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A sufficient condition for null controllability of nonlinear control systems*

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Abstract

Classical control methods such as Pontryagin Maximum Principle and Bang-Bang Principle and other methods are not usually useful for solving *optimal control systems* (OCS) specially *optimal control of nonlinear systems* (OCNS). In this paper, we introduce a new approach for solving OCNS by using some combination of atomic measures. We define a criterion for controllability of lumped nonlinear control systems and when the system is nearly null controllable, we determine controls and states. Finally we use this criterion to solve some numerical examples.

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1 Introduction

We consider a nonlinear time-variant system as follows:

$$\dot{x} = g(t, x(t), u(t)), \qquad \forall t \in J, \tag{1}$$

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$$x(t_0) = x_0, x(t_f) = x_f, \tag{2}$$

where $\Omega_1 = J \times A \times U \times D$, here J is a known closed interval $[t_0, t_f]$, A and D are compact and peicewise connected sets in R^n such that $x(t) \in A$ and $\dot{x}(t) \in D, \forall t \in J$, and U is a compact set in R^m such that $u(t) \in U, \forall t \in J$, and g is continuous on J. If there are $u(.)$ and $x(.)$ that satisfy equation (1)-(2) we call the system is controllable.

In the following, by means of a process of embedding and using measure theory, this problem is replaced by another one in the space of Borel measures, that we seek to minimize to a linear form over a compact subset of the measure space. The theory allows us to convert the new problem to an infinite-dimensional linear programming problem. Later on the infinite-dimensional linear programming problem is approximated by a finite dimensional one. Then by the solution of the linear programming problem one can find approximate functions for states $x(\cdot)$ and control $u(\cdot)$.

If the system has an objective function we can use this process for solving the systems defined by multi-objective control systems.

There are some literature on nonlinear optimal control for lumped and distributed parameter systems, see for example, [2]–[12].

2 Defining the problem

Let us define in (1), for all t in $J = [t_0, t_f]$

$$y(t) \triangleq \dot{x}(t), \tag{3}$$

Then the equations can be rewritten as

$$y(t) = g(t, x(t), u(t)), \tag{4}$$

$$x(t_0) = x_0, x(t_f) = x_f. \tag{5}$$

Now we define the function $h : \Omega_1 \to R$ as

$$h(t, x(t), u(t), y(t)) \triangleq \|y(t) - g(t, x(t), u(t))\|, \tag{6}$$

and let the functional $I(., x(.), u(.), y(.))$ be as follows:

$$I(., x(.), u(.), y(.)) \triangleq \int_J h(t, x(t), u(t), y(t))dt.$$

Now, we investigate a necessary and sufficient condition for controllability of control system (1)-(2).

Theorem 1. *A necessary and sufficient condition for controllability of control system (1)-(2) is*

$$\text{Min} \quad I(\cdot, \cdot, \cdot, \cdot) = 0,$$

that is equation (1) and boundary conditions (2) are valid on Ω_1.

Proof. Since $h \geq 0$ and it is continuous, h is Rimman integrable. If

$$\text{Min} \quad I(\cdot, \cdot, \cdot, \cdot) = 0,$$

$u^*(.)$ and $x^*(.)$ are the corresponding control and trajectory and $x(t_0) = x_0, x(t_f) = x_f$, then

$$\int_J h(t, x^*(t), u^*(t), y^*(t))dt = 0$$

and we will have $h = 0$. So

$$y^*(t) = g(t, x^*(t), u^*(t)),$$

or

$$\dot{x}^*(t) = g(t, x^*(t), u^*(t)).$$

In other words, in this case $u^*(.)$ and $x^*(.)$ satisfy equations (1)-(2) and the system will be controllable.

Conversely, if the system is controllable; that is, if (1)-(2) are satisfied, then $h = 0$, for all t in J. So

$$\int_J h(t, x(t), u(t), y(t))dt = 0$$

and then $I(., x, u, y) = 0$, hence Min $I(., x, u, y) = 0$.

Note In practice we usually obtain suboptimal solution for $I(.,.,.,.)$ in Theorem 1, that is we have many errors for controllability of the system, for example computational errors. So usually I is not exactly equal to zero, in this case let the total permissible errors be at most $\epsilon > 0$, where ϵ is a known positive number. If

$$e(t) \triangleq \|y^*(t) - g(t, x^*(t), u^*(t))\|_{L_2} = (\int_J \|y^*(t) - g(t, x^*(t), u^*(t))\|^2 dt)^{1/2} \quad (7)$$

and $e < \epsilon$, then the system is almost controllable, so we define fuzzy controllability.

Fuzzy controllability Let $\widetilde{C}$ be the fuzzy set of permissible controls and trajectories as follows:

$$\widetilde{C} \cong \{(x, u, y) : C(x, u, y) \text{ is as follows}\}$$

$$C(x, u, y) = \begin{cases} \frac{(\epsilon - e)}{\epsilon} & , e < \epsilon, \\ 0 & , \text{ otherwise.} \end{cases}$$

Then we say the system is controllable of grade C.

Controllability of multi-objective systems

Let our Multi-Objectives System be the minimization of

$$I_i(t, x(t), u(t), \dot{x}(t)) = \int_J f_i(t, x(t), u(t), \dot{x}(t))dt, \quad i = 1, 2, \cdots, k,$$

subject to the conditions (1)-(2), also we would like to be sure that our system is controllable or fuzzy controllable.

If we consider $y(t)$ as before and

$$w(t) \triangleq (w_1(t), w_2(t), \cdots, w_{k+1}(t))$$

such that

$$w(t) \in E, \text{ where } E \subset R^{k+1} \text{ and } E = [0, 1] \times [0, 1] \times \cdots \times [0, 1],$$

and also we define an objective function which is a convex combination of the above objectives, that is, we assume the weights for objectives, as follows:

$$I(\cdot,\cdot,\cdot,\cdot,\cdot) \cong \sum_{i=1}^{k} w_i(t) \int_J f_i(t,x(t),u(t),y(t))dt + w_{(k+1)}(t) \int_J h(t,x(t),u(t),y(t))dt,$$

$$\sum_{i=1}^{k+1} w_i(t) = 1, \quad 0 \leq w_i(t) \leq 1, \quad i = 1,2,...,k+1,$$

and we consider f in the following way

$$f(t,x(t),u(t),y(t),w(t)) \triangleq \sum_{i=1}^{k} w_i(t) f_i(.,x,u,y) + w_{(k+1)}(t) h(.,x,u,y),$$

then $I(.,.,.,.,.)$ will be

$$I(.,x,u,y,w) \triangleq \int_J f(t,x(t),u(t),y(t),w(t))dt,$$

then the minimization of $I(.,x,u,y,w)$ will be a criterion for controllability and also multi-objective performances functional. In the special case, when $w_i(t) = 0, \quad i = 1,2,...,k$ and $w_{(k+1)}(t) = 1$, it is just a criterion for controllability.

3 Metamorphism

We define $[x(\cdot),u(\cdot),y(\cdot)]$ to be an *admissible triple,* provided for all t in J,

- the function $x(\cdot)$ is continuous, and $x(t) \in A$;
- the function $x(\cdot)$ is continuous, and $y(t) \in D$;
- the function $u(\cdot)$ is Lebesgue measurable, and $u(t) \in U$;
- the triple satisfies the system of differential equations (4)-(5) and a.e. on $J^\circ = (t_0, t_f)$ in the sense of Caratheodory.

We denote the set of admissible triples by V. The problem has no solution unless $V \neq \emptyset$.

Using the above assumption, the problem is now as follows:
Find an optimal admissible triple $v \in V$ which minimizes the functional

$$I(\cdot,\cdot,\cdot,\cdot,\cdot) = \int_J f(t, x(t), u(t), y(t), w(t))dt. \tag{8}$$

Assume that B is an open ball in R^{n+1} containing $J \times A$, denote the space of all differentiable functions on B by $C'(B)$, and for $\phi \in C'(B)$ define

$$\phi^g(t, x, u, y) \cong \phi_x(t, x).g + \phi_t(t, x), \tag{9}$$

where $\phi(\cdot)$ and $g(\cdot)$ are n-vectors and the first term in the right-hand side of (9) is an inner product and ϕ^g is in the space $C(\Omega)$ of real-valued continuous functions defined on the compact set Ω, where $\Omega = \Omega_1 \times E$. Then by the definitions of g and ϕ and using the chain rule we have

$$\int_J \phi^g(t, x(t), u(t), y(t))dt = \int_J \dot{\phi}(t, x(t))dt$$
$$= \phi(t_f, x(t_f)) - \phi(t_0, x(t_0)) = \delta\phi.$$

Therefore

$$\int_J \phi^g(t, x(t), u(t), y(t))dt = \delta\phi, \forall \phi \in C'(B). \tag{10}$$

Since A may have an empty interior in R^n, we need to introduce the set B and space $C'(B)$. Suppose $D(J^\circ)$, is the space of infinitely differentiable real valued functions with compact support in J° and each x and g have n components such as x_j and $g_j, j = 1, 2, \ldots$,n. For each $\psi \in D(J^\circ)$, define

$$\psi_j(t, x, u, y) \cong x_j\psi'(t) + g_j\psi(t), j = 1, 2, \ldots, n. \tag{11}$$

If w is an admissible pair, then for any $\psi \in D(J^\circ)$ we have

$$\int_J \psi_j(t, x(t), u(t), y(t))dt = \int_J x_j\psi'(t)dt + \int_J g_j\psi(t)dt$$
$$= x_j(t)\psi(t) \mid_{t_0}^{t_f} - \int_J \{x_j' - g_j(t, x(t), u(t))\}\psi(t)dt,$$

Since ψ has compact support on J°, it follows that

$$\psi(t_0) = \psi(t_f) = 0,$$

and hence

$$\int_J \psi_j(t, x(t), u(t), y(t))dt = 0. \tag{12}$$

Now, we choose those functions in $C'(B)$ which depend on the time variable only and denote this subspace by $C_1(\Omega)$. Set

$$\beta(t, x, u, y, w) = \beta(t), (t, x, u, y) \in \Omega.$$

Thus

$$\int_J \beta^g(t, x(t), u(t), y(t))dt = a_\beta, \beta \in C_1(\Omega),$$

where a_β is the Lebegue integral of $\beta(t, x, u, y)$ on J.

In a given classical problem, the set of admissible triples is fixed. If we add some elements to it, we have changed the problem and considered a new one, inspired classically, but a different formulation nevertheless.
Consider the mapping

$$\Lambda_w : F \in C(\Omega) \to \int_J F(t, x(t), u(t), y(t), w(t))dt,$$

which is a linear and positive functional. Let us rewrite (8) subject to the conditions (4)-(5) in the new representation as follows:

$$\text{Minimize} \quad \Lambda_v(f) \tag{13}$$

subject to

$$\Lambda_v(\phi^g) = \delta\phi, \phi \in C'(B)$$

$$\Lambda_v(\psi_j) = 0, j = 1, 2, ..., n; \psi \in D(J^\circ) \tag{14}$$

$$\Lambda_v(\beta) = a_\beta, \beta \in C_1(\Omega).$$

We mention that Λ_w is a positive Radon measure on the set $C(\Omega)$. We denote the space of all positive Radon measures on Ω by $M^+(\Omega)$. A Radon measure on Ω can be identified with a regular Borel measure on this set (see [13], Riesz

Representation Theorem). Thus, for a given positive functional on $C(\Omega)$, there is a positive Borel measure on Ω such that

$$\Lambda_v(F) = \int_\Omega F d\mu = \mu(F).$$

Now, the problem (13)-(14) can be replaced by a new problem as follows. We seek a measure in $M^+(\Omega)$ which minimizes the functional

$$\mu \in M^+(\Omega) \to \mu(f) \in R \tag{15}$$

and satisfies the following constraints:

$$\begin{aligned} &\mu(\phi^g) = \delta\phi, \phi \in C'(B) \\ &\mu(\psi_j) = 0, j = 1, 2, ..., n; \psi \in D(J^\circ) \\ &\mu(\beta) = a_\beta, \beta \in C_1(\Omega). \end{aligned} \tag{16}$$

Thus, we consider the extension of our problem: Minimization of (15) over the set Q of all positive Radon measures on Ω satisfying (16). Considering such measure theoretic form of the problem has teo main adventages, namely

- The existence of an optimal measure in the set Q, which satisfies (16) can be studied in a straightforward manner without having to impose conditions such as convexity, which may be artificial.
- The functional in (15) is linear, although $f(\cdot,\cdot,\cdot,\cdot,\cdot)$ may be nonlinear.

By the Proposition II.1, Theorem II.1 and Proposition II.3 of [14], we are able to prove the existence of the optimal measure.

4 First approximation

The problem (15)-(16) is an infinite dimensional linear programming(LP) problem, because all the functionals in (15)-(16) are linear in the variable μ even if the original problem is nonlinear and furthermore, the measure μ is required to be positive. Of course, (15)-(16) is an infinite dimensional LP problem, because $M^+(\Omega)$ is an infinite dimensional space. It is possible to approximate the solution of this problem by the solution of a finite-dimensional LP of sufficiently large dimension. Also, from the solution of this new finite dimensional LP we induce an approximated admissible triple in a suitable manner. We shall first develop an intermediate problem, still infinite-dimensional, by considering the minimization (15), not over the set Q but over a subset of $M^+(\Omega)$ defined by requiring that only a finite number of the constraints in (16) are satisfied. This will be achieved by choosing countable sets of functions whose linear combinations are dense in the set Q, and then selecting a finite number of them. Consider the first set of equalities in (16). Let the set $\{\phi_i, i = 1, 2, ...\}$ be such that the linear combinations of the functions $\phi_i \in C'(B)$ are uniformly dense. For instance, these functions can be taken to be monomials in the components of the n-vectors x and variable t.

Now, we consider the functions in $D(J^\circ)$ defined as below

$$\sin[\frac{2\pi r(t-t_0)}{\delta t}], \quad 1-\cos[\frac{2\pi r(t-t_0)}{\delta t}], r = 1, 2, ... \tag{17}$$

where $\delta t = t_f - t_0$, if ψ's are chosen as (17), and the sequence $\{\chi_l\}, l = 1, 2, ...$ is of type ψ_j in (11). Then the first approximation will be completed by using the above subjects and Proposition III.1 of [14].

5 Second approximation

By Proposition III.2 of [14] the optimal measure has the form

$$\mu^* = \sum_{k=1}^{N} \alpha_k^* \delta(z_k^*), \tag{18}$$

where $z_k^* \in \Omega$ and $\alpha_k^* \geq 0, k = 1, 2, ..., N$, where $\delta(\cdot)$ is unitary atomic measure with the support being the singleton set $\{z_k^*\}$, characterized by

$$\delta(z)(F) = F(z), z \in \Omega.$$

This structural result points the way towards a nonlinear problem in which the unknowns are the coefficients α_k^* and supports $\{z_k^*\}, k = 1, 2, ..., N$.

To change this problem to a LP problem, we use another approximation. If ω^N is a countable dense subset of Ω, we can approximate μ^* by a measure $\nu \in M^+(\Omega)$ such that

$$\nu = \sum_{k=1}^{N} \alpha_k^* \delta(z_k),$$

where $z_k \in \omega^N = \{z_1, z_2, ..., z_N\}$(Proposition III.3 of [14]).

This result suggests the following LP problem

Given $\epsilon > 0$ and $z_j \in \omega^N,\ j = 1, 2, ..., N,$

$$\text{Minimize} \qquad \sum_{j=1}^{N} \alpha_j f(z_j) \tag{19}$$

subject to

$$\begin{aligned} &\left| \sum_{j=1}^{N} \alpha_j \phi_i^g(z_j) - \delta\phi_i \right| \leq \epsilon, i = 1, 2, \cdots, M_1, \\ &\left| \sum_{j=1}^{N} \alpha_j \chi_l(z_j) \right| \leq \epsilon, l = 1, 2, ..., M_2, \\ &\left| \sum_{j=1}^{N} \alpha_j \beta_s(z_j) - a_{\beta_s} \right| \leq \epsilon, s = 1, 2, \cdots, L, \\ &\qquad \alpha_j \geq 0, j = 1, 2, ..., N. \end{aligned} \tag{20}$$

Assume $P(M_1, M_2, L)^\epsilon$ in R^N is defined by $\alpha_j \geq 0, j = 1, 2, ..., N$ satisfies (20), then by Theorem III.1 of [14], for every $\epsilon \geq 0$ the problem of minimizing the functional (19) on the set $P(M_1, M_2, L)^\epsilon$ has a solution for $N = N(\epsilon)$ sufficiently large, and the solution satisfies

$$\eta(M_1, M_2, L) + \rho(\epsilon) \leq \sum_{j=1}^{N} \alpha_j f(z_j) \leq \eta(M_1, M_2, L) + \epsilon,$$

where $\rho(\epsilon) \to 0$ as $\epsilon \to 0$.

Let $\theta_r \in C_1(\Omega)$,

$$\theta_r(t, x, u, y, w) = t^r, r = 0, 1, ..., \tag{21}$$

then the set of θ_r's is dense in $C_1(\Omega)$. Assume that there are a number L of them in the set $\{\phi_i^g\}_{i=1}^{M_1}$. It is necessary to choose L number of functions of the time only, to replace the functions $\theta_r, r = 0, 1, ...$ which were not found suitable, so we have chosen some suitable functions, to be denoted by f_s, $s = 1, 2, ...L$, as follows:

$$f_s(t) = \begin{cases} 1 & \text{if } t \in J_s \\ 0 & \text{otherwise,} \end{cases}$$

where $J_s = (t_0 + (s-1)d, t_0 + sd)$, $d = \frac{\delta t}{L}$ Since every continuous function can be written as a linear combination of monomials of type $1, x, x^2, ...$ We assume

$$\phi_1 = x_1, \phi_2 = x_2, ..., \phi_n = x_n,$$

$$\phi_{n+1} = {x_1}^2, \phi_{n+2} = {x_2}^2, ..., \phi_{2n} = {x_n}^2,$$

until M_1 functions are chosen, also assume

$$\psi^r(t) = \sin[\frac{2\pi r(t - t_0)}{\delta t}], r = 1, 2, ..., M_{21}$$

or

$$\psi^r(t) = 1 - \cos[\frac{2\pi r(t - t_0)}{\delta t}], r = M_{21} + 1, M_{21} + 2, ..., 2M_{21},$$

where χ_l are chosen as ψ_j^r in (11), then we have $M_2 = 2nM_{21}$ number of type χ_l.

Now, if in the problem (19)-(20), $\epsilon \to 0$ and $z_j \in \omega^N, j = 1, 2, ..., N$, then we have

$$\text{Minimize} \quad \sum_{j=1}^{N} \alpha_j f(z_j) \tag{22}$$

subject to

$$\sum_{j=1}^{N} \alpha_j \phi_i^g(z_j) = \delta\phi_i, i = 1, 2, \cdots, M_1,$$

$$\sum_{j=1}^{N} \alpha_j \chi_l(z_j) = 0, l = 1, 2, ..., M_2, \tag{23}$$

$$\sum_{j=1}^{N} \alpha_j f_s(z_j) = a_s, s = 1, 2, ..., L,$$
$$\alpha_j \geq 0, j = 1, 2, ..., N,$$

where a_s is the integral of f_s on J. By solving this finite dimensional LP problem we obtain the nearly optimal α^*'s.

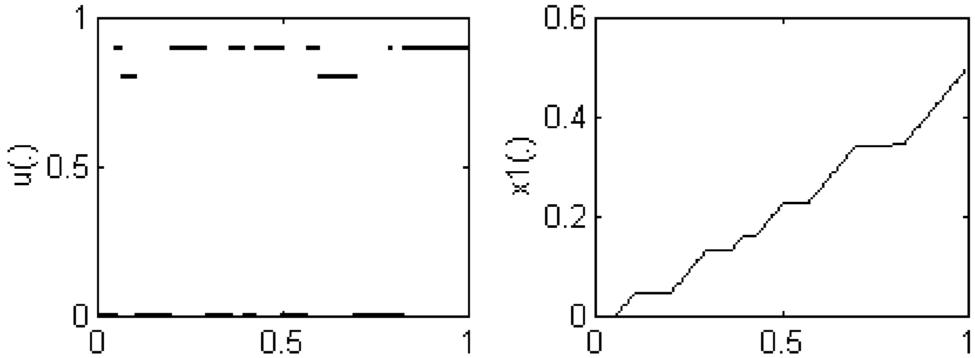
6 Numerical examples

Example 1. Consider the nonlinear time-variant problem

$$\dot{x} = x^2 sin(x) + u$$

$$x(0) = 0, x(1) = 0.5.$$

We let $\epsilon = 0.1$ and partition respectively the sets $J = [0, 1], A = [0, 0.5], D = [0, 1]$, and $U = [0, 1]$ into $p_t = p_A = p_D = p_u = 10$ and $M_1 = 6, M_2 = 4$, and $L = 10$.



Optimal Control State

We used Revised Simplex method to solve such problem and found $f^* = 0.0065$, $x^*(0) = 0$ and $x^*(1) = 0.4995$, and degree of controllability of this example is $C = 0.9349$. Below, the figures of $x(\cdot)$ and $u(\cdot)$ are given.

Example 2. Consider the nonlinear time-variant optimal control problem

$$\text{Minimize} \quad \int_0^1 u^2(t)dt,$$

subject to the conditions

$$\dot{x} = x^2 sin(x) + u$$

$$x(0) = 0, x(1) = 0.5.$$

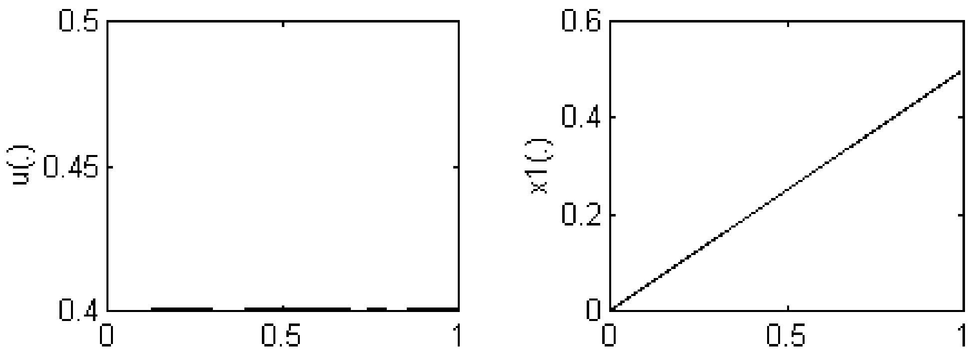
Then

$$h(t, x(t), u(t), y(t)) = \|y(t) - (x^2 sin(x) + u)\|, \qquad \forall t \in J$$

and

$$I(\cdot,\cdot,\cdot,\cdot) = w_1(t)\int_0^1 u^2(t)dt + w_2(t)\int_0^1 h(t, x(t), u(t), y(t))dt.$$

We let $w_1(t) = w_2(t) = \frac{1}{2}$ and $\epsilon = 0.1$ and divide respectively the sets $J = [0,1], A = [0,0.5], D = [0,1]$, and $U = [0,1]$ into $p_t = p_A = p_D = p_u = 10$ and $M_1 = 6, M_2 = 4$, and $L = 10$.
We used Revised Simplex method to solve this problem and found $f^* = 0.1133$, $x^*(0) = 0$ and $x^*(1) = 0.4981$. Below, the figures of $x(\cdot)$ and $u(\cdot)$ are given.



Optimal Control State

Example 3 (A system of coupled hydraulic tanks [1]) A state-space model can be set up with the inlet flow-rate u as input, the depths of liquid (x_1, x_2) in the respective tanks as state variables and the output taken as x_2, since the objective is to control the level in tank2. With tanks of the same dimensions, and orifices of equal size, the state-space equations expressed in suitably normalized variables become

$$\dot{x_1} = u - \sqrt{x_1 - x_2},$$

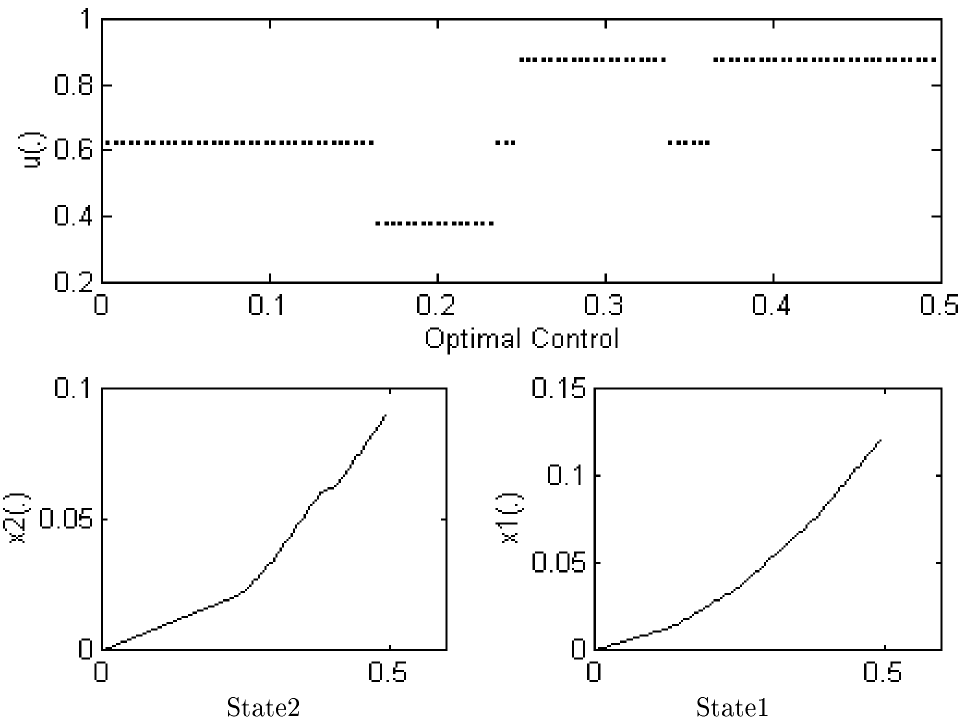
$$\dot{x_2} = \sqrt{x_1 - x_2} - \sqrt{x_2},$$

where it is understood that the system operates only in the region

$$x_1 > x_2 > 0.$$

We assume $\epsilon = 0.1$ and divide respectively the sets $J = [0, 0.5], A1 = [0.05, 0.45], D1 = [0.05, 0.45], A2 = [0.05, 0.35], D2 = [0.05, 0.35]$, and $U = [0, 1]$ into $p_t = p_{A_1} = p_{D_1} = p_{A_2} = p_{D_2} = p_u = 4$ and $M_1 = 2, M_2 = 8$, and $L = 4$.

We solve this problem and found $f^* = 0.0465$, $x_1^*(0) = 0.02, x_1^*(0.5) = 0.1214, x_2^*(0) = 0.02$ and $x_2^*(0.5) = 0.0902$, and degree of controllability of this example is $C = 0.5355$. Below, the figures of $u(\cdot), x_1(\cdot)$, and $x_2(\cdot)$ are given.



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