Some efficient Nordsieck integration methods for IVPs

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Abstract

In this paper, in continuation of the construction of efficient numerical methods for stiff IVPs, we construct type two Nordsieck second derivative general linear methods with order \( p = s \), where \( s \) is the number of internal stages, and stage order \( q = p \). Implementation of the constructed methods with fixed and variable stepsize is discussed which verifies their efficiency.

Keywords: Stiff differential equations; Nordsieck second derivative general linear methods; \( A \)- and \( L \)-stability; Variable stepsize implementation.

1 Introduction

Second derivative general linear methods (SGLMs) for the numerical solution of autonomous ordinary differential equations (ODEs) with initial value problem

\[
\begin{align*}
y'(x) &= f(y(x)), \quad x \in [x_0, x], \\
y(x_0) &= y_0,
\end{align*}
\]

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where \( f : \mathbb{R}^m \rightarrow \mathbb{R}^m \) and \( m \) is the dimensionality of the system, have been studied in recent years. These methods were introduced by Butcher and Hojjati in [13] and investigated more in [2,9] by Abdi and Hojjati. In the construction of SGLMs which are extension of general linear methods (GLMs) ([11,12,13,14]), there are a lot of free parameters which allow us to construct high order methods with a small number of internal stages to reduce computational cost [1].

We recall that SGLMs are characterized by four integers, \( (p, q, r, s) \) where \( p \) and \( q \) are, respectively, order and stage order of the method, \( r \) is the number of input and output approximations, and \( s \) is the number of internal stages. Let \( Y^{[n]} = [Y_i^{[n]}]_{i=1}^s \) be an approximation of stage order \( q \) to the vector \( y(x_{n-1} + ch) = [y(x_{n-1} + c_i h)]_{i=1}^s \), and let the vectors \( f(Y^{[n]}) = [f(Y_i^{[n]})]_{i=1}^s \) and \( g(Y^{[n]}) = [g(Y_i^{[n]})]_{i=1}^s \) denote the stage first and second derivative values, where \( g(\cdot) = f'(\cdot) f(\cdot) \), respectively. If \( r = p + 1 \), we can assume the input and output vectors at the step number \( n \), \( y^{[n-1]} \) and \( y^{[n]} \), are approximations of order \( p \) to the Nordsieck vectors

\[
\begin{bmatrix}
y(x_{n-1}) \\
h y'(x_{n-1}) \\
\vdots \\
h^p y^{(p)}(x_{n-1})
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
y(x_n) \\
h y'(x_n) \\
\vdots \\
h^p y^{(p)}(x_n)
\end{bmatrix},
\]

respectively. In an SGLM used for the numerical solution of (II), these values are related by

\[
\begin{align*}
Y^{[n]} &= h(A \otimes I_m) f(Y^{[n]}) + h^2(\vec{A} \otimes I_m) g(Y^{[n]}) + (U \otimes I_m) y^{[n-1]}, \\
y^{[n]} &= h(B \otimes I_m) f(Y^{[n]}) + h^2(\vec{B} \otimes I_m) g(Y^{[n]}) + (V \otimes I_m) y^{[n-1]},
\end{align*}
\]

(2)

where \( n = 1, 2, \ldots, N \), \( Nh = \pi - x_0 \), \( h \) is the stepsize, and \( \otimes \) is the Kronecker product of two matrices. Here \( A, \vec{A} \in \mathbb{R}^{r \times s}, U \in \mathbb{R}^{s \times r}, B, \vec{B} \in \mathbb{R}^{r \times s}, \) and \( V \in \mathbb{R}^{r \times r} \). The coefficients matrix \( V \) in the Nordsieck SGLM (2) has the form

\[
V = \begin{bmatrix} 1 \\ v \end{bmatrix}^T \hat{V},
\]

\( v = [v_1 \; v_2 \; \cdots \; v_{r-1}]^T, \hat{V} \in \mathbb{R}^{(r-1) \times (r-1)} \). In this paper, the methods will be restricted to the case where \( p = q = r - 1 = s \) and the eigenvalues of \( \hat{V} \) are zeros. The latter condition ensures zero-stability of the method.

An SGLM in Nordsieck form has order \( p \) and stage order \( q = p \) if and only if [14].
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\[ U = C - ACK - \overline{A}CK^2, \]
\[ V = E - BCK - \overline{B}CK^2, \]

where the matrices \( C \in \mathbb{R}^{s \times (p+1)} \), \( K \in \mathbb{R}^{(p+1) \times (p+1)} \), and \( E \in \mathbb{R}^{(p+1) \times (p+1)} \) are defined by

\[ C := \begin{bmatrix} 1 & c & \frac{c^2}{2} & \cdots & \frac{c^p}{p!} \end{bmatrix}, \quad K := \begin{bmatrix} 0 & e_1 & e_2 & \cdots & e_p \end{bmatrix}, \]

and

\[ E := \exp(K) = \begin{bmatrix} 1 & \frac{1}{1!} & \frac{1}{2!} & \cdots & \frac{1}{p!} \\ 0 & 1 & \frac{1}{1!} & \cdots & \frac{1}{(p-1)!} \\ 0 & 0 & 1 & \cdots & \frac{1}{1!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \]

with \( e_j \) as the \( j \)th vector of canonical basis in \( \mathbb{R}^{p+1} \), respectively.

The stability behavior of SGLMs is defined using the standard test problem of Dahlquist \( y' = \xi y \), where \( \xi \) is a complex number. If method (2) is applied to this problem, then the stability matrix is

\[ M(z) = V + (zB + z^2\overline{B})(I - zA - z^2\overline{A})^{-1}U, \]

where \( z = h\xi \) and the stability function of the method is defined as the characteristic polynomial of \( M(z) \); that is,

\[ p(w, z) = \det (wI - M(z)). \]

If \( p(w, z) = w^{r-1}(w - R(z)) \), the method is said to possess “Runge–Kutta stability (RKS)”. GLMs and SGLMs with RKS property were studied by Butcher and Jackiewicz in [9, 12, 14] and by Abdi and Hojjati in [2, 7], respectively.

For economical implementation, it is assumed that the matrices \( A \) and \( \overline{A} \) have a lower triangular form with the same diagonal entries \( \lambda \) and \( \mu \), respectively. SGLMs are divided into four types, depending on the nature of the differential system to be solved and the computer architecture that is used to implement these methods. Types 1 and 2 are those with arbitrary \( a_{ij} \) and \( \overline{a}_{ij} \), where \( \lambda = \mu = 0 \) and \( \lambda > 0, \mu < 0 \), respectively. Such methods are appropriate, respectively, for nonstiff and stiff differential systems in a sequential computing environment. Requiring \( a_{ij} = \overline{a}_{ij} = 0 \), cases \( \lambda = \mu = 0 \) and \( \lambda > 0, \mu < 0 \) lead, respectively, to types 3 and 4 methods which can
be useful, respectively, for nonstiff and stiff systems in a parallel computing environment.

The construction and implementation of type 2 SGLMs with $p = s + 1$ were discussed in [3]. Also, the construction of parallel Nordsieck SGLMs with $p = s$ and their order barriers were studied in [4]. These order barriers were obtained under the assumption of RKS property. In this paper, we are going to construct type 2 Nordsieck SGLMs with $p = s$, which are efficient methods for stiff systems. Also, efficiency of the constructed methods are shown by their implementation in a variable stepsize environment.

Next sections of this paper are organized as follows: In section 2, we construct $A$- and $L$-stable SGLMs in the Nordsieck form with RKS property of orders 2, 3, and 4. Considering Nordsieck SGLMs in the variable stepsize mode, implementation issues including local error estimation, and stepsize control are discussed in section 3. Finally, in section 4, some results of numerical experiments on some stiff test systems are presented and compared with those obtained by Nordsieck SGLMs of the same order.

2 Construction of type 2 methods

In this section, the construction of type 2 Nordsieck SGLMs of order $p$ and stage order $q = p$ with some desired stability properties is explained.

2.1 Methods with $p = q = s = r - 1 = 2$

We construct methods with $p = q = s = r - 1 = 2$ and RKS property. We look for methods which their stability function has the form [5]

$$R(z) = \frac{1 + n_1z + n_2z^2 + n_3z^3}{(1 - \lambda z - \mu z^2)^2},$$

where

$$1 + \sum_{k=1}^{3} n_kz^k = \exp(z)(1 - \lambda z - \mu z^2)^2 - C z^3 + O(z^4)$$

with $C$ as the error constant of the method. For this method to be $A$-stable, it is necessary and sufficient that $\lambda > 0$, $\mu < 0$, and so that the $E(y)$ is non-negative for all real $y$, where the $E$-polynomial is defined by

$$E(y) = |1 - \lambda iy + \mu y^2|^4 - |1 + n_1iy - n_2y^2 - n_3iy^3|^2.$$

By choosing $C = 10^{-4}$, a detailed calculation shows that
\[ E(y) = y^4 (E_0 + E_1 y^2 + E_2 y^4), \]

where
\[
E_0 = \frac{1247}{15000} + 2\mu^2 + \lambda^2 - 2\mu - \frac{4997}{7500}\lambda + 4\lambda\mu, \\
E_1 = -\frac{2497009}{90000000} - \lambda^4 + 2\lambda^3 - \frac{19997}{15000}\lambda^2 - 4\mu^2 + \frac{4997}{7500}\mu + \frac{4997}{15000}\lambda \]
\[+ 4\mu^3 - 2\lambda^2\mu^2 - 4\lambda^3\mu + 8\lambda\mu^2 + 8\lambda^2\mu - \frac{34997}{7500}\lambda\mu, \]
\[E_2 = 7\mu^4. \]

Pairs of \((\lambda, \mu)\) with values in domain \([0, 2] \times [-2, 0]\) giving \(L\)-stability are shown in Figure 1.

![Figure 1: \(L\)-stable choices of \((\lambda, \mu)\) for \(p = s = 2\) corresponding to \(C = 10^{-4}\).](image)

We select a single example, characterized by \(\lambda = \frac{4}{5}, \mu = -\frac{1}{5}\) and \(c = [\frac{1}{2}, 1]^T\). The coefficients of the method are
\[
\begin{bmatrix}
A & \bar{A} & U \\
B & \bar{B} & V
\end{bmatrix} = 
\begin{bmatrix}
\frac{4}{5} & 0 & -\frac{1}{5} & 0 & 1 & -\frac{3}{10} & -\frac{3}{40} \\
-\frac{967}{18750} & 4 & \frac{494}{3375} & -\frac{1}{8} & 1 & \frac{4717}{18750} & -\frac{253}{12500} \\
-\frac{967}{18750} & 4 & \frac{494}{3375} & -\frac{1}{8} & 1 & \frac{4717}{18750} & -\frac{253}{12500} \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}.
\]
2.2 Methods with \( p = q = s = r - 1 = 3 \)

We construct methods with \( p = q = s = r - 1 = 3 \) and RKS property. We look for methods which their stability function has the form

\[
R(z) = \frac{1 + n_1 z + n_2 z^2 + n_3 z^3 + n_4 z^4 + n_5 z^5}{(1 - \lambda z - \mu z^2)^3},
\]

where

\[
1 + \sum_{k=1}^{5} n_k z^k = \exp(z)(1 - \lambda z - \mu z^2)^3 - C_1 z^4 - C_2 z^5 + O(z^6).
\]

Here \( C_1 \) is the error constant of the method and \( C_2 \) is an arbitrary number. The E-polynomial has the form

\[
E(y) = y^6 \left(E_0 + E_1 y^2 + E_2 y^4 + E_3 y^6\right).
\]

For these methods to be \( A \)-stable, it is necessary and sufficient that \( \lambda > 0 \), \( \mu < 0 \), and so that \( E_0 + E_1 x + E_2 x^2 + E_3 x^3 + E_4 x^4 \) is non-negative for all positive real numbers \( x \), where \( E_0, E_1, E_2, E_3, \) and \( E_4 \) are complicated expressions in terms of \( \lambda \) and \( \mu \). By choosing \( C_1 = C_2 = 10^{-4} \), pairs of \( (\lambda, \mu) \) with values in domain \([0, 2] \times [-2, 0]\) giving \( L \)-stability are shown in Figure 2.

![Figure 2: L-stable choices of \((\lambda, \mu)\) for \( p = s = 3 \) corresponding to \( C_1 = C_2 = 10^{-4} \).](image)

Here, we represent an example, characterized by

\[
\lambda = \frac{1}{2}, \quad \mu = -\frac{1}{15}, \quad c = \begin{bmatrix} \frac{1}{3} & 2/3 \end{bmatrix}^T.
\]
The coefficients of the method are

\[
A = \begin{bmatrix}
0.5000000000000000 & 0 & 0 \\
1.4279081052775164 & 0.5000000000000000 & 0 \\
1.0000000000000000 & -0.3168631901664915 & 0.5000000000000000 \\
\end{bmatrix},
\]

\[
\overline{A} = \begin{bmatrix}
-0.6666666666666667 & 0 & 0 \\
-0.3067166674763493 & -0.6666666666666667 & 0 \\
-0.60208272123515 & 0.0288951398441268 & -0.6666666666666667 \\
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
1.0000000000000000 & -0.3168631901664915 & 0.5000000000000000 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
84.134011152194390 & -15.9895441299910120 & -37.951133307057703 \\
\end{bmatrix},
\]

\[
\overline{B} = \begin{bmatrix}
-0.60208272123515 & 0.0288951398441268 & -0.6666666666666667 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1.7866934603873189 & 20.045815971414841 \\
\end{bmatrix},
\]

\[
U = \begin{bmatrix}
1.0000000000000000 & -0.1666666666666667 & -0.0444444444444444 & 0.0086172839506173 \\
1.0000000000000000 & -1.2612414386108497 & -0.21369715393340 & 0.0056267104705261 \\
1.0000000000000000 & -0.1831368098335086 & -0.0241114076097811 & -0.0019021824846360 \\
\end{bmatrix},
\]

\[
V = \begin{bmatrix}
1.0000000000000000 & -0.1831368098335086 & -0.0241114076097811 & -0.0019021824846360 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -30.1933336017226565 & 2.3070365964725901 & 0 \\
\end{bmatrix}.
\]

### 2.3 Methods with \(p = q = s = r - 1 = 4\)

In this part, we construct methods of the Nordsieck SGLMs of type 2 with \(p = q = s = r - 1 = 4\) and \(c = [0 \ 0 \ 1]^T\). Setting some free parameters in order to make calculation easier, RKS conditions make the coefficients matrices of the method to take the following forms
For this method the only nonzero eigenvalue of $M(z)$ is

$$R(z) = 1 + \frac{z}{2 + \frac{z^2}{12}},$$

consequently, by the Ehle conjecture [11], the method is $A$-stable. The error constant of the method is $C = \frac{1}{170}$.

### 3 Implementation of the methods in a variable stepsize mode

In this section, we recall some implementation strategies given in [12], to apply the constructed methods in variable stepsize environment.

A Nordsieck SGLM in the variable stepsize mode takes the form

$$Y^{[n]} = h_n (A \otimes I_m) f(Y^{[n]}) + h_n^2 (A \otimes I_m) g(Y^{[n]}) + (UD(\delta_n) \otimes I_m)y^{[n-1]},$$

$$y^{[n]} = h_n (B \otimes I_m) f(Y^{[n]}) + h_n^2 (B \otimes I_m) g(Y^{[n]}) + (VD(\delta_n) \otimes I_m)y^{[n-1]},$$

where

$$D(\delta_n) := \text{diag}(1, \delta_n, \delta_n^2, ..., \delta_n^p), \quad \delta_n = h_n / h_{n-1}.$$

Due to the structure of the matrices $V$ and $D(\delta_n)$, the matrix $VD(\delta_n)$ has one eigenvalue with magnitude one and a zero eigenvalue with multiplicity $p$, for any value of $\delta_n$.

To achieve a suitable choice of stepsize for the next step, we first need to estimate the leading term in the local truncation error. To do this, we approximate $h^{p+1}y^{[p+1]}(x_n)$; so that $\text{LTE}(x_n) = C_p h^{p+1}y^{[p+1]}(x_n)$ can be calculated as an approximation to the local truncation error.

For the methods with $p = s = 2$ and abscissas $c = [\frac{1}{2}, 1]^T$, we use the linear combination of the form

$$\begin{bmatrix}
\frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 1 & \frac{1}{2} & 0 \\
\frac{1}{2} & 1 & 1 & \frac{1}{2} \\
\frac{1}{2} & 1 & 1 & 1 \\
6 & 0 & 0 & 6 \\
12 & 0 & 0 & 12 \\
7 & -1 & 0 & 6
\end{bmatrix}.$$
est\( (x_n) := C_p \left( \alpha_1 hf(Y_1) + \alpha_2 hf(Y_2) + \beta h^2 g(Y_1) \right), \)

with \( \alpha_1 = -8, \quad \alpha_2 = 8, \quad \beta = -4. \)

For the methods with \( p = s = 3 \) and abscissas \( c = [\frac{1}{3} \quad \frac{2}{3} \quad 1]^T \), we use the linear combination of the form

\[
est(x_n) := C_p \left( \alpha_1 hf(Y_1) + \alpha_2 hf(Y_2) + \alpha_3 hf(Y_3) + \beta h^2 g(Y_1) \right),
\]

with \( \alpha_1 = \frac{243}{2}, \quad \alpha_2 = -162, \quad \alpha_3 = \frac{81}{2}, \quad \beta = 27. \)

For the methods with \( p = s = 4 \) and abscissas \( c = [0 \quad 0 \quad 0 \quad 1]^T \), we use the linear combination of the form

\[
est(x_n) := C_p \left( \alpha_1 hf(Y_3) + \alpha_2 hf(Y_4) + \beta_1 h^2 g(Y_3) + \beta_2 h^2 g(Y_4) + \gamma y[n-1] \right),
\]

with \( \alpha_1 = 72, \quad \alpha_2 = -72, \quad \beta_1 = 48, \quad \beta_2 = 24, \quad \gamma = 12. \)

To control the stepsize, we use the following strategy

\[
est(x_n) \leq Rtol \cdot \max\{\|y_n\|, \|y_{n+1}\|\} + Atol, \tag{4}
\]

where \( Atol \) and \( Rtol \) are given absolute and relative tolerances, respectively. If the control (4) is not satisfied, the current step is repeated with a halved stepsize. Otherwise, the current step is accepted and we carry out the next step with the new stepsize defined as

\[
h_{n+1} = \delta_{n+1} h_n,
\]

where

\[
\delta_{n+1} = \min\left\{ \text{facmax}, \left( \frac{\text{fac} \cdot \text{tol}}{\|\text{est}(x_n)\|} \right)^{\frac{1}{p}} \right\}.
\]

Here, \( \text{facmax} \) and \( \text{fac} \) are safety factors built into a code to prevent the step from increasing too rapidly and to avoid an excessive number of rejected steps. We have chosen \( Atol = Rtol = \text{tol}, \text{facmax} = 2, \) and \( \text{fac} = 0.9. \)

### 4 Numerical result

In this section we present the results of numerical experiments to show efficiency of the constructed methods in section 2 for fixed and variable stepsize.
mode using the provided techniques in section 3. We consider the following test problems:

- Problem 1. The nonlinear stiff system of ODEs

\[
\begin{align*}
  y'_1(x) &= -1002y_1(x) + 1000y_2^2(x), \\
  y'_2(x) &= y_1(x) - y_2(x)(1 + y_2(x)),
\end{align*}
\]

whose exact solution is \( [y_1(x), y_2(x)]^T = [\exp(-2x), \exp(-x)]^T \) and \( x \in [0, 2] \). This problem is stiff with an approximate stiffness ratio of 10^3 near to \( x = 0 \).

- Problem 2. A system of differential equation, is called CUSP, resulting from discretization of the diffusion terms by the method of line the periodic boundary-value problem [14], is as below

\[
\begin{align*}
  \frac{\partial y}{\partial t} &= -\frac{1}{\epsilon}(y^3 + ay + b) + \sigma \frac{\partial^2 y}{\partial x^2}, \\
  \frac{\partial a}{\partial t} &= b + 0.07b + \sigma \frac{\partial^2 a}{\partial x^2}, \\
  \frac{\partial b}{\partial t} &= (1 - a^2)b - a - 0.4y + 0.035b + \sigma \frac{\partial^2 b}{\partial x^2},
\end{align*}
\]

where

\( \nu = \frac{u}{0.1 + u}, \quad u = (y - 0.7)(y - 1.3). \)

This problem takes the form

\[
\begin{align*}
  \dot{y}_i &= -\epsilon^{-1}(y_i^3 + a_iy_i + b_i) + D(y_{i-1} - 2y_i + y_{i+1}), \\
  \dot{a}_i &= b_i + 0.07nu_i + D(a_{i-1} - 2a_i + a_{i+1}), \quad i = 1, 2, \ldots, N, \\
  \dot{b}_i &= (1 - a_i^2)b_i - a_i - 0.4y_i + 0.035nu_i + D(b_{i-1} - 2b_i + b_{i+1}),
\end{align*}
\]

where

\( \nu_i = \frac{u_i}{0.1 + u_i}, \quad u_i = (y_i - 0.7)(y_i - 1.3), \quad D = \sigma N^2, \)

with periodic boundary condition

\[
\begin{align*}
  y_0 &= y_N, \quad a_0 := a_N, \quad b_0 := b_N, \\
  y_{N+1} &= y_1, \quad a_{N+1} := a_1, \quad b_{N+1} := b_1.
\end{align*}
\]

We take \( \sigma = \frac{1}{10^3}, \epsilon = 10^{-4}, N = 32, \) and the initial values as

\[
\begin{align*}
  y_i(0) &= 0, \quad a_i(0) = -2 \cos(\frac{2i\pi}{N}), \quad a_i(0) = 2 \sin(\frac{2i\pi}{N}), \quad i = 1, 2, \ldots, N,
\end{align*}
\]

with \( t_{out} = 1.1. \)
Problem 3. For the ODE case, the Ring modulator problem originates from electrical circuit analysis is of the form [15]

\[
\begin{align*}
y'_1 &= C^{-1}(y_8 - 0.5y_{10} + 0.5y_{11} + y_{14} - R^{-1}y_1), \\
y'_2 &= C^{-1}(y_9 - 0.5y_{12} + 0.5y_{13} + y_{15} - R^{-1}y_2), \\
y'_3 &= C_s^{-1}(y_{10} - q(U_{D1}) + q(U_{D4})), \\
y'_4 &= C_s^{-1}(-y_{11} + q(U_{D2}) - q(U_{D3})), \\
y'_5 &= C_s^{-1}(y_{12} + q(U_{D1}) - q(U_{D3})), \\
y'_6 &= C_s^{-1}(-y_{13} - q(U_{D2}) + q(U_{D4})), \\
y'_7 &= C_p^{-1}(R_p^{-1}y_7 + q(U_{D1}) + q(U_{D2}) - q(U_{D3}) - q(U_{D4})), \\
y'_8 &= -L_h^{-1}y_1, \\
y'_9 &= -L_h^{-1}y_2, \\
y'_{10} &= L_{s2}^{-1}(0.5y_1 - y_3 - R_{g2}y_{10}), \\
y'_{11} &= L_{s2}^{-1}(-0.5y_1 + y_4 - R_{g3}y_{11}), \\
y'_{12} &= L_{s2}^{-1}(0.5y_2 - y_5 - R_{g2}y_{12}), \\
y'_{13} &= L_{s3}^{-1}(-0.5y_2 + y_6 - R_{g3}y_{13}), \\
y'_{14} &= L_{s1}^{-1}(-y_1 + U_{in1}(t) - (R_i + R_{g1})y_{14}), \\
y'_{15} &= L_{s1}^{-1}(-y_2 - (R_e + R_{g1})y_{15}).
\end{align*}
\]

The auxiliary functions \( U_{D1}, U_{D2}, U_{D3}, U_{D4}, q, U_{in1}, \) and \( U_{in2} \) are given by

\[
\begin{align*}
U_{D1} &= y_3 - y_5 - y_7 - U_{in2}(t), \\
U_{D2} &= -y_4 + y_6 - y_7 - U_{in2}(t), \\
U_{D3} &= y_4 + y_5 + y_7 + U_{in2}(t), \\
U_{D4} &= -y_3 - y_6 + y_7 + U_{in2}(t), \\
q(U) &= \gamma(e^{8U} - 1), \\
U_{in1}(t) &= 0.5\sin(2000\pi t), \\
U_{in2}(t) &= 2\sin(20000\pi t).
\end{align*}
\]

The values of the parameters are
\[ C = 1.6 \times 10^{-8}, \quad C_s = 2 \times 10^{-12}, \quad C_p = 10^{-12}, \quad L_h = 4.45, \]
\[ L_{s1} = 0.002, \quad L_{s2} = 5 \times 10^{-4}, \quad L_{s3} = 5 \times 10^{-4}, \]
\[ R = 25000, \quad R_p = 50, \quad R_{g1} = 36.3, \quad R_{g2} = 17.3, \]
\[ R_{g3} = 50, \quad R_i = 50, \quad R_c = 600, \]
\[ \delta = 17.74933332, \quad \gamma = 40.67286402 \times 10^{-9}, \]

with the initial vector \( y_0 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T \) and \( t \in [0, 10^{-3}] \).

### 4.1 Fixed stepsize experiments

We first present fixed stepsize numerical results in order to show accuracy of constructed Nordsieck SGLMs and validate the order of these methods in the integration of stiff differential systems. To do this, we have applied the methods of order 2, 3, and 4 to the Problem 1. We have implemented the methods with a fixed stepsize \( h = 1/2^k \), with several integer values of \( k \). The results of numerical experiments for type 2 Nordsieck SGLMs are shown in the Tables 1, 2, and 3. In these tables, we have listed the norm of error \( \|e_h(\bar{x})\| \) at the endpoint of integration \( \bar{x} = 2 \) and numerical estimate to the order of convergence, \( p \), computed by the formula

\[
p = \frac{\log(\|e_h(\bar{x})\|/\|e_{h/2}(\bar{x})\|)}{\log(2)},
\]

where \( e_h(\bar{x}) \) and \( e_{h/2}(\bar{x}) \) are errors corresponding to stepsizes \( h \) and \( h/2 \) for Nordsieck SGLMs. Also, to show that the constructed methods are competitive with the efficient existing methods, we have reported the numerical results of type 2 methods which have been constructed in [2].

<table>
<thead>
<tr>
<th>( k )</th>
<th>Type 2 method of order 2 [2]</th>
<th>Nordsieck SGLM of order 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( e_h(\bar{x}) ) ( p )</td>
<td>( e_h(\bar{x}) ) ( p )</td>
</tr>
<tr>
<td>10</td>
<td>( 1.78 \times 10^{-11} )</td>
<td>( 1.35 \times 10^{-10} )</td>
</tr>
<tr>
<td>11</td>
<td>( 5.44 \times 10^{-12} )</td>
<td>( 1.71 ) ( 3.31 \times 10^{-11} )</td>
</tr>
<tr>
<td>12</td>
<td>( 1.43 \times 10^{-12} )</td>
<td>( 1.93 ) ( 8.18 \times 10^{-12} )</td>
</tr>
<tr>
<td>13</td>
<td>( 2.64 \times 10^{-13} )</td>
<td>( 2.44 ) ( 2.03 \times 10^{-12} )</td>
</tr>
</tbody>
</table>
Table 2: Numerical results for Nordsieck SGLMs of order \( p = q = 3 \).

<table>
<thead>
<tr>
<th>( k )</th>
<th>Type 2 method of order 3 ([2])</th>
<th>Nordsieck SGLM of order 3</th>
<th>( e_h(\bar{x}) )</th>
<th>( p )</th>
<th>( e_h(\bar{x}) )</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>( 4.07 \times 10^{-6} )</td>
<td></td>
<td>( 6.58 \times 10^{-8} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>( 5.66 \times 10^{-7} )</td>
<td>2.85</td>
<td>( 8.66 \times 10^{-9} )</td>
<td>2.93</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>( 7.43 \times 10^{-8} )</td>
<td>2.92</td>
<td>( 1.11 \times 10^{-9} )</td>
<td>2.96</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>( 9.50 \times 10^{-9} )</td>
<td>2.96</td>
<td>( 1.40 \times 10^{-10} )</td>
<td>2.99</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3: Numerical results for Nordsieck SGLMs of order \( p = q = 4 \).

<table>
<thead>
<tr>
<th>( k )</th>
<th>Type 2 method of order 4 ([2])</th>
<th>Nordsieck SGLM of order 4</th>
<th>( e_h(\bar{x}) )</th>
<th>( p )</th>
<th>( e_h(\bar{x}) )</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>( 6.22 \times 10^{-8} )</td>
<td></td>
<td>( 5.81 \times 10^{-9} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>( 4.21 \times 10^{-9} )</td>
<td>3.88</td>
<td>( 3.63 \times 10^{-10} )</td>
<td>4.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>( 2.74 \times 10^{-10} )</td>
<td>3.94</td>
<td>( 2.27 \times 10^{-11} )</td>
<td>4.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>( 1.75 \times 10^{-11} )</td>
<td>3.97</td>
<td>( 1.42 \times 10^{-12} )</td>
<td>4.00</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### 4.2 Variable stepsize experiments

We first investigate the potential for efficient implementation in a variable stepsize environment by the reliability of the estimation error for Problem 1. The obtained results for the methods of order 2, 3, and 4 have plotted in Figures \([3]\), \([4]\), and \([5]\) respectively. These figures confirm efficiency of the used estimation for the local truncation error. To compare, we also present the results of numerical experiments of the \( L \)-stable Nordsieck SGLM given in \([3]\). In the implementation of considered SGLMs, we apply the same introduced implementation strategies, including the starting procedures, stage predictors, local error estimation, and the changing stepsize. In our numerical results, we use the following abbreviations:

- \( ns \): the number of steps
- \( nrs \): the number of rejected steps
- \( nfe \): the number of function evaluations
- \( nJe \): the number of Jacobian evaluations
- \( ge \): the global error
- \( tol \): given tolerance
- \( \text{NSGLM}2p \): type 2 Nordsieck SGLM of order \( p \)
- \( \text{NSGLM}4p \): type 4 Nordsieck SGLM of order \( p \)
Figure 3: Local errors and local error estimates versus $x$ of the method of order 2 for problem 1 with $h_0 = 10^{-5}$ and $tol = 10^{-8}$.

Figure 4: Local errors and local error estimates versus $x$ of the method of order 3 for problem 1 with $h_0 = 10^{-5}$ and $tol = 10^{-8}$.

Figure 5: Local errors and local error estimates versus $x$ of the method of order 4 for problem 1 with $h_0 = 10^{-5}$ and $tol = 10^{-8}$.
Table 4: Numerical results for Problem 2 solved by NSGLM2\(p\) and NSGLM4\(p\) of orders 2, 3, and 4 with \(h_0 = 10^{-3}\).

<table>
<thead>
<tr>
<th>tol</th>
<th>Method</th>
<th>ge</th>
<th>ns</th>
<th>nrs</th>
<th>nfe</th>
<th>nJe</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10^{-6})</td>
<td>NSGLM22</td>
<td>(3.61 \times 10^{-5})</td>
<td>169</td>
<td>26</td>
<td>1644</td>
<td>1256</td>
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<tr>
<td></td>
<td>NSGLM42</td>
<td>(2.02 \times 10^{-4})</td>
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<td>11</td>
<td>6693</td>
<td>4047</td>
</tr>
<tr>
<td>(10^{-8})</td>
<td>NSGLM22</td>
<td>(1.07 \times 10^{-6})</td>
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<td>3718</td>
<td>2316</td>
</tr>
<tr>
<td></td>
<td>NSGLM42</td>
<td>(8.84 \times 10^{-6})</td>
<td>5997</td>
<td>14</td>
<td>24187</td>
<td>12167</td>
</tr>
<tr>
<td>(10^{-10})</td>
<td>NSGLM22</td>
<td>(4.58 \times 10^{-8})</td>
<td>3154</td>
<td>15</td>
<td>12847</td>
<td>6511</td>
</tr>
<tr>
<td></td>
<td>NSGLM42</td>
<td>(3.99 \times 10^{-7})</td>
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<td>14</td>
<td>110499</td>
<td>55323</td>
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<tr>
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<td>(2.95 \times 10^{-5})</td>
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<td>36</td>
<td>2339</td>
<td>1700</td>
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<tr>
<td></td>
<td>NSGLM43</td>
<td>(5.80 \times 10^{-5})</td>
<td>426</td>
<td>17</td>
<td>3248</td>
<td>1922</td>
</tr>
<tr>
<td>(10^{-8})</td>
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<td>474</td>
<td>14</td>
<td>4047</td>
<td>2586</td>
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<td>(1.28 \times 10^{-6})</td>
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<td>NSGLM23</td>
<td>(1.47 \times 10^{-8})</td>
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<td></td>
<td>NSGLM43</td>
<td>(5.59 \times 10^{-9})</td>
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<td>646</td>
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<td>82581</td>
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<td>(9.16 \times 10^{-5})</td>
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<td>25</td>
<td>3308</td>
<td>1864</td>
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<td></td>
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<td>14</td>
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<td>14</td>
<td>6599</td>
<td>2399</td>
</tr>
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<td>NSGLM24</td>
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<td>141</td>
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<td>1700</td>
<td>18</td>
<td>13934</td>
<td>7066</td>
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</tbody>
</table>

Some numerical results for Problem 2 and 3 demonstrating the computational cost of NSGLM2\(p\) are given in Tables 4 and 5 and compared with those in NSGLM4\(p\) for \(p = 2, 3, 4\). Also, in Figures 6 and 7, we compare the accepted stepsizes of NSGLM23 with those in NSGLM24 and the accepted stepsizes of NSGLM43 with those in NSGLM44 through integration for Problem 2 and 3, respectively. The numerical results show that the proposed methods are capable in solving stiff problems and competitive with the existing methods.

5 Conclusion

We constructed type 2 Nordsieck SGLMs of orders 2, 3, and 4 with RKS property. Order 2 and 3 methods are \(L\)-stable and order 4 method is \(A\)-stable. These methods have been equipped to the variable stepsize using Nordsieck technique. The capability of the proposed methods in solving stiff problems with long interval of integration and badly scaled solution have been validated by some numerical experiments and comparisons.
Table 5: Numerical results for Problem 3 solved by NSGLM2p and NSGLM4p of orders 2, 3, and 4 with $h_0 = 10^{-6}$.

<table>
<thead>
<tr>
<th>tol</th>
<th>Method</th>
<th>ge</th>
<th>ns</th>
<th>nrs</th>
<th>nfe</th>
<th>nJc</th>
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<td>NSGLM42</td>
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<td>$5.94 \times 10^{-7}$</td>
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<td>486</td>
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<tr>
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<td>396</td>
<td>1092456</td>
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<tr>
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<td>2790</td>
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<td>30994</td>
<td>21199</td>
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<tr>
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<td>499</td>
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<td>219980</td>
<td>109992</td>
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</tbody>
</table>

Figure 6: Accepted stepsizes versus $x$ for Problem 2 with $h_0 = 10^{-3}$ and tol = $10^{-6}$; (a) NSGLM23 and NSGLM43, (b) NSGLM24 and NSGLM44.
Figure 7: Accepted stepsizes versus $x$ for Problem 3 with $h_0 = 10^{-6}$ and $tol = 10^{-6}$: (a) NSGLM23 and NSGLM43, (b) NSGLM24 and NSGLM44.

References


برخی روش‌های انگرالگیری نرمالی‌کارا برای مسایل مقدار اولیه

نتایج بررسی اسکوئی به علی عبدي و علامت‌خ حجتی

دانشکده علوم ریاضی، دانشگاه تبریز

دریافت مقاله 9 مرداد 1395، دریافت مقاله اصلاح شده 20 آبان 1395، پذیرش مقاله 1 آذر 1396

چکیده: در این مقاله، در ادامه ساخت روش های عددی کارای حل مسایل مقدار اولیه سخت، روش های مقدار اولیه سخت، روش های خطي عمومی با مشتق دوم نرمالی‌کارا برای حل مسایل مقدار اولیه سخت، روش های خطي عمومی با مشتق دوم نرمالی‌کارا، روش های خطي عمومی با مشتق دوم نرمالی‌کارا را می‌نامیم، که در آن S تعداد مراحل می‌باشد، است و مراحل می‌باشند. پیاده‌سازی روش های ساخته شده با طول گام ثابت و متغیر بحث قرار گرفته است که کارایی روش‌ها را تأیید می‌کند.

کلمات کلیدی: معادلات دیفرانسیل سخت، روش‌های خطي عمومی با مشتق دوم نرمالی‌کارا، A، L و - پایداری پیاده‌سازی با طول گام متغیر.