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Research Article

A shifted fractional-order Hahn functions Tau method for time-fractional PDE with nonsmooth solution

N. Mollahasani*,

Abstract

In this paper, a new orthogonal system of nonpolynomial basis functions is introduced and used to solve a class of time-fractional partial differential equations that have nonsmooth solutions. In fact, unlike polynomial bases, such basis functions have singularity and are constructed with a fractional variable change on Hahn polynomials. This feature leads to obtaining more accurate spectral approximations than polynomial bases. The introduced method is a spectral method that uses the operational matrix of fractional order integral of fractional-order shifted Hahn functions and finally converts the equation into a matrix equation system. In the introduced method, no collocation method has been used, and initial and boundary conditions are applied during the execution of the method. Error and convergence analysis of the numerical method has been investigated in a Sobolev space. Finally, some numerical experiments are considered in the form of tables and figures to demonstrate the accuracy and capability of the proposed method.

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*Corresponding author

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Nasibeh Mollahasani
Department of applied mathematics, Graduate university of advanced technology, Kerman, Iran.
e-mails: n.mollahasani@math.uk.ac.ir, nasibmo62@gmail.com

1 Introduction

In recent works, science and engineering researchers found that the use of fractional calculus in modeling gives a more realistic description of various complex phenomena with long-range temporal cumulative memory. Fractional order operators have nonlocal and memory features. Therefore, these two important properties simulate and describe a variety of engineering and scientific problems with memory characteristics and inheritance more appropriately than integer order differential equations, such as finance [30], physics [36], and hydrology [3], by using fractional differential equations. Many analytical methods have been used to solve fractional differential equations, such as the Green function method, Fourier, Laplace, and Mellin transform methods [24]. The complexity of integral and fractional differential operators and also the nonobservance of many properties expected in classic calculus encouraged researchers to study effective and reliable numerical methods for solving fractional differential equations. These numerical methods mainly include finite element and finite difference methods, spectral methods, and so on [31, 9, 8, 29, 34, 11, 1, 13, 22, 5, 23, 19]. In solving fractional order differential equations, two basic features that make classical methods not efficient and accurate are that fractional order operators have nonlocal properties and the other is the singularity of the solutions of fractional equations. Therefore, spectral methods based on ordinary polynomials, which have high accuracy for solving problems with smooth solutions (see, for example, [33, 12]), are not suitable for solving fractional differential equations with nonsmooth solutions since they do not have high expected accuracy. Concerning the numerical solution of partial differential equations dependent on time, one of the most common approaches is to use the finite difference approximation together with the spectral approximation for time and spatial derivatives, respectively. One of the main drawbacks of this approach is that the temporal discretization error may overcome the spatial discretization error, and the unknowns have to be solved simultaneously at all times [20]. As emphasized above, fractional differential equations mostly have nonsmooth solutions. It is also possible to encounter coefficients in terms of the given fractional equation in a nonsmooth case. On the other hand, in most of the spectral-introduced solving methods, in order to achieve high accuracy, they raise unrealistic assumptions. For instance, one of the assumptions in most of them is the smoothness of the unique regular solution of the fractional differential equation at the initial time $t = 0$ [32, 21, 35, 16]. So far, very few works have been done to solve fractional differential and integral equations with nonsmooth solutions, numerically, some of which can be seen in [15, 25, 26]. Due to their high accuracy, spectral methods have become one of the first choices researchers study to solve fractional differential equations with nonsmooth solutions. Among these techniques, we can refer to the methods available in [37, 38, 7]. Analytical and numerical studies indicate the exponential convergence of these methods for nonsmooth solutions in certain situations, and by using specific techniques, though, the exact solution of fractional time differential equations does not generally follow the mentioned form [27, 14].

This paper is organized as follows: fractional Hahn functions and their properties are defined in section 2, and also, function approximation and the operational matrix of fractional integration are introduced. In section 3, our proposed method is described. An error analysis is presented in section 4, and finally, some numerical examples are depicted in section 5.

2 Fractional-order shifted Hahn functions approximation

The main goal of this section is to introduce a new class of fractional basis functions, which are defined using shifted Hahn polynomials (SHPs) and applied to calculating their operational matrix of fractional integration.

Definition 1. For given constants $\sigma_1, \sigma_2 > -1$, and $M \in \mathbb{N}$, Hahn polynomials on $[0, M]$ are defined as [17]

$$h_k(x; \sigma_1, \sigma_2, M) = \sum_{i=0}^{k} \frac{(-k)_i (k+\sigma_1+\sigma_2+1)_i (-x)_i}{(\sigma_1+1)_i (-M)_i i!}, \quad k = 0, 1, 2, \ldots, M, \tag{1}$$

where $(\cdot)_i$ is the Pochhammer notation, which is defined as

$$\begin{cases} (\zeta)_0 = 1, \\ (\zeta)_i = \zeta(\zeta+1)\cdots(\zeta+i-1), \quad i \in \mathbb{N}, \quad \text{for } \zeta \in \mathbb{R}^+. \end{cases} \tag{2}$$

Remark 1. The relationship between Stirling numbers and Pochhammer notation is as follows:

$$(-k)_i = (-1)^i \sum_{l=0}^{i} S_i^{(l)} k^l, \tag{3}$$

where $S_i^{(l)}$ are Stirling numbers of the first kind defined as

$$S_i^{(l)} = \sum_{r=0}^{i-l} (-1)^r \binom{i-1+r}{i-l+r} \binom{2r-l}{i-l-r} s_{i-l+r}^{(r)},$$

in which $s_i^{(l)}$ are Stirling numbers of the second kind in the form

$$s_i^{(l)} = \frac{1}{l!} \sum_{r=0}^{l} (-1)^{l-r} \binom{r}{l} r^i.$$

Now, by using (3) in (1) and the changing of variables as $x = \frac{Mt}{L}$, we can achieve the following standard polynomial form of SHPs on $[0, L]$ as

$$\begin{aligned}\overline{h_k}(t;\sigma_1,\sigma_2,M,L) &= h_k(\frac{Mt}{L};\sigma_1,\sigma_2,M)\\ &= \sum_{i=0}^{k}\sum_{l=0}^{i}(-1)^i\frac{(-k)_i(k+\sigma_1+\sigma_2+1)_i}{(\sigma_1+1)_i(-M)_i i!}\times S_i^{(l)}(\frac{M}{L})^l t^l\\ &= \sum_{i=0}^{k}\sum_{l=0}^{i}\Delta_{i,k,l}t^l,\end{aligned}$$

for $k = 0, 1, 2, \ldots, M$, where $\Delta_{i,k,l} = (-1)^i\frac{(-k)_i(k+\sigma_1+\sigma_2+1)_i}{(\sigma_1+1)_i(-M)_i i!}\times S_i^{(l)}(\frac{M}{L})^l$.

SHPs are orthogonal on $[0, L]$ via the inner product in the following form [28]:

$$\langle f,g\rangle_{\tilde{\omega}} := \sum_{r=0}^{M} f(\frac{L}{M}r)g(\frac{L}{M}r)\tilde{\omega}(r), \tag{4}$$

where $\tilde{\omega}(r)$ is a real nonnegative weight function defined by

$$\tilde{\omega}(x;\sigma_1,\sigma_2,M) = \binom{\sigma_1+x}{x}\binom{\sigma_2+M-x}{M-x}. \tag{5}$$

The orthogonal relationship of SHPs is as follows:

$$\langle\overline{h_k},\overline{h_j}\rangle_{\tilde{\omega}} := \begin{cases}\sum_{r=0}^{M}\overline{h_k}^2(\frac{L}{M}r,\sigma_1,\sigma_2,M,L)\tilde{\omega}(r), & k=j,\\ 0, & k\neq j.\end{cases} \tag{6}$$

To define fractional-order shifted Hahn functions (FOSHFs), t is substituted by t^α in SHPs such that α is a positive real number. Therefore, FOSHFs can be defined in the following form:

$$\begin{aligned}\overline{h_k^\alpha}(t;\sigma_1,\sigma_2,M,L) &= \sum_{i=0}^{k}\sum_{l=0}^{i}(-1)^i\frac{(-k)_i(k+\sigma_1+\sigma_2+1)_i}{(\sigma_1+1)_i(-M)_i i!}\times S_i^{(l)}(\frac{M}{L})^l t^{\alpha l}\\ &= \sum_{i=0}^{k}\sum_{l=0}^{i}\Delta_{i,k,l}t^{\alpha l}, \qquad k=0,1,2,\ldots,M.\end{aligned} \tag{7}$$

Proposition 1. FOSHFs are orthogonal on $[0, L]$ via the inner product in the following form:

$$\langle f,g\rangle_{\tilde{\omega}}^{\alpha} := \sum_{r=0}^{M} f((\frac{L}{M}r)^{\frac{1}{\alpha}})g((\frac{L}{M}r)^{\frac{1}{\alpha}})\tilde{\omega}(r), \tag{8}$$

where $\tilde{\omega}(r)$ is defined in (5).

Proof. Replacing f, g by $\overline{h_k^{\alpha}}, \overline{h_j^{\alpha}}$ in (8) and then using the orthogonality property (6), the assertion is available. $\square$

Definition 2. Associated with the FOSHFs, the orthonormal FOSHFs can be defined as

$$\begin{aligned}&\overline{\mathcal{H}_k^{\alpha}}(t;\sigma_1,\sigma_2,M,L)\\&\quad=\frac{1}{\sqrt{\langle\overline{h_k^{\alpha}}(t;\sigma_1,\sigma_2,M,L),\overline{h_k^{\alpha}}(t;\sigma_1,\sigma_2,M,L)\rangle_{\tilde{\omega}}^{\alpha}}}\overline{h_k^{\alpha}}(t;\sigma_1,\sigma_2,M,L).\end{aligned} \tag{9}$$

2.1 Function approximation

For an integer $m \geq 0$, the Sobolev space $H_{\tilde{\omega}}^{m}\ [a,b]$ is

$$H_{\tilde{\omega}}^{m}[a,b]=\{u\in L_{\tilde{\omega}}^{2}[a,b]:\ 0\leq j\leq m,\ u^{(j)}(x)\in L_{\tilde{\omega}}^{2}[a,b]\},$$

where $L_{\tilde{\omega}}^{2}$ is the space of all square-integrable functions with respect to the weight function $\tilde{\omega}$. Indeed, $H_{\tilde{\omega}}^{m}[a,b]$ is defined as the vector space of functions $u \in L_{\tilde{\omega}}^{2}\ [a,b]$ such that all derivatives of u of order up to m can be represented by functions in $L_{\tilde{\omega}}^{2}\ [a,b]$.
Goertz and Öffner described the expansion of a function by Hahn polynomials and concluded that the series expansion of a function by Hahn polynomials converges pointwise under some assumptions (for more details, see [10]). Therefore, any function $u(t) \in L_{\tilde{\omega}}^{2}[0,L]$ can be expanded in terms of FOSHFs basis. In practice, only the first $(M+1)$ terms of FOSHFs are considered. Hence

$$u(t)\simeq\sum_{i=0}^{M}u_i\overline{\mathcal{H}_i^{\alpha}}(t;\sigma_1,\sigma_2,M,L)=u_M(t)=\mathbf{U}^T\mathcal{H}^{(\alpha)}(t;\sigma_1,\sigma_2,M,L), \tag{10}$$

where $\mathbf{U}^T=[u_0,u_1,\ldots,u_M]$ is the vector of FOSHFs coefficients, which can be derived as

$$\begin{aligned}u_i&=\left\langle u(t),\overline{\mathcal{H}_i^{\alpha}}(t;\sigma_1,\sigma_2,M,L)\right\rangle_{\tilde{\omega}}^{\alpha}\\&:=\sum_{r=0}^{M}u((\frac{L}{M}r)^{\frac{1}{\alpha}})\overline{\mathcal{H}_i^{\alpha}}((\frac{L}{M}r)^{\frac{1}{\alpha}};\sigma_1,\sigma_2,M,L)\tilde{\omega}(r),\quad i=0,1,\ldots,M,\end{aligned} \tag{11}$$

and $\mathcal{H}^{(\alpha)}(t;\sigma_1,\sigma_2,M,L)$ is the vector of FOSHFs defined as follows:

$$\mathcal{H}^{(\alpha)}(t;\sigma_1,\sigma_2,M,L):=[\overline{\mathcal{H}_0^{\alpha}}(t;\sigma_1,\sigma_2,M,L),\overline{\mathcal{H}_1^{\alpha}}(t;\sigma_1,\sigma_2,M,L),$$

$$\ldots, \overline{\mathcal{H}^{\alpha}_{M}}(t;\sigma_1,\sigma_2,M,L)]^T. \tag{12}$$

For simplicity, from now on, $\mathcal{H}^{(\alpha)}(t;\sigma_1,\sigma_2,M,L)$ is presented by $\mathcal{H}^{(\alpha)}_{M}(t)$.

Similarly, any two variables function $f(x,t)\in L^2_{\tilde{\omega}}([0,L]\times[0,T])$ can be approximated by the FOSHFs as follows:

$$\begin{aligned} f(x,t) &\simeq \sum_{i=0}^{M}\sum_{j=0}^{N} f_{i,j}\overline{\mathcal{H}^{\alpha}_{i}}(x;\sigma_1,\sigma_2,N,L)\overline{\mathcal{H}^{\beta}_{j}}(t;\sigma_1,\sigma_2,N,L)\\ &=: f_{M,N}(x,t) = (\mathcal{H}^{(\alpha)}_{M}(x))^T F \mathcal{H}^{(\beta)}_{N}(t), \end{aligned} \tag{13}$$

where $F=[f_{i,j}]$ is an $(M+1)\times(N+1)$ matrix that its entries are

$$\begin{aligned} f_{i,j} = \sum_{r_1=0}^{M}\sum_{r_2=0}^{N} & u((\frac{L}{M}r_1)^{\frac{1}{\alpha}},(\frac{T}{N}r_2)^{\frac{1}{\beta}})\overline{\mathcal{H}^{\alpha}_{i}}((\frac{L}{M}r_1)^{\frac{1}{\alpha}};\sigma_1,\sigma_2,M,L)\\ & \overline{\mathcal{H}^{\beta}_{j}}((\frac{T}{N}r_2)^{\frac{1}{\beta}};\sigma_1,\sigma_2,N,T)\tilde{\omega}(r_1)\tilde{\omega}(r_2), \end{aligned} \tag{14}$$

for $i=0,1,\ldots,M$ and $j=0,1,\ldots,N$.

Theorem 1. Let $M,N\in\mathbb{N}$, $\Lambda=[0,L]\times[0,T]$ and let $f\in H^2_{\tilde{\omega}}(\Lambda)$. Suppose that $f_{M,N}(x,t) = (\mathcal{H}^{(\alpha)}_{M}(x))^T F\mathcal{H}^{(\beta)}_{N}(t)$ is the best approximation of f in $\Omega = span\{\overline{\mathcal{H}^{\alpha}_{i}}(x;\sigma_1,\sigma_2,M,L)\overline{\mathcal{H}^{\beta}_{j}}(t;\sigma_1,\sigma_2,N,T)|\ i=0,1,\ldots,M,\ j=0,1,\ldots,N\}$. We will have

$$\|f(x,t)-f_{M,N}(x,t)\|^2_{L^2_{\tilde{\omega}}(\Lambda)} \le \frac{L^{M+2}T^{M+2}}{2^{2(M+N+1)}(M+1)!(N+1)!}\tilde{F},$$

where $\tilde{F}=\max_{(x,t)\in\Lambda}|\frac{\partial^{M+N}g(x,t)}{\partial x^M\partial t^N}|$ such that $g(x,t)=f(x^{\frac{1}{\alpha}},t^{\frac{1}{\alpha}})$.

Proof. Let $\phi_{M,N}(\eta,\xi)$ be the interpolation polynomial of $g(\eta,\xi)=f(\eta^{\frac{1}{\alpha}},\xi^{\frac{1}{\alpha}})$ at $(M+1)(N+1)$ shifted Chebyshev points in Λ. Then

$$|g(\eta,\xi)-\phi_{M,N}(\eta,\xi)| \le \frac{1}{2^{M+N}(M+1)!}(\frac{L}{2})^{M+1}(\frac{T}{2})^{N+1}\max_{(\eta,\xi)\in\Lambda}|\frac{\partial^{M+N}g(\eta,\xi)}{\partial\eta^M\partial\xi^N}|.$$

If $\tilde{F}=\max_{(\eta,\xi)\in\Lambda}|\frac{\partial^{M+N}g(\eta,\xi)}{\partial\eta^M\partial\xi^N}|$ and $\eta=x^\alpha,\ \xi=t^\beta$ are sets, then we get

$$|g(x^\alpha,t^\beta)-\phi_{M,N}(x^\alpha,t^\beta)| \le \frac{1}{2^{M+N}(M+1)!}(\frac{L}{2})^{M+1}(\frac{T}{2})^{N+1}\tilde{F}. \tag{15}$$

It is obvious that $\phi_{M,N}(x^\alpha,t^\beta)\in\Omega$. So, since $f_{M,N}(x,t)$ is the best approximation of f concerning L^2-norm, we have

$$\|f(x,t)-f_{M,N}(x,t)\|_2 \le \|f(x,t)-\phi_{M,N}(x,t)\|_2$$

$$= (\int_0^L \int_0^T (f(x,t) - \phi_{M,N}(x,t))^2 dtdx)^{\frac{1}{2}}.$$

Thus, from (15) the assertion is derived. □

2.2 FOSHFs operational matrix of fractional integration

Here, an operational matrix of fractional integration for FOSHFs is going to be obtained. Note that the Riemann–Liouville fractional integration of order β for a function f is defined as

$$I^{\vartheta} f(x) = \frac{1}{\Gamma(\vartheta)} \int_a^x (x-t)^{\vartheta-1} f(t)dt, \qquad x > a, \quad \vartheta \geq 0. \tag{16}$$

For this special type of fractional integration, there are some particular properties. The most useful of which is

$$I^{\vartheta} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\vartheta+1)} x^{\vartheta+\gamma}. \tag{17}$$

Using the above concepts, the following lemma states the FOSHFs operational matrix of fractional integration.

Lemma 1. The fractional integration of order β of the vector $\mathbf{H}_M^{(\alpha)}(t)$ can be expanded by itself as follows:

$$I^{\vartheta} \mathcal{H}_M^{(\alpha)}(t) \simeq \mathfrak{P}_{\vartheta} \mathcal{H}_M^{(\alpha)}(t), \tag{18}$$

where $\mathfrak{P}_{\vartheta} = [\mathfrak{p}_{kj}]_{(M+1)\times(M+1)}$, which is called the FOSHFs operational matrix of fractional integration with

$$\mathfrak{p}_{kj} = \sum_{r=0}^{M} \sum_{i=0}^{k} \sum_{l=0}^{i} \sum_{i_1=0}^{j} \sum_{l_1=0}^{i_1} \tilde{\omega}(r) \overline{\Delta}_{i,k,l} \Delta_{i_1,j,l_1} \frac{\Gamma(\alpha l+1)}{\Gamma(\alpha l+\vartheta+1)} (\frac{L}{M} r)^{\frac{\vartheta+l\alpha}{\alpha}+l_1}.$$

Proof. According to (12), we have

$$I^{\vartheta} \mathcal{H}_M^{(\alpha)}(t) = \begin{bmatrix} I^{\vartheta} \overline{\mathcal{H}_0^{\alpha}}(t; \sigma_1, \sigma_2, M, L) \\ I^{\vartheta} \overline{\mathcal{H}_1^{\alpha}}(t; \sigma_1, \sigma_2, M, L) \\ \vdots \\ I^{\vartheta} \overline{\mathcal{H}_k^{\alpha}}(t; \sigma_1, \sigma_2, M, L) \\ \vdots \\ I^{\vartheta} \overline{\mathcal{H}_M^{\alpha}}(t; \sigma_1, \sigma_2, M, L) \end{bmatrix}. \tag{19}$$

By using (7), Proposition 1, the linear property of operator I, and (17) for each entry in (19), we will have

$$\begin{aligned} I^{\vartheta}\overline{\mathcal{H}_k^{\alpha}}(t;\sigma_1,\sigma_2,M,L) &= \sum_{i=0}^{k}\sum_{l=0}^{i}\overline{\Delta}_{i,k,l}I^{\beta}t^{\alpha l} \\ &= \sum_{i=0}^{k}\sum_{l=0}^{i}\overline{\Delta}_{i,k,l}\frac{\Gamma(\alpha l+1)}{\Gamma(\alpha l+\vartheta+1)}t^{\vartheta+l\alpha} \\ &\simeq \sum_{j=0}^{M}\mathfrak{p}_{kj}\overline{\mathcal{H}_j^{\alpha}}(t;\sigma_1,\sigma_2,M,L), \quad k=0,1,\ldots,M, \end{aligned}$$

where

$$\begin{aligned} \mathfrak{p}_{kj} &= \langle\sum_{i=0}^{k}\sum_{l=0}^{i}\overline{\Delta}_{i,k,l}\frac{\Gamma(\alpha l+1)}{\Gamma(\alpha l+\vartheta+1)}t^{\vartheta+l\alpha},\overline{\mathcal{H}_j^{\alpha}}(t;\sigma_1,\sigma_2,M,L)\rangle_{\tilde{\omega}}^{\alpha} \qquad (20) \\ &= \sum_{r=0}^{M}\tilde{\omega}(r)\sum_{i=0}^{k}\sum_{l=0}^{i}\overline{\Delta}_{i,k,l}\frac{\Gamma(\alpha l+1)}{\Gamma(\alpha l+\vartheta+1)}(\frac{L}{M}r)^{\frac{\vartheta+l\alpha}{\alpha}}\overline{\mathcal{H}_j^{\alpha}}((\frac{L}{M}r)^{\frac{1}{\alpha}};\sigma_1,\sigma_2,M,L) \\ &= \sum_{r=0}^{M}\sum_{i=0}^{k}\sum_{l=0}^{i}\tilde{\omega}(r)\overline{\Delta}_{i,k,l}\frac{\Gamma(\alpha l+1)}{\Gamma(\alpha l+\vartheta+1)}(\frac{L}{M}r)^{\frac{\vartheta+l\alpha}{\alpha}}\overline{\mathcal{H}_j^{\alpha}}((\frac{L}{M}r)^{\frac{1}{\alpha}};\sigma_1,\sigma_2,M,L), \end{aligned}$$

where $\overline{\Delta}_{i,k,l} = \frac{\Delta_{i,k,l}}{\sqrt{\langle\overline{h_k^{\alpha}}(t;\sigma_1,\sigma_2,M,L),\overline{h_k^{\alpha}}(t;\sigma_1,\sigma_2,M,L)\rangle_{\tilde{\omega}}^{\alpha}}}$. substituting (7) in (20) instead of $\overline{\mathcal{H}_j^{\alpha}}((\frac{L}{M}r)^{\frac{1}{\alpha}};\sigma_1,\sigma_2,M,L)$ finishes the proof. □

3 Description of method

The main aim of this section is to approximate the solution of the following equation:

$$D_t^{\vartheta}u(x,t) = -au(x,t) + b\frac{\partial u(x,t)}{\partial x} + c\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \qquad (21)$$

subject to the initial and boundary conditions

$$u(x,0)=g(x), \quad u(0,t)=\lambda(t), \quad u(L,t)=\eta(t) \quad \text{for } 0\le x\le L,\ 0\le t\le T.$$

Let

$$\frac{\partial^2 u(x,t)}{\partial x^2} \simeq (\mathcal{H}_M^{\alpha}(x))^T\mathbf{U}\mathcal{H}_N^{\beta}(t). \qquad (22)$$

Applying the integration operator I^{ϑ} on both sides of (22) and using the operational matrix of integration (18), for $\vartheta=1$ and $\vartheta=2$, respectively, yield

$$\frac{\partial u(x,t)}{\partial x} \simeq (\mathcal{H}_M^\alpha(x))^T(\mathfrak{P})^T\mathbf{U}\mathcal{H}_N^\beta(t) + xh(t), \tag{23}$$

$$u(x,t) \simeq (\mathcal{H}_M^\alpha(x))^T(\mathfrak{P}^2)^T\mathbf{U}\mathcal{H}_N^\beta(t) + xh(t) + \lambda(t), \tag{24}$$

in which the function $h(t)$ is calculated by putting $x = L$ in (24) and then applying the final condition $u(L,t) = \eta(t)$ as follows:

$$h(t) = \frac{1}{L}(\eta(t) - \lambda(t) - (\mathcal{H}_M^\alpha(L))^T(\mathfrak{P}^2)^T\mathbf{U}\mathcal{H}_N^\beta(t)).$$

Therefore (25) and (24) can be rewritten as follows:

$$\begin{aligned}\frac{\partial u(x,t)}{\partial x} \simeq\ & (\mathcal{H}_M^\alpha(x))^T(\mathfrak{P}^2)^T\mathbf{U}\mathcal{H}_N^\beta(t)\\ &+ \frac{1}{L}(\eta(t) - \lambda(t) - (\mathcal{H}_M^\alpha(L))^T(\mathfrak{P}^2)^T\mathbf{U}\mathcal{H}_N^\beta(t)), \qquad (25)\\ u(x,t) \simeq\ & (\mathcal{H}_M^\alpha(x))^T(\mathfrak{P}^2)^T\mathbf{U}\mathcal{H}_N^\beta(t)\\ &+ \frac{x}{L}(\eta(t) - \lambda(t) - (\mathcal{H}_M^\alpha(L))^T(\mathfrak{P}^2)^T\mathbf{U}\mathcal{H}_N^\beta(t)) + \lambda(t). \qquad (26)\end{aligned}$$

By substituting (22), (25), and (26) in (21), we get

$$\begin{aligned}D_t^\alpha u(x,t) \simeq\ & -a[(\mathcal{H}_M^\alpha(x))^T(\mathfrak{P}^2)^T\mathbf{U}\mathcal{H}_N^\beta(t)\\ &+ \frac{x}{L}(\eta(t) - \lambda(t) - (\mathcal{H}_M^\alpha(L))^T(\mathfrak{P}^2)^T\mathbf{U}\mathcal{H}_N^\beta(t)) + \lambda(t)]\\ &+ b[(\mathcal{H}_M^\alpha(x))^T(\mathfrak{P}^2)^T\mathbf{U}\mathcal{H}_N^\beta(t)\\ &+ \frac{1}{L}(\eta(t) - \lambda(t) - (\mathcal{H}_M^\alpha(L))^T(\mathfrak{P}^2)^T\mathbf{U}\mathcal{H}_N^\beta(t))]\\ &+ c(\mathcal{H}_M^\alpha(x))^T\mathbf{U}\mathcal{H}_N^\beta(t) + f(x,t)\\ =\ & (\mathcal{H}_M^\alpha(x))^T\mathfrak{A}\mathcal{H}_N^\beta(t), \qquad (27)\end{aligned}$$

where

$$\begin{aligned}\mathfrak{A} = & -a(\mathfrak{P}^2)^T\mathbf{U} + a\frac{\mathcal{X}}{L}(\mathcal{H}_M^\alpha(L))^T(\mathfrak{P}^2)^T\mathbf{U}\\ &+ b(\mathfrak{P})^T\mathbf{U} - b\frac{\infty}{L}(\mathcal{H}_M^\alpha(L))^T(\mathfrak{P}^2)^T\mathbf{U} + c\mathbf{U} + \mathbf{K}_1,\end{aligned}$$

and $\mathbf{K}_1$, $\mathbf{X}$, and $\mathbf{1}$ are the matrix and vector coefficient of FOSHF-approximation related to the following relations:

$$\begin{aligned}k_1(x,t) &= f(x,t) - a(\frac{x}{L}(\eta(t) - \lambda(t)) + \lambda(t)) + \frac{b}{L}(\eta(t) - \lambda(t))\\ &\simeq (\mathcal{H}_M^\alpha(x))^T\mathbf{K}_1\mathcal{H}_N^\beta(t),\\ x &\simeq (\mathcal{H}_M^\alpha(x))^T\mathbf{X},\end{aligned}$$

$$1 \simeq (\mathcal{H}_M^\alpha(x))^T \mathbf{1}.$$

Again, by applying the integration operator I_t^ϑ on both sides of (27) and using the operational matrix of integration $\mathfrak{P}^\vartheta$, we will have

$$u(x,t) \simeq (\mathcal{H}_M^\alpha(x))^T \mathfrak{A}\mathfrak{P}^\vartheta \mathcal{H}_N^\beta(t) + g(x). \tag{28}$$

Equalizing the right sides of (26) and (28), we get

$$(\mathcal{H}_M^\alpha(x))^T [(\mathfrak{P}^2)^T \mathbf{U} - \frac{\mathbf{X}}{L}(\mathcal{H}_M^\alpha(L))^T (\mathfrak{P}^2)^T \mathbf{U} - \mathfrak{A}\mathfrak{P}^\vartheta]\mathcal{H}_N^\beta(t) = (\mathcal{H}_M^\alpha(x))^T \mathbf{K}_2 \mathcal{H}_N^\beta(t),$$

where $\mathbf{K}_2$ is the matrix coefficient of FOSHF-approximation related to the following relation:

$$k_2(x,t) = g(x) - \frac{x}{L}(\eta(t) - \lambda(t)) - \lambda(t) \simeq (\mathcal{H}_M^\alpha(x))^T \mathbf{K}_2 \mathcal{H}_N^\beta(t).$$

Thus

$$(\mathfrak{P}^2)^T \mathbf{U} - \frac{\mathbf{X}}{L}(\mathcal{H}_M^\alpha(L))^T (\mathfrak{P}^2)^T \mathbf{U} - \mathfrak{A}\mathfrak{P}^\vartheta(t) = \mathbf{K}_2,$$

which can be rewritten as

$$\mathfrak{B}\mathbf{U} + \mathfrak{C}\mathbf{U}\mathfrak{D} = \mathfrak{E}, \tag{29}$$

where $\mathfrak{B} = (I - \frac{\mathbf{X}}{L}(\mathcal{H}_M^\alpha(L))^T)(\mathfrak{P}^2)^T$,
$\mathfrak{C} = [aI - \frac{a}{L}\mathbf{X}(\mathcal{H}_M^\alpha(L))^T + \frac{b}{L}\mathbf{1}(\mathcal{H}_M^\alpha(L))^T](\mathfrak{P}^2)^T - b\mathfrak{P}^T - cI$, $\mathfrak{D} = \mathfrak{P}^\vartheta$, and $\mathfrak{E} = \mathbf{K}_1\mathfrak{P}^\vartheta + \mathbf{K}_2$. Equation (29) is a matrix equation with the unknown matrix U. It can be solved by the global GMRES method. After solving the equation, by placing the obtained matrix U in (24), the approximate solution of the problem is obtained.

4 Error analysis

In this section, the convergence of the introduced method in a Sobolev space is considered. An upper bound is derived for the absolute error of the proposed method. To this end, some bounds are obtained for the approximations of different parts of the mentioned equation. First, the basic definitions and concepts related to Sobolev spaces are from the books [4, 18], with a slight change in symbols.

Let Λ be an open subset of $\mathbb{R}^n$ and let $L^2_{\tilde{\omega}}(\Lambda)$ be the space of all square-integrable functions concerning the weight function $\tilde{\omega}$. For an integer $m \geq 0$, the Sobolev space $H^m_{\tilde{\omega}}(\Lambda)$ is

$$H^m_{\tilde{\omega}}(\Lambda) = \{u |\ u \in L^2_{\tilde{\omega}}(\Lambda),\ \partial^\nu u \in L^2_{\tilde{\omega}}(\Lambda) \text{ for all } |\nu| \leq m\},$$

where ∂^{ν} is called the distributional derivatives and defined as the following form:

$$\partial^{\nu}u=\frac{\partial^{|\nu|}u}{\partial x_1^{\nu_1}\partial x_2^{\nu_2}\cdots\partial x_n^{\nu_n}},\quad |\nu|=\nu_1+\nu_2+\cdots+\nu_n.$$

For all $u,\ \nu\in H^m_{\tilde{\omega}}(\Lambda)$, the inner product is given as

$$\langle u,\nu\rangle_{H^m_{\tilde{\omega}}(\Lambda)}=\langle u,\nu\rangle_{L^2_{\tilde{\omega}}(\Lambda)}+\sum_{1\le|\nu|\le m}\langle\partial^{\nu}u,\partial^{\nu}\nu\rangle_{L^2_{\tilde{\omega}}(\Lambda)}.$$

The corresponding norm and seminorm are defined as

$$\|u\|_{H^m_{\tilde{\omega}}(\Lambda)}=(\|u\|^2_{L^2_{\tilde{\omega}}(\Lambda)}+\sum_{1\le|\nu|\le m}\|\partial^{\nu}u\|^2_{L^2_{\tilde{\omega}}(\Lambda)})^{\frac{1}{2}},$$

$$|u|_{H^m_{\tilde{\omega}}(\Lambda)}=(\sum_{|\nu|=m}\|\partial^{\nu}u\|^2_{L^2_{\tilde{\omega}}(\Lambda)})^{\frac{1}{2}}.$$

It is obvious to see that if $m\geqslant 0$, then $\|u\|_{L^2_{\tilde{\omega}}(\Lambda)}\leqslant\|u\|_{H^m_{\tilde{\omega}}(\Lambda)}$. In a special case, for $m=0$, it yields $\|u\|_{H^m_{\tilde{\omega}}(\Lambda)}=\|u\|_{L^2_{\tilde{\omega}}(\Lambda)}$. Also, for $m=0$, we have $|u|_{H^m_{\tilde{\omega}}(\Lambda)}=\|u\|_{L^2_{\tilde{\omega}}(\Lambda)}$.

Suppose that $u\in H^m_{\tilde{\omega}}(\Lambda)$ and $\mathcal{P}^{\alpha,\beta}_{M,N}$ are the orthogonal projection operator, where $\Lambda=[0,L]\times[0,T]$ and

$$\mathcal{P}^{\alpha,\beta}_{M,N}u:=\sum_{i=0}^{M}\sum_{j=0}^{N}u_{i,j}\overline{\mathcal{H}^{\alpha}_i}(x;\sigma_1,\sigma_2,N,L)\overline{\mathcal{H}^{\beta}_j}(t;\sigma_1,\sigma_2,N,L).$$

In other words, $\mathcal{P}^{\alpha,\beta}_{M,N}u=u_{M,N}(x,t)=(\mathcal{H}^{(\alpha)}_M(x))^TU\mathcal{H}^{(\beta)}_N(t)$.

In the following, for simplicity and brevity, $M=N$, $\alpha=\beta$, and $\mathcal{P}_M:=\mathcal{P}^{\alpha,\beta}_{M,N}$ are stated. According to [6], for all $u\in H^m_{\tilde{\omega}}(\Lambda)$, we have

$$\|u-\mathcal{P}_Mu\|_{H^j_{\tilde{\omega}}(\Lambda)}\le CM^{\rho(j)-m}|u|_{H^{m;M}_{\tilde{\omega}}(\Lambda)},\quad 0\le j\le m,\tag{30}$$

where C is a constant independent of M and only depends on m,

$$\rho(j)=\begin{cases}0, & j=0,\\ 2j-\frac{1}{2}, & j>0,\end{cases}$$

and

$$|u|_{H^{m;M}_{\tilde{\omega}}(\Lambda)}=(\sum_{k=\min(m,M+1)}^{m}\sum_{i=1}^{2}\|D^k_iu\|^2_{L^2_{\tilde{\omega}}})^{\frac{1}{2}}.$$

Theorem 2. Suppose that $u(x,t)\in H^m_{\tilde{\omega}}(\Lambda)$, $m\geqslant 0$ and that $u_{N,M}(x,t)$ is the best approximation of u. Then

$$\|u(x,t)-u_{N,M}(x,t)\|_{L^2_{\tilde{\omega}}(\Lambda)}\leqslant\|u(x,t)-u_{N,M}(x,t)\|_{H^j_{\tilde{\omega}}(\Lambda)}(\Lambda)$$

$$\leqslant CM^{\rho(j)-m}|u|_{H_{\tilde{\omega}}^{m;M}(\Lambda)}, \quad 0 \leq j \leq m. \tag{31}$$

Proof. Considering this fact that $\|\cdot\|_{L^2_{\tilde{\omega}}(\Lambda)} \leqslant \|\cdot\|_{H^m_{\tilde{\omega}}(\Lambda)}$, inequality (30), and the uniqueness of the best approximation, the proof of the theorem is easily done. □

Lemma 2. Suppose that the assumptions of Theorem 2 are true, that $u(x,t) \simeq u_{M,N}(x,t) = (\mathcal{H}_M^{(\alpha)}(x))^T U \mathcal{H}_N^{(\beta)}(t)$, and that $\mathfrak{P}_\vartheta$ is the FOSHF-operational matrix of fractional integration. Then

$$\begin{aligned}
&\|I_x^\vartheta u(x,t) - (\mathcal{H}_M^{(\alpha)}(x))^T \mathfrak{P}_\vartheta^T U \mathcal{H}_N^{(\beta)}(t)\|_{L_2(I)} \\
&\quad \leq \frac{L^\vartheta}{\Gamma(\vartheta+1)} CM^{\rho(j)-m}|u|_{H_{\tilde{\omega}}^{m;M}(\Lambda)}, \quad 0 \leq j \leq m.
\end{aligned}$$

Proof. According to (16), we have

$$\begin{aligned}
&\|I_x^\vartheta u(x,t) - (\mathcal{H}_M^{(\alpha)}(x))^T \mathfrak{P}_\vartheta^T U \mathcal{H}_N^{(\beta)}(t)\|_{L^2_{\tilde{\omega}}(\Lambda)} \\
&\quad = \|I_x^\vartheta u(x,t) - I_x^\vartheta u_{M,N}(t)\|_{L^2_{\tilde{\omega}}(\Lambda)} \\
&\quad = \|I_x^\vartheta (u(x,t) - u_{M,N}(x,t))\|_{L^2_{\tilde{\omega}}(\Lambda)} \\
&\quad = \|\frac{1}{\Gamma(\vartheta)} \int_0^x (x-\xi)^{\vartheta-1}(u(\xi,t) - u_{M,N}(\xi,t)) d\vartheta\|_{L^2_{\tilde{\omega}}(\Lambda)} \\
&\quad = \frac{1}{\Gamma(\vartheta)} \|x^{\vartheta-1} * (u(x,t) - u_{M,N}(x,t))\|_{L^2_{\tilde{\omega}}(\Lambda)}.
\end{aligned} \tag{32}$$

Now, by using this fact that $\|f*g\|_\rho \leq \|f\|_1.\|g\|_\rho$, and Theorem 2, respectively, we get

$$\begin{aligned}
&\|I_x^\vartheta u(x,t) - (\mathcal{H}_M^{(\alpha)}(x))^T \mathfrak{P}_\vartheta^T U \mathcal{H}_N^{(\beta)}(t)\|_{L^2_{\tilde{\omega}}(\Lambda)} \\
&\quad \leq \frac{L^\vartheta}{\vartheta\Gamma(\vartheta)} \|u(x,t) - u_M(x,t)\|_{L^2_{\tilde{\omega}}(\Lambda)} \\
&\quad \leq \frac{L^\vartheta}{\Gamma(\vartheta+1)} CM^{\rho(j)-m}|u|_{H_{\tilde{\omega}}^{m;M}(\Lambda)}, \quad 0 \leq j \leq m.
\end{aligned} \tag{33}$$

□

To get an error bound for derived approximation in the proposed method, which has been introduced in section 3, without losing the generality, we suppose that

$$\frac{\partial^2 u(x,t)}{\partial x^2} \simeq (\mathcal{H}_M^{(\alpha)}(x))^T U \mathcal{H}_N^{(\beta)}(t) =: \phi_{M,N}(x,t), \tag{34}$$

$$\frac{\partial u(x,t)}{\partial x} \simeq (\mathcal{H}_M^{(\alpha)}(x))^T W \mathcal{H}_N^{(\beta)}(t) = \varphi_{M,N}(x,t), \tag{35}$$

$$u(x,t) \simeq (\mathcal{H}_M^{(\alpha)}(x))^T V \mathcal{H}_N^{(\beta)}(t) =: \psi_{M,N}(x,t). \tag{36}$$

As it can be seen from the process of the presented method in section 3 that relation (26) has appeared in applying the operator I_x^2 on the sides of (34) and (36) can be derived by expanding (26) in terms of FOSHFs basis. It is easy to see that

$$\|u(x,t) - \psi_{M,N}(x,t)\|_{L^2_{\tilde{\omega}}(\Lambda)} = \|I_x^2 \frac{\partial^2 u(x,t)}{\partial x^2} - I_x^2 \phi_{M,N}(x,t)\|_{L^2_{\tilde{\omega}}(\Lambda)}. \tag{37}$$

So, considering (37) and applying Lemma 2, the following corollary is obtained.

Corollary 1. If relation (34) is true, then

$$\|u(x,t)-\psi_{M,N}(x,t)\|_{L^2_{\tilde{\omega}}(\Lambda)} \leq \frac{L^2}{\Gamma(3)} C M^{\rho(j)-m} |\frac{\partial^2 u(x,t)}{\partial x^2}|_{H^{m;M}_{\tilde{\omega}}(\Lambda)}, \quad 0 \leq j \leq m. \tag{38}$$

Consider the main equation (21) and the presented method in section 3, by substituting (34)–(36) on the right side of (21) and applying the operator I_t^{ϑ} on it, we get

$$u(x,t) \simeq -aI_t^{\vartheta}\psi(x,t) + bI_t^{\vartheta}\varphi(x,t) + cI_t^{\vartheta}\phi(x,t) + I_t^{\vartheta} f(x,t) + g(x). \tag{39}$$

On the other hand, we have

$$u(x,t) = -aI_t^{\vartheta}u(x,t) + bI_t^{\vartheta}\frac{\partial u(x,t)}{\partial x} + cI_t^{\vartheta}\frac{\partial^2 u(x,t)}{\partial x^2} + I_t^{\vartheta} f(x,t) + g(x). \tag{40}$$

Putting the right side of (39) and (40) as equivalent, we define perturbation term as follows:

$$\begin{aligned}\mathfrak{R}_{M,N}(x,t) := -aI_t^{\vartheta}(u(x,t) - \psi_{M,N}(x,t)) + bI_t^{\vartheta}(\frac{\partial u(x,t)}{\partial x} \\ - \varphi_{M,N}(x,t)) + cI_t^{\vartheta}(\frac{\partial^2 u(x,t)}{\partial x^2} - \phi_{M,N}(x,t)).\end{aligned} \tag{41}$$

Theorem 3. Suppose that, $u(x,t) \in H^m_{\tilde{\omega}}(\Lambda)$ for $m \geqslant 0$ is the exact solution of (21). If $\psi_{M,N}(x,t)$ is the approximate solution, obtained by applying the presented method, then $\mathfrak{R}_{M,N}(x,t) \longrightarrow 0$ as $M,\ N \longrightarrow \infty$.

Proof. According to (41), we have

$$\begin{aligned}\|\mathfrak{R}_{M,N}(x,t)\|_{L^2_{\tilde{\omega}}(\Lambda)} \leq {} & |a| \|I_t^{\vartheta}(u(x,t) - \psi_{M,N}(x,t))\|_{L^2_{\tilde{\omega}}(\Lambda)} \\ & + |b| \|I_t^{\vartheta}(\frac{\partial u(x,t)}{\partial x} - \varphi_{M,N}(x,t))\|_{L^2_{\tilde{\omega}}(\Lambda)} \\ & + |c| \|I_t^{\vartheta}(\frac{\partial^2 u(x,t)}{\partial x^2} - \phi_{M,N}(x,t))\|_{L^2_{\tilde{\omega}}(\Lambda)}.\end{aligned} \tag{42}$$

Now, by applying Lemma 2 in approximations (34)–(36), respectively, we get

$$\|I_t^{\vartheta}(u(x,t)-\psi_{M,N}(x,t))\|_{L^2_{\tilde{\omega}}(\Lambda)} \leq \frac{T^{\vartheta}}{\Gamma(\vartheta+1)} CM^{\rho(j)-m}|u|_{H^{m;M}_{\tilde{\omega}}(\Lambda)}, \quad 0\leq j\leq m, \tag{43}$$

$$\|I_t^{\vartheta}(\frac{\partial u(x,t)}{\partial x}-\phi_{M,N}(x,t))\|_{L^2_{\tilde{\omega}}(\Lambda)} \leq \frac{T^{\vartheta}}{\Gamma(\vartheta+1)} CM^{\rho(j)-m}|\frac{\partial u}{\partial x}|_{H^{m;M}_{\tilde{\omega}}(\Lambda)}, \quad 0\leq j\leq m, \tag{44}$$

$$\|I_t^{\vartheta}(\frac{\partial^2 u(x,t)}{\partial x^2}-\phi_{M,N}(x,t))\|_{L^2_{\tilde{\omega}}(\Lambda)} \leq \frac{T^{\vartheta}}{\Gamma(\vartheta+1)} CM^{\rho(j)-m}|\frac{\partial^2 u}{\partial x^2}|_{H^{m;M}_{\tilde{\omega}}(\Lambda)}, \quad 0\leq j\leq m. \tag{45}$$

So, by using (43)–(45) in (42), it yields

$$\|\mathfrak{R}_{M,N}(x,t)\|_{L^2_{\tilde{\omega}}(\Lambda)} \leq \frac{T^{\vartheta}}{\Gamma(\vartheta+1)} CM^{\rho(j)-m}(|a||u|_{H^{m;M}_{\tilde{\omega}}(\Lambda)}+|b||\frac{\partial u}{\partial x}|_{H^{m;M}_{\tilde{\omega}}(\Lambda)}+|c||\frac{\partial^2 u}{\partial x^2}|_{H^{m;M}_{\tilde{\omega}}(\Lambda)}). \tag{46}$$

Hence, it is concluded that $\mathfrak{R}_{M,N}(x,t)\longrightarrow 0$ as $M,\ N\longrightarrow\infty$. □

5 Numerical results

In this section, the introduced method in section 3 is utilized to approximate the solutions to problems. It should be mentioned that the maximum of absolute error is the infinity norm of the error function and

$$L_{\infty}=\max_{1\leq j\leq N}|e(x_j,T)|.$$

All numerical experiments have been performed using MATLAB R2017a on a Core(TM)2 laptop with 4GB RAM and a speed of 2.00 GHz.

Example 1. Consider the following time-fractional diffusion differential equation:

$$D_t^{\vartheta}u(x,t)=-u(x,t)+\frac{\partial^2 u(x,t)}{\partial x^2}+f(x,t), \quad 0<\vartheta<1,\ (x,t)\in[0,1]\times[0,1], \tag{47}$$

where $f(x,t)=sin(\pi x)(1+\frac{t^{\vartheta}}{\Gamma(\vartheta+1)})+\frac{\pi^2 t^{\vartheta}}{\Gamma(\vartheta+1)})$, subject to the initial and boundary conditions:

$$u(x,0)=0, \quad u(0,t)=0, \quad u(1,t)=0.$$

The exact solution is $u(x,t) = \frac{t^{\vartheta}}{\Gamma(\vartheta+1)} sin(\pi x)$. Table 1 shows the L_∞-norm of absolute error for fixed $N = 1$ and some M and β in comparison to [2]. In Figure 1, the L_∞-norm of absolute error for fixed $N = 1,\ \vartheta = 0.9$, and some $M = 4, 5, \ldots, 10$ is shown, which demonstrates that the approximate solution converges to the exact solution as M increases. Finally, Figure 2 shows the absolute error functions for fixed $N = 1,\ \vartheta = 0.9$, and some $M = 6,\ 8,\ 10$.

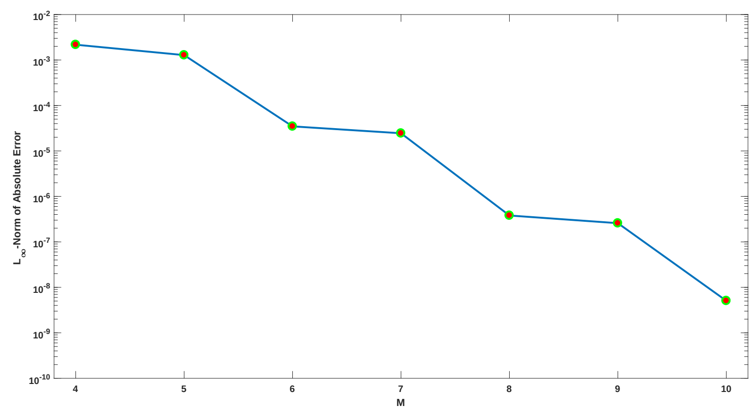

Figure 1: L_∞-norm of the absolute error function for fixed $N = 1,\ \vartheta = 0.9$, and some $M = 4, 5, \ldots, 10$ (Example 1)

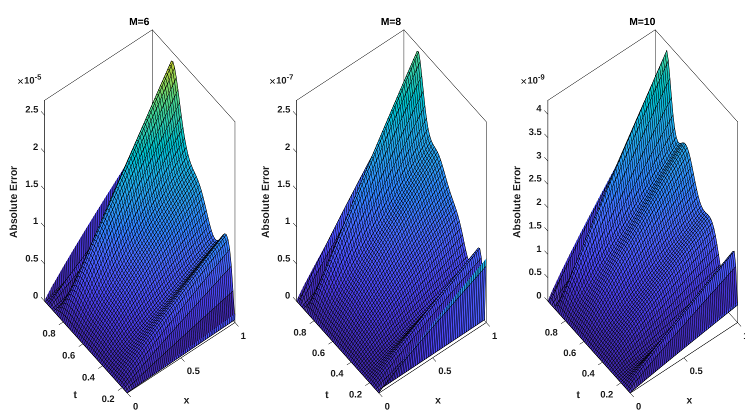

Figure 2: Absolute error functions for fixed $N = 1,\ \vartheta = 0.9$, and $M = 6,\ 8,\ 10$ (Example 1)

Example 2. Consider the following inhomogeneous fractional-order Burger's equation:

Table 1: L_∞-norm of absolute error for fixed $N = 1$ and some M and β in comparison to [2] (Example 1)

M	$\vartheta = 0.25$	[2]	$\vartheta = 0.5$	[2]	$\vartheta = 0.75$	[2]
4	2.301e-3	1.690e-3	2.389e-4	4.979e-3	2.287e-4	2.918e-3
6	3.638e-5	5.764e-4	3.721e-6	3.331e-5	3.589e-6	2.752e-5
8	4.517e-7	1.761e-6	4.621e-8	1.754e-7	4.455e-8	1798e-7
10	7.101e-10	3.127e-9	7.263e-10	8.428e-10	7.003e-10	8.116e-10

$$D_t^{\vartheta} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial x} + f(x,t), \quad 0 < \vartheta \le 1, \ (x,t) \in [0,1] \times [0,1], \tag{48}$$

where $f(x,t) = \frac{2t^{2-\vartheta}}{\Gamma(3-\vartheta)} + 2x - 2$, subject to the initial and boundary conditions:

$$u(x,0) = x^2, \quad u(0,t) = t^2, \quad u(1,t) = 1 + t^2.$$

The exact solution is $u(x,t) = x^2 + t^2$. Figures 3 and 4 show the absolute error functions after solving the problem by using the presented method with $M = 2, \ N = 4, \ \alpha = 1, \ \beta = 0.5$ for the fractional-order derivative $\vartheta = 0.5$ and $M = 2, \ N = 2, \ \alpha = 1, \ \beta = 1$, and $\vartheta = 1$, respectively.

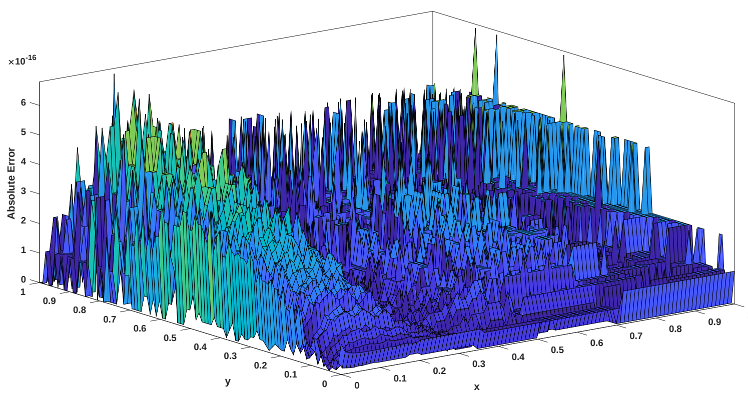

Figure 3: Absolute error function for $M = 2, \ N = 4, \ \alpha = 1, \ \beta = 0.5$, and $\vartheta = 0.5$ (Example 2)

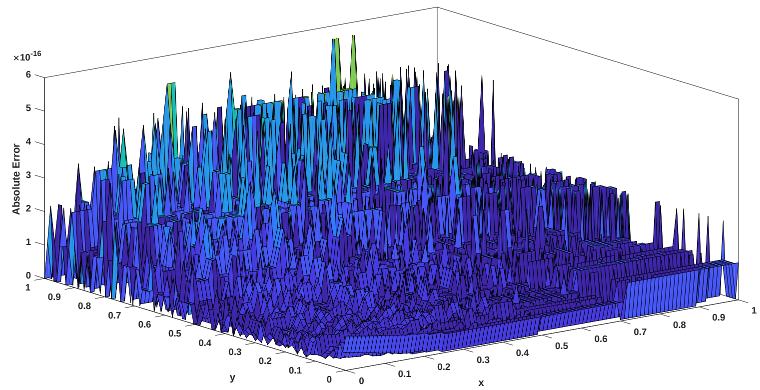

Figure 4: Absolute error function for $M = 2,\ N = 2,\ \alpha = 1,\ \beta = 1$, and $\vartheta = 1$ (Example 2)

Example 3. Consider the following transformed time-fractional Black–Scholes model with homogeneous boundary conditions:

$$D_t^{\vartheta} u(x,t) - \frac{\sigma^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2} - (r - \frac{\sigma^2}{2})\frac{\partial u(x,t)}{\partial x} + ru(x,t) = f(x,t),$$
$$0 < \vartheta \leq 1,\ (x,t) \in (0,1) \times (0,1], \tag{49}$$

where

$$f(x,t) = \frac{6t^{3-\vartheta}}{\Gamma(4-\vartheta)}(x^5 - x^4) - (t^3+1)[\frac{\sigma^2}{2}(20x^3 - 12x^2) + (r - \frac{\sigma^2}{2})(5x^4 - 4x^3) - r(x^5 - x^4)],$$

subject to the initial and boundary conditions:

$$u(x,0) = x^5 - x^4, \quad u(0,t) = 0, \quad u(1,t) = 0.$$

The exact solution is $u(x,t) = (t^3+1)(x^5 - x^4)$. Let $r = 0.02$ and let $\sigma = 0.8$. Figure 5 shows the absolute error function obtained by applying the presented method for $\vartheta = 0.5\ \alpha = 1,\ \beta = 0.5,\ M = 5$, and $N = 6$. Also, Figure 6 shows the absolute error after solving the problem by using the presented method with $M = 5,\ N = 3,\ \alpha = 1$, and $\beta = 1$ for $\vartheta = 1$.

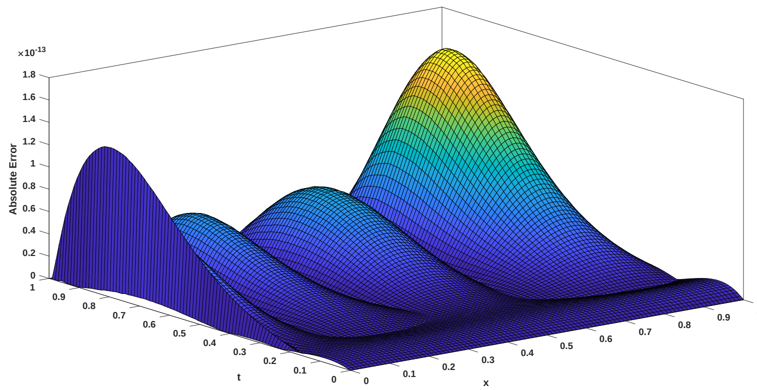

Figure 5: Absolute error function for $\vartheta = 0.5$ $\alpha = 1,\ \beta = 0.5,\ M = 5$, and $N = 6$ (Example 3)

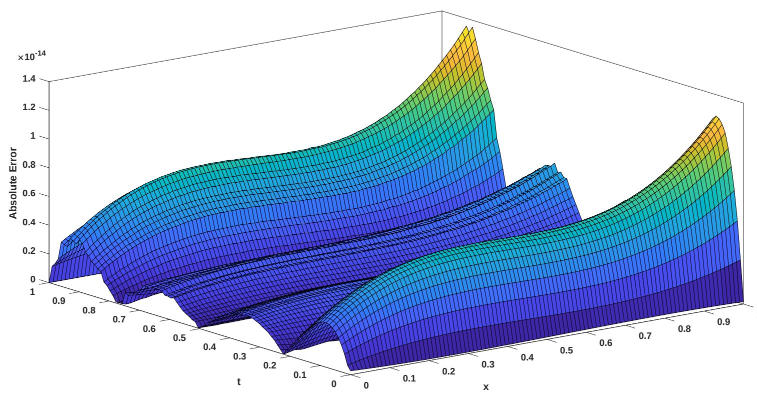

Figure 6: Absolute error function for $\vartheta = 0.5$ $\alpha = 1,\ \beta = 1,\ M = 5$, and $N = 3$ (Example 3).

Example 4. Consider the following time-fractional equation:

$$D_t^{\vartheta} u(x,t) = -au(x,t) + b\frac{\partial u(x,t)}{\partial x} + c\frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t),$$
$$0 < \vartheta \leq 1,\ (x,t) \in (0,L) \times (0,T], \quad (50)$$

subject to the initial and boundary conditions:

$$u(x,0) = x^{\frac{5}{2}}, \quad u(0,t) = 0, \quad u(L,t) = L^{\frac{5}{2}} e^{-t},$$

where in the case of $\vartheta = 1$ and the function f is chosen as $f(x,t) = -e^{-t}\frac{5}{2}(x^{\frac{3}{2}} + \frac{3}{2}\sqrt{x})$, the exact solution is $u(x,t) = e^{-t}x^{\frac{5}{2}}$. It is notable that in other cases of $0 < \vartheta < 1$, the exact solution is unknown. Figure 7 shows the absolute error functions obtained by applying the presented method for $\vartheta = 0.5,\ \alpha = 0.5,\ \beta = 1,\ M = 5$, and $N = 4,\ 6,\ 8$. Also, Figure 8 shows the L_∞-norm of the absolute error function for fixed $M = 5,\ \vartheta = 0.5$, and some $N = 2, 3, \ldots, 8$, which demonstrates that the L_∞-norm of the absolute error function converges to zero as N increases. Finally, Figure 9 depicts approximate solutions for different $0 < \vartheta \leq 1,\ M = 5,\ N = 7$, which shows that as $\vartheta \to 1$, the approximate solution converges to the exact solution when $\vartheta = 1$.

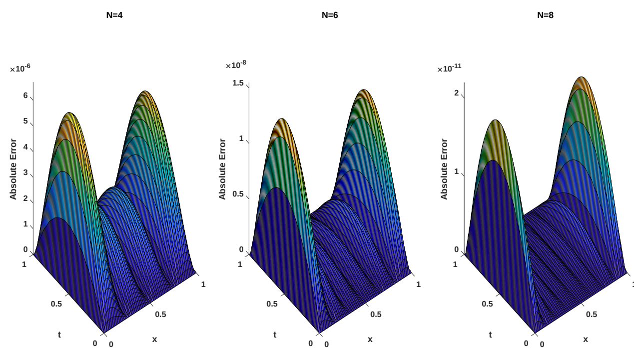

Figure 7: Absolute error functions for $\vartheta = 0.5,\ \alpha = 0.5,\ \beta = 1,\ M = 5$, and $N = 4,\ 6,\ 8$. (Example 4)

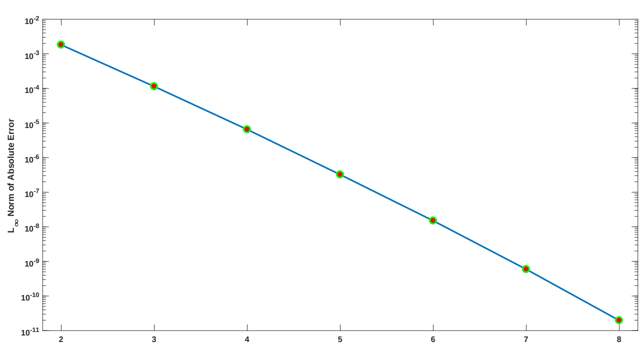

Figure 8: L_∞-norm of absolute error function for fixed $M = 5,\ \vartheta = 0.5$, and $N = 2, 3, \ldots, 8$. (Example 4)

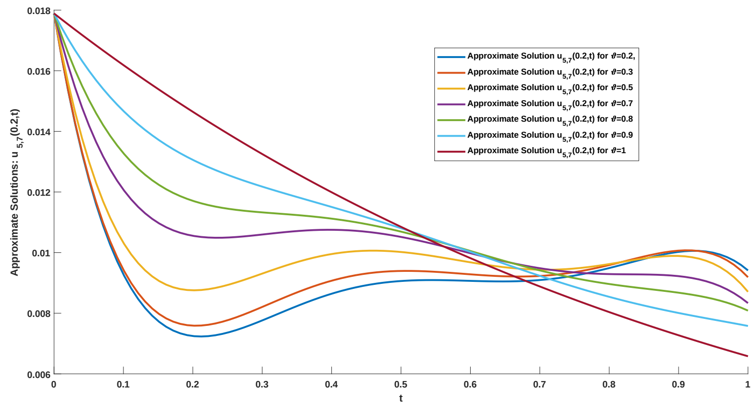

Figure 9: Approximate solutions for different $0 < \vartheta \leq 1,\ M = 5,\ N = 7$. (Example 4)

6 Conclusion

In this paper, a new orthogonal system of nonpolynomial basis functions, named FOSHFs, has been introduced and used to solve a class of fractional-time partial differential equations with nonsmooth solutions. For introducing the method, an operational matrix of fractional order integral of Hahn functions has been used for the first time as basis functions here. Furthermore, the convergence of FOSHFs approximation has been proved. In numerical examples, the obtained results have demonstrated the efficiency and convergence of the proposed method for the cases of nonsmooth solutions.

7 Declarations

The author declares that there is no conflict of interest.

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