.

5 Conclusion

In this paper, a new Chebyshev wavelet-based algorithm, which is the state-control parameterization method for solving problems, has been presented. The state and control process were parameterized via orthogonal Chebyshev polynomial basis functions. Therefore, the SOC problem was converted to an SO, and an optimal approximation of the solution was produced without requiring compute the operational matrix of the derivative. An advantage of the proposed method is that an SOC problem is solved without encountering a partial differential HJB equation. The convergence of the proposed method was proved via a new theorem. One of the well-known problems in mathematical finance, the Merton portfolio allocation problem in finite horizon, was simulated by the proposed method. The simulation results showed the capability and efficiency of the proposed method in comparison with other similar works. For instance, the return of the constructed portfolio was improved by the proposed method.

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