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Research Article

Differential transform method: A comprehensive review and analysis

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Abstract

The complexity of solving differential equations in real-world applications motivates researchers to extend numerical methods. Among different numerical and semi-analytical methods for solving initial and boundary value problems, the differential transform method (DTM) has received notable attention. It has developed and experienced generalizations for implementing other types of mathematical problems such as optimal control, calculus of variations, and integral equations. This review aims to provide insight into DTM. History, theoretical base, applications, computational aspects, and its revisions are reviewed. The present study helps to understand the theory, capabilities, and features of the DTM, as well as its drawbacks and limitations.

AMS subject classifications (2020): Primary 34A25; Secondary 65L10, 65N99.

Keywords: Boundary value problems; Initial value problems; Differential Transform Method; Semi-analytical methods.

1 Introduction

There are many practical problems, which are formulated as boundary value problems (BVPs). They have appeared, for example, in studying the boundary layer flow [61], the squeezing nanofluids [62], the formation of rogue waves in the ocean [20], electrical heating of conductors [26], and modeling the behavior of induction motors [6]. In addition, other types of important problems, such as calculus of variations or optimal control problems, are reduced to a set of BVPs or initial value problems (IVPs). The multitude of such applications and the complexity of solving BVPs motivated the researcher to develop solving methods. Solving strategy has three categories as follows:

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- Analytical methods: Methods that find the exact or analytical solution of the BVPs as a function or closed-form are known as analytical methods. Direct integration [44], method of images [44], separation of variables, and Green's function method [37] are some examples of analytical methods. Despite exact results, the analytical methods are usually restricted to simple or special forms of BVPs. Moreover, they require manual calculations that make their implementation on computers difficult.

- Numerical algorithms: Numerical methods are suitable for computerization while their results contain errors and convergences issues should be checked. These types of methods are based on the numerical approximation of derivatives like finite difference [51] and shooting method [52]. Some of the numerical methods have also motivated from the physics of the problem, such as lattice Boltzmann [83].

- Semi-analytical methods: When the result of a method is a function or a sequence of functions converging to the exact solution, the method is semi-analytical. Collocation finite element [52], Galerkin finite element [52], Adomian decomposition [22], and iterative approximations [55] are some examples of semi-analytical methods. They have the advantage of finding function answers and computer implementation while they have errors in results.

Among semi-analytical methods, the differential transform method (DTM) is one of the most popular and practical algorithms. This method was introduced by Zhou [95] in 1986 for solving IVPs in the field of electrical circuits. The method is based on the calculation of the coefficients of the Taylor series of the problem's solution. The method has been developed for solving BVPs in one and more dimensions, integral equations, calculus of variations, and optimal control. Especially in the last decade, it was used for analyzing several physical phenomena with stochastic and fractional behavior.

These applications and implementations of the DTM are motivations for reviewing the method in the present work. The aim is to give a comprehensive review of the method, including the theory, improvements, and applications. Some examples related to the method are explained to show its accuracy and benefits. Finally, the restrictions and drawbacks of the method are noted.

The present review helps researchers who are attending to use the method for solving a practical problem to be familiar with this method and its limitations.

The review is organized as follows: after introductions in Section 1, the literature review and history are given in Section 2. Section 3 assigns the method description, benefits, and drawbacks. Finally, some concluding remarks are expressed in Section 4.

2 Literature review

This section is divided into two subsections: the historical and application reviews. A graphical review based on the research' subjects is also included.

2.1 Historical review

In this subsection, the papers with independent research on extending or improving the DTM have been reviewed in historical order.

- 1986: The concept of differential transform was established by Zhou, a Chinese researcher in the field of electrical engineering. The method was originally explained in [95] for IVPs.
- 1996: DTM has been extended to cover the hybrid boundary conditions for eigenvalue problems in [23].
- 1998: The method was improved in the case of BVPs in an infinite horizon. As a practical implementation, it has been implemented to solve the Blasius problem efficiently in [94].
- 1999: The two-dimensional differential transform has been proposed for solving IVPs with partial differential equations in [24].
- 2003: Following the extensions for two-dimensional DTM, new theorems were given in [14] with applications of the method for diffusion equation.
- 2004: The three-dimensional DTM was introduced for solving systems of partial differential equations (PDEs) accompanied by the initial conditions in [15]. The DTM has also been applied to find accurate solutions for algebraic differential equations of ordinary type in [16].
- 2005: The integro-differential equations with boundary conditions were the next type of problems examined for their solution by DTM in [9]. General theorems were derived, and the method was successfully applied to solve examples of linear and nonlinear integro-differential problems.
- 2006: The DTM was extended to solve difference equations with different types and orders in [10]. Solving differential-difference equations with boundary conditions with DTM have been also reported in [11].
- 2007: The concept of fractional derivatives and the growing topic of fractional differential equations cause to define the fractional differential transform. In [65], the theory of fractional DTM was established for

solving ordinary fractional with initial conditions. The method was later proposed using generalized Taylor series and Caputo fractional derivative.

- 2008: The fractional DTM for ordinary equations has been more generalized for equations with multi-order in [31]. The method of fractional DTM has been extended to linear PDEs of fractional order in [69]. The extension of the method to systems of fractional PDEs with initial conditions was also given in [32]. A modified DTM based on Laplace transform and Padé approximation was introduced to find oscillatory solutions.

- 2009: The two-dimensional DTM was implemented to solve a class of linear and nonlinear Volterra integral equations in [87]. To resolve the complexity of computation in a multidimensional DTM, a reduced method was introduced in [53]. The method is based on the separation of variables. The efficiency of the method has been demonstrated by its application on several IVPs. In the follow-up to the fractional derivatives, the DTM was examined for fractional integro-differential equations in [12]. Another notable work is [76], where the DTM is combined with Padé approximation to solve BVPs with infinite horizon. The fuzzy DTM was introduced in [7] to solve fuzzy differential equations. The method is based on the generalized H-differentiability. The DTM was also applied to solving nonlinear optimal control problems in [43]. Two approaches for finite and infinite horizon problems were proposed based on the minimum principle and the dynamic programming on Hamilton–Jacobi–Bellman equations, respectively, in combination with DTM.

- 2010: To accelerate the convergence of the DTM solution, a multi-step DTM was proposed in [70]. In this version of DTM, the solution is a piecewise function consisting of a finite number of DTM solutions for consecutive time intervals. Another derivation of DTM called projected DTM, was also proposed in [45]. In this method, the solution of two-dimensional PDEs is obtained with DTM for one variable while the coefficients are functions of the other variable.

- 2011: The piecewise DTM has been further extended for solving fractional chaotic dynamical systems in [8]. It is indeed the extension of [70] in fractional cases.

- 2012: Random DTM is another version of DTM for solving random differential equations based on the mean fourth calculus proposed in [90]. The results of the implementation of the method for Riccati differential problems show the efficiency of this approach. A combination of the Adomian decomposition method and DTM was proposed in [29] for solving fractional differential equations.

- 2013: The fuzzy DTM method [7] has been extended to cover solving Volterra integral equations in [81]. A combination of DTM with Adomian polynomial was proposed in [34] to overcome the problem of nonlinear terms in ordinary differential equations (ODEs) when DTM is used. For the calculus of variation problem with a differentiable solution, there exists a two-point boundary problem obtained by the Euler–Lagrange equation. Using the method proposed in [67], this problem was solved by the DTM to derive a numerical method for finding semi-analytical stationary functions. A DTM for solving linear optimal control problems with a quadratic performance index was introduced in [80]. The method uses the Pontryagin maximum principle to obtain a BVP, which is finally solved by DTM. Reduced DTM is examined for solving two-dimensional Volterra integral equations in [2]. Based on the simulations, the results of reduced MTD are more accurate in comparison with traditional DTM.

- 2014: In [3], the nonlinear integro-differential equations with proportional delay are under investigation with DTM. Some theorems related to the delayed functions and their transforms were also proved in addition to the numerical simulation. To enlarge the domain of convergence of DTM, a method was proposed in [17]. The Laplace–Padé resummation was examined to solve partial differential algebraic equations.

- 2015: The generalized DTM method for IVPs on fractional PDEs has been extended to BVPs in [27]. DTM was applied in [35] as a new tool to compute the Laplace transform of real-valued single variable functions. The Cauchy-type singular integral equations are solved by a proposed method based on DTM in [4]. The forms of differential transform of kernel functions were obtained with high-accuracy solutions on several examples with two kernel types. DTM was also used to solve optimal control problems in [66]. The method is based on applying the DTM to the BVPs resulting from sufficient conditions for solving linear and nonlinear optimal control problems.

- 2016: Two-dimensional extended DTM was proposed for solving PDEs with local fractional derivatives in [93]. The concept of this version of DTM for nondifferential functions was analyzed, and basic theorems were proved. The efficiency and accuracy of the method were shown via numerical simulations. A class of BVPs defined for nonlinear singular second-order ODE was examined with DTM in [91]. The method benefits from the Adomian polynomials to overcome the nonlinear terms. In addition, to demonstrate the applicability of the method by some examples, an upper bound for the error was also obtained. Multi-point BVPs also found their DTM solution. The problem of unknown initial conditions in these types of problems was resolved in [92]. The first two DTM coefficients were taken unknown and were determined from a system of algebraic equations.

- 2017: A version of DTM was introduced in [88] for solving comfortable fractional differential equations. The generalized DTM for solving fractional problems was further studied from a theoretical perspective in [71]. The sufficient conditions for convergence of the method and estimation of truncation error were obtained. An efficient version of multi-step DTM was addressed in [70]. The method reduces the number of subintervals and consequently improves the computational complexity of multi-step DTM.

- 2018: The projected DTM was combined with integral transform to provide an efficient method for solving fractional PDEs in [82]. The results showed that the method is accurate and fast convergent. Fuzzy DTM was extended for solving fuzzy Volterra integro-differential equations in [19]. The method is based on a generalization of Seikkala differentiability for fuzzy functions.

- 2019: The switching DTM was introduced in [61] to cover infinite horizons, that is, boundary conditions at infinity. In the proposed approach, the solution has two parts: a DTM solution and an analytical solution that matches the condition at infinity.

- 2020: In [30], the method of [29] was applied for computing two-dimensional DTM solutions of PDE problems. The method reduced the computational complexity of traditional two-dimensional DTM. A combination of differential transform and smoothed particle hydrodynamics was proposed in [57] for solving transient heat conduction problems. Numerical simulations showed that the method is robust and accurate. Tarig transform was combined with projected DTM to develop an effective method for solving fractional nonlinear PDEs in [60].

- 2021: As recent applications of DTM to practical problems, we can address [46], where the problem of thermal distribution through a longitudinal trapezoidal moving fin has been investigated using one-dimensional Padé-DTM. Similar work for a moving rod was reported in [84], where two-dimensional Padé-DTM is implemented.

- 2022: A comparison between sinc approximation and DTM on nonlinear Hammerstein integral equations has been made in [50]. In the case of separable kernels, the DTM performs more accurately and faster than the sinc approximation. Integro-differential equations with a retarded argument have notable engineering applications. In [42], these types of problems have been solved by DTM with satisfactory and applicable results.

To demonstrate the review of fulfilled research on the concept of DTM, a graphical tree of a subject is given in Figure 1. There are four main blocks determining the top subjects, along with subblocks indicating the details. The related references to each subject are written close to the related box.

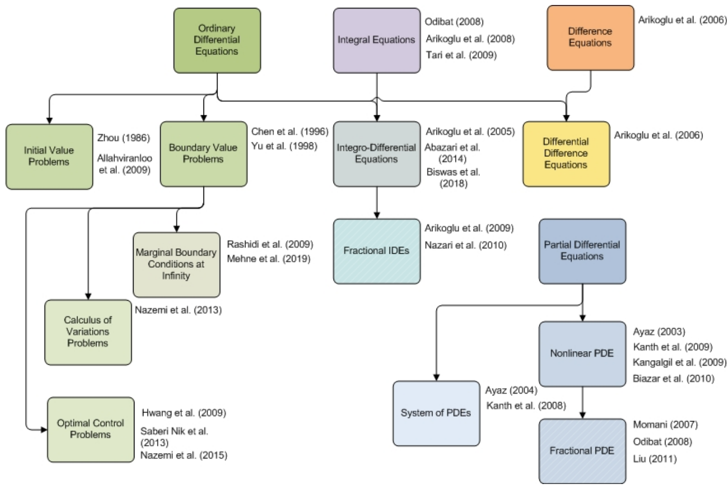

Figure 1: Subjective review of researches on DTM

2.2 Application review

In this subsection, some notable applications of DTM in the real world and practical problems are listed.

- Fluid mechanics: Fractional coupled Burgers' equations [58], Blssius equation of boundary layer flow [94], nanoparticle migration [56], nano boundary-layers over-stretching surfaces [77], magnetohydrodynamics (MHD) boundary-layer equations [76], MHD in a laminar liquid film [78], parametric investigation of the thermal analysis for solar collectors [28], the study of time-dependent MHD heat transfer flow of Jeffrey fluid [54], and analysis an unsaturated single-phase fluid flow in porous media [25].

- Electrical engineering: Solving telegraph equation by DTM [18] and reduced DTM [86, 85], solving Thomas-Fermi equation by the improved DTM [38], and dynamic simulation of power systems [59].

- Acoustics: KdV and modified KdV equations [48], two-dimensional fractional Helmholtz equation [5], and Kadomtsev–Petviashvili equations [63].

- Physics: Solving a model of fractional telegraph point reactor kinetics [39] and solving Fokker–Planck equation [41].

- Quantum Mechanics: Klein–Gordon equation [79] and Burgers–Huxley equations [1].

- Structures and vibration: Vibration analysis of a rotating tapered cantilever Bernoulli–Euler beam [72], nonlinear oscillators [64], analysis and prediction of vibration of a nanobeam [40], investigation of flapwise bending free vibration of isotropic rotating Timoshenko microbeams [13], analyzing the thermal buckling of a functionally graded circular plate [33], solving nonlinear Duffing oscillator [68], and buckling analysis of nanobeams [47].

- Miscellaneous applications: Population growth estimation [73], solving a typhoid fever model [74], solving tumor-immune system [49], analysis of fish-farm model [89], solving the model of pollution for a system of lakes [17], and modeling of jamming transition problem in traffic flow [36].

3 How does DTM work?

In this section, the basic definitions and fundamental properties of the differential transform method are presented. Let us consider the following ordinary differential equation:

$$T(x, u(x), u'(x), u''(x), \dots, u^{(n)}(x)) = 0, \tag{1}$$

where T is a transformation on a class of sufficiently differentiable functions $u(x)$. Assume that under specific conditions, the above-mentioned differential equation has a unique solution $u(x)$ satisfying in

$$u(0) = u_0, u'(0) = u'_0, \dots, u^{(n)}(0) = u_0^{(n)}, \tag{2}$$

where $u_0, u'_0, \dots, u_0^{(n)}$ are given. Now, the aim of DTM in the simplest case is to solve the IVP (1)–(2). Let us consider the Taylor series of the solution in a neighborhood of $x = 0$:

$$u(x) = \sum_{k=0}^{\infty} \frac{u^{(k)}(0)}{k!} x^k. \tag{3}$$

It may be also rewritten as

$$u(x)=\sum_{k=0}^{\infty}U(k)x^k, \tag{4}$$

where

$$U(k)=\frac{u^{(k)}(0)}{k!} \tag{5}$$

Therefore, if the values of $U(k)$ are available, then the solution may be constructed from (4). This is the key of the DTM method that defines a transformation from $u(x)$ to the set of coefficients $\{U(1),U(2),\ldots\}$ and vice versa. This transformation is called the differential transformation. Now, in DTM, $U(k)$'s are substituted in (1) converting it to a system of algebraic equations. This will be performed using the basic properties of the differential transform. Some of these properties are listed below:

Let us assume, for simplicity, that $\xrightarrow{\text{DT}}$ denotes the differential transform. If λ is a constant scalar, $u(x)\xrightarrow{\text{DT}}U(k)$, and $v(x)\xrightarrow{\text{DT}}V(k)$, where $U(k)$ and $V(k)$ are differential transformations of $u(x)$ and $v(x)$, respectively, then

- $u(x)+v(x)\xrightarrow{\text{DT}}U(k)+V(k)$;
- $\lambda u(x)\xrightarrow{\text{DT}}\lambda U(k)$;
- $u(x)v(x)\xrightarrow{\text{DT}}\sum_{i=0}^{k}U(i)V(k-i)$;
- $u'(x)\xrightarrow{\text{DT}}(k+1)U(k+1)$;
- $u''(x)\xrightarrow{\text{DT}}(k+1)(k+2)U(k+2)$;
- $u^{(n)}(x)\xrightarrow{\text{DT}}(k+1)(k+2)\cdots(k+n)U(k+n)$;
- $u(x)=\int_0^x v(s)ds\xrightarrow{\text{DT}}U(k)=\begin{cases}\frac{V(k-1)}{k}, & k\geq 1,\\ 0, & k=0;\end{cases}$
- $u(x)=x^n\xrightarrow{\text{DT}}U(k)=\delta(k-n)=\begin{cases}1, & k=n,\\ 0, & k\neq n;\end{cases}$
- $u(x)=e^{\lambda x}\xrightarrow{\text{DT}}U(k)=\frac{\lambda^k}{k!}$.

These properties will be obtained directly for the definition of the differential transform given by (5).

In what follows, the implementation of the method for solving a simple IVP is given.

Example 1. Let us consider the following IVP:

$$(x^2+9)u''+2xu'=0, \tag{6}$$

$$u(0)=\pi, \qquad u'(0)=\frac{4}{3}. \tag{7}$$

The problem has the following unique solution:

$$u(x) = 4\tan^{-1}\left(\frac{x}{3}\right) + \pi. \tag{8}$$

To implement the DTM on this problem, let us assume that $u(x) \xrightarrow{\text{DT}} U(k)$. Then, by the above-mentioned properties of differential transform and substituting the corresponding transforms of individuals terms of (6), we have

$$\begin{aligned} &\sum_{i=0}^{k} \left(\delta(i-2) + 9\delta(i)\right)(k-i+1)(k-i+2)U(k-i+2) \\ &\quad +2\sum_{i=0}^{k} \delta(i-1)(k-i+1)U(k-i+1) = 0. \end{aligned} \tag{9}$$

Regarding the initial conditions in (7) and the definition of Dirac delta function, the transformed problem is defined by the following recursive converted equation:

$$U(0) = \pi, \tag{10}$$

$$U(1) = \frac{4}{3}, \tag{11}$$

$$U(k+2) = \frac{-k}{9(k+2)}U(k), \quad k \geq 0. \tag{12}$$

The unknown coefficients will be calculated from the above relations, and then the solution of the problem in the form of an infinite series is determined by (4). One can truncate this series with n terms as

$$u_n(x) = \sum_{k=0}^{n} U(k)x^k \tag{13}$$

to approximate the solution. For example, the solution for $n = 8$ with 4-digit accuracy is calculated as follows:

$$u_8(x) = 3.1416 + 1.3333x - 0.04934x^3 + 0.0033x^5 - 0.0003x^7. \tag{14}$$

The approximate solution (14) has decreasing coefficients and indicates that $\{u_n(x)\}$ converges pointwise to the solution of the problem when $0 \leq x < 1$. As also depicted in Figure 2, the obtained DTM solution (14) is very close to the exact one. However, when $x > 1$, the convergence of the sequence of approximations does not guarantee. For instance, in Figure 3, the exact and the DTM solutions have been drawn for $0 \leq x \leq 4$. As it can be seen, despite the coincidence of the DTM and exact solutions in $[0, 1]$, the DTM solution diverges for $x > 1$. Therefore, when using the DTM, we have to check the

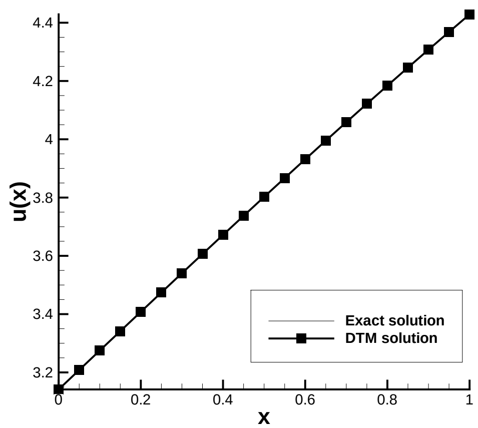

Figure 2: Exact and numerical solutions of Example 1 for $0 \leq x \leq 1$.

range of validity of the solution. In the next example, the implementation of the method on a BVP is discussed.

Example 2. Let us consider the following BVP:

$$(1+x^2)u'' + xu' - u = x^2, \tag{15}$$

$$u(0) = 1, \qquad u(1) = -\frac{\sqrt{5}}{6} + \frac{\sqrt{2}}{3} + 1. \tag{16}$$

The problem has the following unique solution:

$$u(x) = -\frac{\sqrt{5}}{6}x + \frac{1}{3}\sqrt{1+x^2} + \frac{1}{3}(2+x^2). \tag{17}$$

The corresponding equation in the transform space has the following form:

$$\sum_{i=0}^{k} \left(\delta(i-2) + \delta(i)\right)(k-i+1)(k-i+2)U(k-i+2) + \sum_{i=0}^{k} \delta(i-1)(k-i+1)U(k-i+1) - U(k) = \delta(k-2). \tag{18}$$

The first boundary condition leads to $U(0) = 1$, however the value of $U(1)$ is unknown and will be found by using the second boundary condition. Let us assume temporary that $U(1) = \alpha$. Then implementing the conditions and properties of Dirac delta function results in

$$U(0) = 1, \tag{19}$$

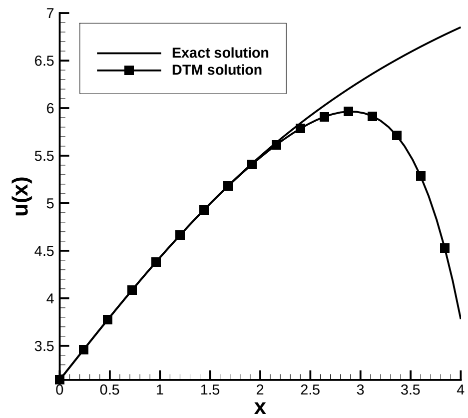

Figure 3: Exact and numerical solutions of Example 1 for $0 \leq x \leq 4$.

$$U(1) = \alpha, \tag{20}$$

$$U(2) = \frac{1}{2}, \tag{21}$$

$$U(3) = 0, \tag{22}$$

$$U(4) = -\frac{1}{24}, \tag{23}$$

$$U(k+2) = -\frac{k-1}{k+2}U(k), \qquad k \geq 3. \tag{24}$$

Therefore, the solution with $n = 8$ terms has the following form:

$$u_8(x) = 1.0000 + \alpha x + 0.5000x^2 - 0.0417x^4 + 0.0208x^6 - 0.0130x^8. \tag{25}$$

Now, we implement the second boundary condition at $x = 1$ to the above solution and find $\alpha = -0.3674$. The exact value of α, which is obtained from the exact solution, is -0.3727. The resulting DTM solution in this case is close to the exact one as depicted in Figure 4.

If the interval of the solution is extended to $[0, 2]$, with boundary condition $u(2) = 2$, then the exact solution remains unchanged and $\alpha = 0.8320$ differs from -0.3727, for the exact value of $u(2)$. Therefore, a large deviation from the exact solution is anticipated. When the two curves are compared (Figure 4), we can see that the behavior of the DTM solution differs from the exact one due to the power of x above 1. This is similar to the case of IVPs, except that here the constraint on the second point forces the solution to prevent large oscillations.

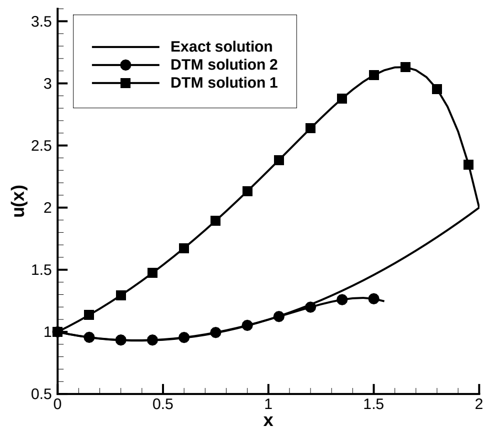

Figure 4: Exact and numerical solutions of Example 2 for $0 \leq x \leq 1$ and $0 \leq x \leq 2$.

3.1 Extensions and improvements

After the early implementation of DTM to initial and boundary value problems for ODEs, the researchers extended the method for other types of mathematical problems as encountered in Section 2. In the present section, some of these modifications are explained.

3.2 Multi-step DTM

As indicated in Example 2, the domain where the DTM solution is valid is usually narrow. In order to extend the solution for large intervals of independent variables, the multi-step DTM is proposed. The method is applied in sub-intervals instead of the entire domain. The solution is, indeed, a piecewise function of particular DTM solutions of the following form:

$$u(x) = \begin{cases} \sum_{k=0}^{N} U_0(k)x^k, & x \in [0, x_1], \\ \sum_{k=0}^{N} U_1(k)(x-x_1)^k, & x \in [x_1, x_2], \\ \vdots & \vdots \\ \sum_{k=0}^{N} U_p(k)(x-x_{p-1})^k, & x \in [x_{p-1}, x_p]. \end{cases} \tag{26}$$

The initial condition for each piece is obtained from the previous stage. Therefore, the method has errors but leads to better results compared to the traditional one. As an example, the multi-step DTM solution has been obtained for Example 1 as follows:

$$u(x) = \begin{cases} 3.1416 + 1.3333x - 0.0494x^3 + 0.0033x^5 - 0.0003x^7, & x \in [0,1], \\ 3.2705 + 1.0797x + 0.1086x^2 - 0.0230x^3 - 0.00066x^4 & \\ \quad - 0.0020x^5 + 0.0016x^6 - 0.0002x^7, & x \in [1,2], \\ 3.6890 + 0.7183x + 0.1688x^2 - 0.0518x^3 + 0.0326x^4 & \\ \quad - 0.0151x^5 + 0.0030x^6 - 0.0002x^7, & x \in [2,3], \\ 4.5191 + 0.6477x^2 - 0.3598x^3 + 0.1439x^4 - 0.0336x^5 & \\ \quad + 0.0040x^6 - 0.0002x^7, & x \in [3,4]. \end{cases} \tag{27}$$

Figure 5 Demonstrates the resulting multi-step DTM solution, the exact

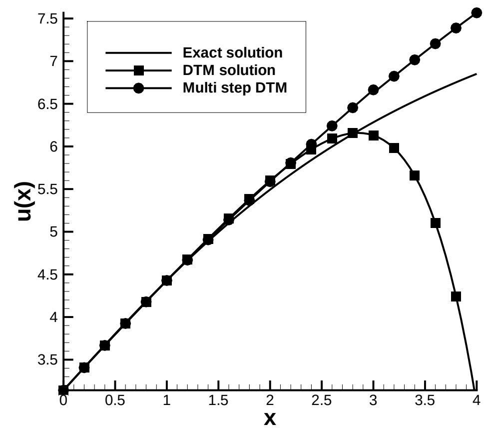

Figure 5: Exact, DTM, and multi-step DTM solutions of Example 1 for $0 \le x \le 4$.

and the one-step DTM solution. Comparing these curves elucidate that the multi-step DTM is more close to the exact solution and does not diverge like the traditional DTM solution.

3.3 Infinite horizons

There are BVPs with some conditions at infinity; that is, the domain of the independent variable is not bounded. In this subsection, two approaches of DTM when facing this situation are reviewed.

3.3.1 Padé Approximation

One of the well-known methods to approximate a real-valued function as a rational function is Padé approximation, which is usually used when simulating the behavior of a function at infinity is desired. This method has been combined with DTM to solve the infinite horizon BVP for the first time in [76]. Despite the application of this method in solving problems with conditions at infinity, such as [78, 75], it seems that this approach is not applicable. To illustrate this issue, let us consider the following rational function:

$$R_{L,M}(x) = \frac{p_0 + p_1 x + p_2 x^2 + \cdots + p_L x^L}{1 + q_1 x + q_2 x^2 + \cdots + q_L x^M}. \tag{28}$$

Moreover, $R_{L,M}$ is the Padé approximation of $u(x)$ if its value and derivatives coincide with those of $u(x)$ at $x = 0$, that is

$$R_{L,M}(0) = u(0), \tag{29}$$
$$R'_{L,M}(0) = u'(0), \tag{30}$$
$$R''_{L,M}(0) = u''(0), \tag{31}$$
$$\vdots$$
$$R^{(L+M)}_{L,M}(0) = u^{(L+M)}(0). \tag{32}$$

Therefore, in approximating $u(x) \approx R_{L,M}(x)$, the rational function has the initial value and derivatives as the main function. Also, far field behavior may be controlled with degrees of $R_{L,M}$.
Let us examine the method on a famous problem in fluid dynamics (see [76]):

$$u''' + uu'' - \beta {u'}^2 - Mu' = 0, \tag{33}$$
$$u(0) = 0, \quad u'(0) = 1, \tag{34}$$
$$u'(+\infty) = 0. \tag{35}$$

Clearly, by the application of DTM, a polynomial approximating the solution in the vicinity of $x = 0$ will result. However, the polynomial does not have marginal behavior at infinity as required by (35). To cope with the problem, after finding the DTM solution $u^n(x)$, a Padé approximation with $L+M = n$ is obtained. Therefore, the following relation should be occurred:

$$U(0)+U(1)x+U(2)x^2+\cdots+U(n)x^n = \frac{p_0 + p_1 x + p_2 x^2 + \cdots + p_L x^L}{1 + q_1 x + q_2 x^2 + \cdots + q_L x^M}. \tag{36}$$

Two initial conditions will translate to $U(0) = 0$ and $U(1) = 1$, however the degree of equation requires another initial condition. Therefore, $U(2)$ is taken as an unknown α, which will be determined from $u'(+\infty) = 0$ after finding the rational approximation. Then $L + M$ unknown coefficients a_0, a_1, ...,

a_L, b_1, b_2, …, b_M will be found by equating two sides up to x^{L+M}.

Studying the results of this method in [78] in detail shows that the method does not lead to valid solutions easily. For example, the following Padé-DTM solution is claimed in [76] for $\beta = 1.5$ and $M = 50$:

$$\begin{aligned} R_{10,10}(x) = (&x - 22.6935x^2 - 20.8798x^3 - 31.0628x^4 - 19.2098x^5 \\ &-0.841719x^6 + 16.6574x^7 + 1.82323x^8 + 5.78142x^9 \\ &-0.00715648x^{10})/(1 - 19.1112x - 97.9261x^2 - 202.307x^3 \\ &-222.828x^4 - 119.697x^5 + 11.8529x^6 + 70.5985x^7 + 55.4051x^8 \\ &+22.1859x^9 + 4.17935x^{10}) \end{aligned}$$

If we draw the above $u(x)$ near the origin with step size larger than 10^{-5}, then the solution agrees with physics as depicted in Figure 7 of [76]. However, when we take distance from $x = 0$, the solution shows different behavior. In Figure 6, the claimed solution is drawn for $0 \leq x \leq 2$ with step size $\Delta x = 0.01$. It has a clear jump near $1 = 1.4$, which is unexpected. Therefore, it cannot be the correct solution. If the step size of the graph is finer, then the amplitude of the jump increases, indicating a singularity in the rational function. When we examine the roots of the denominator, it reveals that it has two real roots, approximately at $x = 0.0423118$ and $x = 1.40647$. The first one is visible when the step size of plotting is smaller than 10^{-5}. Therefore, the resulting solution is not acceptable near the $x = 0$ nor beyond $x = 1$. This is just an example, and there are other examples showing the inefficiency of the Padé-DTM. Because of the following problems, using Padé

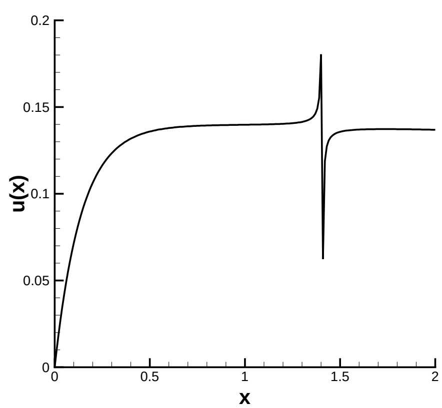

Figure 6: Jump of a Padé-DTM solution.

with DTM is not recommended in general:

- Rational functions may have singularities as indicated in the case study.

- In taking the derivative, the degree of the numerator and denominator will change. This may lead to $R_{L,M}(x) \to 0$ at infinity without obtaining a condition on α. As an example, as indicated in [21], for a Blasius problem, the Padé approximation does not match the required asymptotic behavior.
- Even the marginal condition of u at infinity satisfies, there is no guarantee that the resulting Padé has the same rate of convergence as the exact solution.

3.3.2 Switching DTM

Another DTM-based method for problems with a boundary condition at infinity was proposed in [61]. The method finds a solution consisting of two parts; the first part is a DTM solution, and the second part is a solution of the differential equation satisfying the marginal condition. The method has a successful implementation, but it is case-dependent in finding the second part of the solution.

3.4 Multidimensional DTM

One of the most important and practical extensions of DTM is multidimensional DTM. Let us consider, for example, a two-dimensional BVP or IVP having $u(t,x)$ as the solution. The two-dimensional extension of DTM transform $u(t,x) \xrightarrow{\mathrm{DT}} U(k,h)$ is defined as

$$U(k,h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h}u(t,x)}{\partial t^k x^h}\right]_{(0,0)}, \tag{37}$$

and the inverse transform is

$$u(t,x) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)t^k x^h. \tag{38}$$

Substituting (37) into the equations and applying the boundary-initial conditions will result in a set of algebraic equations. Then the equations are solved for $U(k,h)$, and the truncated inverse transform (38) gives an approximate solution. Some of the properties of the two-dimensional DTM transform used to build the algebraic equations are listed below. Assume that $u(t,x) \xrightarrow{\mathrm{DT}} U(k,h)$, that $w(t,x) \xrightarrow{\mathrm{DT}} W(k,h)$, and that λ is a constant scalar.

- $u(t,x) + w(t,x) \xrightarrow{\mathrm{DT}} U(k,h) + W(k,h)$.

- $\lambda u(t,x) \xrightarrow{\text{DT}} \lambda U(k,h)$.
- $u(x)w(x) \xrightarrow{\text{DT}} \sum_{i=0}^{k} \sum_{j=0}^{k} U(i,h-j)W(k-i,j)$.
- $\frac{\partial u(t,x)}{\partial t} \xrightarrow{\text{DT}} (k+1)U(k+1,h)$.
- $\frac{\partial u(t,x)}{\partial x} \xrightarrow{\text{DT}} (h+1)U(k,h+1)$.
- $\frac{\partial^{i+j} u(t,x)}{\partial t^i \partial x^j} \xrightarrow{\text{DT}} (k+1)(k+2)\cdots(k+i)(h+1)(h+2)\cdots(k+j)U(k+i,h+j)$.
- $t^i x^j \xrightarrow{\text{DT}} \delta(k-i,h-j)$.

Now, the two-dimensional DTM in its traditional form is applied to a problem.

Example 3. Let us consider the following IVP defined on the telegraph equation (see [18]):

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} + u, \tag{39}$$

$$u(0,x) = e^x, \qquad \frac{\partial u}{\partial t}(0,x) = -2e^x. \tag{40}$$

The exact solution is $u(t,x) = e^{x-2t}$.
The corresponding differential transform of (39) is as follows:

$$\begin{aligned}(h+1)(h+2)U(k,h+2) = {}& (k+1)(k+2)U(k+2,h) + 2(k+1)U(k+1,h) \\ & +U(k,h).\end{aligned} \tag{41}$$

Taking differential transform from two sides of initial conditions implies that

$$U(0,h) = \frac{1}{h!}, \tag{42}$$

$$U(1,h) = \frac{-2}{h!}. \tag{43}$$

Now, the following recursive relation is obtained to find the coefficients:

$$U(k+2,h) = \frac{(h+1)(h+2)U(k,h+2) - 2(k+1)U(k+1,h) - U(k,h)}{(k+1)(k+2)}. \tag{44}$$

Setting $k = 0, 1, 2$ in (44) and then $h = 0, 1, \ldots, 4$ in the results, we obtain the coefficients $U(k,h)$ and construct an approximate solution as

$$\begin{aligned}u_{4,4}(t,x) = {}& 1 + x + 0.5x^2 + 0.1667x^3 + 0.0417x^4 \\ & -2t - 2tx - tx^2 - 0.3333tx^3 - 0.0833tx^4 \\ & +2t^2 + 2t^2x + t^2x^2 + 0.3333t^2x^3 + 0.0833t^2x^4\end{aligned}$$

$$-1.3333t^3 - 1.3333t^3x - 0.6667t^3x^2 - 0.2222t^3x^3$$
$$-0.0556t^3x^4 + 0.6667t^4 + 0.6667t^4x + 0.3333t^4x^2$$
$$+0.1111t^4x^3 + 0.0278t^4x^4. \quad (45)$$

The absolute errors of the resulting DTM solution on an 8×8 grid are given in Table 1. The error is small near the initial condition, and the DTM solution approximates the exact solution. However, it grows slowly with t and x. The error may be reduced by increasing the number of terms in (45) since the DTM solution is convergent to the exact one in this case (see [18]).

Table 1: The absolute error of the DTM solution of Example 3

t, x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
0.0	0.0000	0.0000	0.0000	0.0000	0.0002	0.0005	0.0012	0.0027
0.1	0.0000	0.0000	0.0000	0.0000	0.0001	0.0004	0.0010	0.0021
0.2	0.0001	0.0002	0.0002	0.0002	0.0001	0.0001	0.0005	0.0014
0.3	0.0010	0.0011	0.0012	0.0014	0.0015	0.0015	0.0013	0.0008
0.4	0.0040	0.0045	0.0050	0.0056	0.0062	0.0068	0.0073	0.0077
0.5	0.0119	0.0133	0.0148	0.0166	0.0185	0.0205	0.0227	0.0249
0.6	0.0286	0.0320	0.0357	0.0399	0.0446	0.0497	0.0554	0.0615
0.7	0.0599	0.0669	0.0748	0.0836	0.0934	0.1043	0.1163	0.1296

3.4.1 Projected DTM

The projected DTM was introduced in [45] to reduce the computational complexity and simplify the solution in the case of multidimensional DTM. In this approach, the differential transform is applied on only one variable. Therefore, the coefficients are not constant and are functions of the remaining variables. For example, in Example 3, if we take the differential transform with respect to t, then the unknown coefficients are in the form of $U(h, x)$. Consequently, instead of (38), the solution has simpler form as

$$u(t, x) = \sum_{k=0}^{\infty} U(k, x)t^k, \quad (46)$$

which requires lower computational task. If we apply the method to Example 3, then the corresponding equation is changed to

$$\frac{\partial^2 U}{\partial x^2}(k, x) = (k+1)(k+2)U(k+2, x) + 2(k+1)U(k+1, x) + U(k, x) \quad (47)$$

with initial conditions:

$$U(0, x) = e^x, \quad (48)$$
$$U(1, x) = -2e^x. \quad (49)$$

The coefficients are also calculated from

$$U(k+2,x) = \frac{\frac{\partial^2 U}{\partial x^2}(k,x) - 2(k+1)U(k+1,x) - U(k,x)}{(k+1)(k+2)}. \tag{50}$$

Setting $k = 0, 1, \ldots$ and using the initial conditions, will result in $U(2,x) = 2e^x$, $U(3,x) = -\frac{4}{3}e^x$, $U(4,x) = \frac{1}{6}e^x$, $\ldots$. Then, the truncated solution up to 4 terms is

$$u_4^p(t,x) = e^x - 2e^x t + 2e^x t^2 - \frac{4}{3}e^x t^3 + \frac{1}{6}e^x. \tag{51}$$

From a computational viewpoint, calculating each coefficient in (44) requires 13 elementary operations, while in (44), nine operations are required. On the other hand, the number of terms in $u_{4,4}$ is 4 times than in u_4^p. Therefore, in this case, the projected DTM has lower complexity of order 5.78 with respect to the traditional DTM. However, it should be noted that when estimating the solution at a mesh on (t,x), the computation of e^x terms has more complexity than the power of x but is more accurate.

3.4.2 Reduced DTM

Another approach for simplifying and reducing the computational cost of multidimensional DTM is the reduced DTM proposed in [53]. This modification benefits from a separation of variables. The solution $u(t,x)$ in two-dimensional, for example, is written as

$$u(t,x) = f(t)g(x). \tag{52}$$

Then, one-dimensional DTM is applied, and the corresponding differential transform is obtained similar to the projected DTM.

4 Advantages and disadvantages of DTM

In the previous section, the implementation of DTM on a set of different problems was expressed. Based on the results of these examples and other references, the DTM has advantages and disadvantages as a semi-analytical method for solving initial and boundary value problems. In this section, we mention some of these advantages and disadvantages.

4.1 Advantages of DTM

The advantages of DTM may be encountered as follows:

- The solution has a closed form as a series. This enables us to use it quickly for more analysis, such as calculating derivatives, for example.
- DTM usually results in high-accuracy solutions in the domain of convergence.
- Low computational complexity in solving the transformed equations for linear systems.
- The method does not require discretization; therefore, the results are not affected by this type of error.
- Based on the literature review, the method is flexible to be adopted with various kinds of dynamical systems and boundary conditions.

4.2 Disadvantages of DTM

When using DTM, we have to care about the restrictions of the method. Some of the disadvantages of this method that restricts its application are listed below:

- The implementation of the method for nonlinear systems may lead to complex forms of the algebraic system of equations that restrict the implementation of the method to linear systems. There are, however, some approaches, such as the polynomial expansion of nonlinear terms or using Adomian decomposition. Such tricks may reduce the degree of nonlinearity, however, they add additional errors and increase computational complexity.
- The domain of convergence is usually small, and the results are valid close to $x = 0$. The multi-step DTM resolves this problem relatively. However, as inferred from Figure 5, the multi-step solution itself leads to accumulated errors that show the limited use of the method in short ranges.
- Documented efforts to extend DTM to infinite horizon problems, such as Padé approximation and switching DTM, do not guarantee valid and general solutions.

5 Concluding remarks

The method of differential transform was described and reviewed in this paper. Progress in the implementation, application, and improvements of DTM was expressed. The method gives an analytical solution that has advantages in comparison with the numerical methods for boundary/initial value problems. However, detailed investigations showed that the method has convergence restrictions. Indeed, when using DTM, it is important to note that the solution is accurate in an interval close to the initial conditions.

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