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Hopf bifurcation in a general n-neuron ring network with n time delays

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Abstract

In this paper, we consider a general ring network consisting of n neurons and n time delays. By analyzing the associated characteristic equation, a classification according to n is presented. It is investigated that Hopf bifurcation occurs when the sum of the n delays passes through a critical value. In fact, a family of periodic solutions bifurcate from the origin, while the zero solution loses its asymptotically stability. To illustrate our theoretical results, numerical simulation is given.

Keywords: Ring network; Stability; Periodic solution; Hopf bifurcation; Time delay.

1 Introduction

Since Hopfield constructed a simplified neural network (NN) model [7, 15], the dynamic behaviors (such as stability, periodic oscillatory, limit cycles, bifurcation and chaos) of continuous-time neural networks have received much attention due to their applications in optimization, signal processing, image processing, solving nonlinear algebraic equations, pattern recognition, associative memories and so on (see, [3, 10, 11, 18] and references therein). It is well known that time delays exist in the signal transmission, thus Marcus and Westervelt proposed an NN model with delays, based on the Hopfield NN model [12]. The time delays are regarded as parameters; consequently, periodic solutions often appear as solutions of delay differential equations (DDEs) through Hopf bifurcation. For example, existence of periodic solutions, in a special type of DDEs, has been discussed in [9]. In [8, 16, 17], the authors used Hopf bifurcation theory to study some kinds of neural networks.

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A ring network is a network topology in which each node connects to exactly two other nodes, forming a single continuous pathway for signals through each node a ring. Data travel from node to node, with each node along the way handling every packet.

Ring networks have been found in a variety of neural structures such as cerebellum [5], and even in chemistry and electrical engineering. In the field of neural networks, rings are studied to gain insight into the mechanisms underlying the behavior of recurrent networks. In [16], Wang and Han investigated the continuous-time bidirectional ring network model. Many results have been reported in the literature on the dynamics of ring neural networks (see [2, 18]). The local and global stability of ring networks with delays have been discussed in [2, 18]. In [1], stability analysis of a delayed ring network model has been discussed.

In this paper, we study Hopf bifurcation for a kind of DDE system. In fact, we consider a general ring network with n neurons and n time delays, which is described by the following DDE system:

$$\begin{cases} \dot{x}_1(t) = -r_1 x_1(t) + g_1(x_1(t)) + f_1(x_n(t-\tau_n)), \\ \dot{x}_2(t) = -r_2 x_2(t) + g_2(x_2(t)) + f_2(x_1(t-\tau_1)), \\ \dot{x}_3(t) = -r_3 x_3(t) + g_3(x_3(t)) + f_3(x_2(t-\tau_2)), \\ \vdots \\ \dot{x}_{n-1}(t) = -r_{n-1} x_{n-1}(t) + g_{n-1}(x_{n-1}(t)) + f_{n-1}(x_{n-2}(t-\tau_{n-2})), \\ \dot{x}_n(t) = -r_n x_n(t) + g_n(x_n(t)) + f_n(x_{n-1}(t-\tau_{n-1})), \end{cases} \tag{1}$$

where $r_i \geq 0$ $(i = 1, 2, ..., n)$ denotes the stability of internal neuron processes, and $x_i(t)$ $(i = 1, 2, ..., n)$ represents the state of the ith neuron at time t. f_i and g_i $(i = 1, 2, ..., n)$ are the activation function and nonlinear feedback function, respectively. Also, $\tau_i \geq 0$ $(i = 1, 2, ...n)$ describes the synaptic transmission delay. Figure 1 shows system (1) schematically.

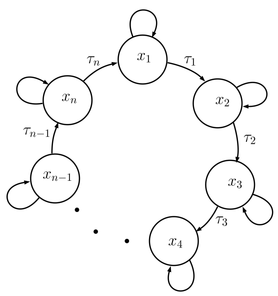

Figure 1: A general n-neuron ring with n delays

The properties of periodic solutions are also important in many applications. In fact, various local periodic solutions can arise from the different equilibrium points of ring networks by applying Hopf bifurcation technique, therefore the study of Hopf bifurcation is very important. In [16], the simplified three-neuron bidirectional ring network has been investigated. Four-neuron and five-neuron networks with multiple delays have been studied in [8, 17].

In this paper, we discuss Hopf bifurcation on system (1) generally, not for a particular n. To the best of our knowledge, it has not been done before. We would like to point out that in [1], although stability analysis of the system has been presented, but the authors didn't discuss Hopf bifurcation on the system in all cases. First, we take the sum of the delays $\tau_1 + \tau_2 + ... + \tau_n$ as a parameter τ. By classifying based on n, we study the associated characteristic equation. To calculate the critical value τ_0 for Hopf bifurcation, we generalize and modify the methods proposed in [8, 9]. Then we will show that the zero solution loses its stability and Hopf bifurcation occurs when τ passes through the critical value τ_0. We would like to point out that it is the first time to deal with Hopf bifurcation analysis of system (1). This paper is organized in five sections. In Sect.2, we give the necessary preliminaries. In Sect.3, we will study Hopf bifurcation on system (1). To illustrate the results, some numerical simulations are presented in Sect.4. Finally, in Sect.5, some main conclusions are stated.

2 Preliminaries

2.1 Delay Differential Equation

There is always a time delay in many natural phenomena, because a finite time is required to sense information and then react to it. DDEs are differential equations in which the derivatives of some unknown functions at present time are dependent on the values of the functions at previous times. A general delay differential equation for $x(t) \in \mathrm{R}^{\mathrm{n}}$ takes the form

$$\dot{x}(t) = f(t, x(t), x_t), \tag{2}$$

where $x_t(\theta) = x(t + \theta)$ and $-\tau \leq \theta \leq 0$. Observe that $x_t(\theta)$ with $-\tau \leq \theta \leq 0$ represents a portion of the solution trajectory in a recent past. In this equation f is a functional operator from $\mathrm{R} \times \mathrm{R}^{\mathrm{n}} \times C^1(\mathrm{R}, \mathrm{R}^{\mathrm{n}})$ to R^{n}. Similar to ODEs, many properties of linear DDEs can be characterized and analyzed using the characteristic equation.

The linearization of system (2) at the equilibrium point x_0 is

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau_1) + \cdots + A_m x(t - \tau_m), \tag{3}$$

where $A_j = D_{j+1}f(x_0, ..., x_0), (j = 0, 1, ..., m)$ and $D_j f$ is the Jacobian of f corresponding to its jth component.

Substitude $x(t) = e^{\lambda t}v$, $v \in R^n$ into (3), we have

$$[\lambda I - A_0 - \sum_{j=1}^{m} A_j e^{-\lambda \tau_j}]e^{\lambda t}v = 0.$$

Therefore the characteristic equation associated with (2) is

$$det(\lambda I - A_0 - \sum_{j=1}^{m} A_j e^{-\lambda \tau_j}) = 0. \tag{4}$$

For further details, see [4].

2.2 Hopf bifurcation

In this section, we study bifurcations that occur in C^1 system

$$\dot{x} = f(x, \mu), \tag{5}$$

depending on a parameter $\mu \in \mathrm{R}$, at nonhyperbolic equilibrium points. In the following, we will give definition of a structurally stable vector field or dynamical system.

Definition 2.1. *Let E be an open subset of* $\mathrm{R^n}$. *A vector field $f \in C^1(E)$ is said to be structurally stable if there is an $\varepsilon > 0$ such that for all $g \in C^1(E)$ with $\|f - g\| < \varepsilon$, f and g are topologically equivalent on E; i.e., there is a homeomorphism $H : E \to E$ which maps trajectories of $\dot{x} = f(x)$ onto trajectories of $\dot{x} = g(x)$, and preserves their orientation by time. In this case, we also say that the dynamical system $\dot{x} = f(x)$ is structurally stable. If a vector field $f \in C^1(E)$ is not structurally stable, then f is said to be structurally unstable.*

The qualitative behavior of the solution set of system (5) depending on a parameter $\mu \in \mathrm{R}$, changes as the vector field f passes through a point in the bifurcation set or as the parameter μ varies through a bifurcation value μ_0. A value μ_0 of the parameter μ in equation (5) for which the C^1 vector field $f(x, \mu_0)$ is not structurally stable is called a bifurcation value. For more information see [13].

Theorem 2.2. *Suppose that system* (5), *has an equilibrium* (x_0, μ_0) *at which the following properties are satisfied:*

a *: $D_x f_{\mu_0}(x_0)$ has a simple pair of pure imaginary eigenvalues and other eigenvalues have negative real parts. Therefore, there exist a smooth* *curve of equilibria* $(x(\mu),\mu)$ *with* $x(\mu_0)=x_0$. *The eigenvalues* $\lambda(\mu)$, $\bar{\lambda}(\mu)$ *of* $D_x f_{\mu_0}(x(\mu))$ *which are imaginary at* $\mu=\mu_0$ *vary smoothly with* μ.

Furthermore, if,

$$\mathbf{b}\ :\ \frac{d}{d\mu}(Re\lambda(\mu))|_{\mu=\mu_0}=d\neq 0,$$

then Hopf bifurcation will occur.

Proof. see [6]. □

When the parameter μ passes through the critical value μ_0, according theorem 2.2, Hopf bifurcation occurs. In fact, a family of periodic solutions appear or disappear. If the equilibrium point is asymptotically stable for $\mu<\mu_0$, when μ passes through μ_0, a family periodic solutions bifurcate from the equilibrium point. In this case, we say that supercritical Hopf bifurcation occurs. But, if a branch of periodic solutions exist for $\mu<\mu_0$ and while μ passes through μ_0, these periodic solutions disappear, we say that subcritical Hopf bifurcation happens. For more details, see [6].

3 Main results

To establish the main results for system (1), it is necessary to make the following assumption

$$f_i, g_i\in C^1,\ f_i(0)=g_i(0)=0,\ \text{for } i=1,2,...,n. \tag{6}$$

It is easily seen that the origin (0,0,...,0) is an equilibrium point of (1). Under the hypothesis (6), the linearization of (1) around the origin gives

$$\begin{cases}\dot{x}_1(t)=-k_1x_1(t)+f_1^{'}(0)x_n(t-\tau_n),\\ \dot{x}_2(t)=-k_2x_2(t)+f_2^{'}(0)x_1(t-\tau_1),\\ \vdots\\ \dot{x}_{n-1}(t)=-k_{n-1}x_{n-1}(t)+f_{n-1}^{'}(0)x_{n-2}(t-\tau_{n-2}),\\ \dot{x}_n(t)=-k_nx_n(t)+f_n^{'}(0)x_{n-1}(t-\tau_{n-1}),\end{cases} \tag{7}$$

where $k_i=r_i-g_i^{'}(0)$, $i=1,2,...,n$. The characteristic equation of (2) is

$$\lambda^n+a_1\lambda^{n-1}+...+a_{n-1}\lambda+a_n+be^{-\lambda\tau}=0, \tag{8}$$

where

$$a_1 = \sum_{i=1}^{n} k_i, \quad a_2 = \sum_{1 \leqslant i < j \leqslant n} k_i k_j, \quad \cdots,$$
$$a_{n-1} = \sum_{1 \leqslant \underbrace{i < j < l < \ldots < m}_{\text{n-1}} \leqslant n} k_i k_j k_l \ldots k_m,$$
$$a_n = \prod_{i=1}^{n} k_i, \quad b = -\prod_{i=1}^{n} f_i^{'}(0), \quad \tau = \sum_{i=1}^{n} \tau_i. \tag{9}$$

Denote

$$p(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n,$$

then equation (8) becomes

$$p(\lambda) + be^{-\lambda\tau} = 0. \tag{10}$$

To study Hopf bifurcation, it is necessary to discuss the existence of pure imaginary roots of (10). Letting $\lambda = i\omega$, and substituting this into (10), we have

$$A + iB + b(cos\omega\tau - isin\omega\tau) = 0, \tag{11}$$

where

$$A = Re\{p(i\omega)\}, \quad B = Im\{p(i\omega)\}. \tag{12}$$

Separating the real and imaginary parts of (11), we get

$$A + bcos\omega\tau = 0, \tag{13}$$

and

$$B - bsin\omega\tau = 0. \tag{14}$$

We rewrite the equations (13) and (14), as follows:

$$A = -bcos\omega\tau, \tag{15}$$

and

$$B = bsin\omega\tau. \tag{16}$$

Squaring both sides of (15) and (16), and adding them up gives

$$A^2 + B^2 = b^2. \tag{17}$$

Now, according to (12), it is easy to use computer to calculate the roots of (17). Then we can get the time delay τ, by substituting ω in (15).

To find the solutions for equation (17), we consider the following cases:

Case (a) : $n = 4k$ $(k \in \mathbb{N})$,

Case (b) : $n = 4k + 1$ $(k \in \mathbb{N})$,

Case (c) : $n = 4k + 2$ $(k \in \mathbb{N} \cup \{0\})$,

Case (d) : $n = 4k + 3$ $(k \in \mathbb{N} \cup \{0\})$,

In case (a), from (12) and the definition of $p(\lambda)$, we can get

$$\begin{cases} A = \omega^n - a_2\omega^{n-2} + a_4\omega^{n-4} - ... + a_n, \\ B = -a_1\omega^{n-1} + a_3\omega^{n-3} - a_5\omega^{n-5} + ... + a_{n-1}\omega. \end{cases} \tag{18}$$

Using (17) and (18), we obtain

$$(\omega^n - a_2\omega^{n-2} + a_4\omega^{n-4} - ... + a_n)^2 + (-a_1\omega^{n-1} + a_3\omega^{n-3} - a_5\omega^{n-5} + ... + a_{n-1}\omega)^2 = b^2. \tag{19}$$

In case (b), the definition of $p(\lambda)$ and (12) lead to

$$\begin{cases} A = a_1\omega^{n-1} - a_3\omega^{n-3} + a_5\omega^{n-5} - ... + a_n, \\ B = \omega^n - a_2\omega^{n-2} + a_4\omega^{n-4} - ... + a_{n-1}\omega. \end{cases} \tag{20}$$

Substituting (20) in (17) gives

$$(a_1\omega^{n-1} - a_3\omega^{n-3} + a_5\omega^{n-5} - ... + a_n)^2 + (\omega^n - a_2\omega^{n-2} + a_4\omega^{n-4} - ... + a_{n-1}\omega)^2 = b^2. \tag{21}$$

In case (c), A and B can be calculated as follows:

$$\begin{cases} A = -\omega^n + a_2\omega^{n-2} - a_4\omega^{n-4} + ... + a_n, \\ B = a_1\omega^{n-1} - a_3\omega^{n-3} + a_5\omega^{n-5} - ... + a_{n-1}\omega. \end{cases} \tag{22}$$

From the equations (17) and (22), we have

$$(-\omega^n + a_2\omega^{n-2} - a_4\omega^{n-4} + ... + a_n)^2 + (a_1\omega^{n-1} - a_3\omega^{n-3} + a_5\omega^{n-5} - ... + a_{n-1}\omega)^2 = b^2. \tag{23}$$

In case (d), from the definition of $p(\lambda)$ and (12), we can compute

$$\begin{cases} A = -a_1\omega^{n-1} + a_3\omega^{n-3} - a_5\omega^{n-5} + ... + a_n, \\ B = -\omega^n + a_2\omega^{n-2} - a_4\omega^{n-4} + ... + a_{n-1}\omega. \end{cases} \tag{24}$$

By using the equations (17) and (24), we can get

$$(-a_1\omega^{n-1} + a_3\omega^{n-3} - a_5\omega^{n-5} + ... + a_n)^2 + (-\omega^n + a_2\omega^{n-2} - a_4\omega^{n-4} + ... + a_{n-1}\omega)^2 = b^2. \tag{25}$$

In all the above cases, after simplification, we can easily see that the equations (19), (21), (23) and (25) lead to

$$\omega^{2n} + e_1\omega^{2n-2} + e_2\omega^{2n-4} + ... + e_{n-1}\omega^2 + e_n = 0, \tag{26}$$

where the coefficients e_i $(i = 1, 2, ..., n)$, can be calculated as follows:

$$e_1 = a_1^2 - 2a_2,\ e_2 = a_2^2 - 2a_1a_3,\ \cdots,\ e_{n-1} = a_{n-1}^2 - 2a_na_{n-2},\ e_n = a_n^2 - b^2. \tag{27}$$

Letting $z = \omega^2$, then equation (26) becomes

$$z^n + e_1z^{n-1} + e_2z^{n-2} + ... + e_{n-1}z + e_n = 0. \tag{28}$$

Thus, the fact that equation (28) has positive roots is a necessary condition for the existence of the pure imaginary roots of equation (8).

Let

$$h(z) = z^n + e_1z^{n-1} + e_2z^{n-2} + ... + e_{n-1}z + e_n. \tag{29}$$

In the following, we will give lemma to establish the distribution of positive real roots of equation (28).

Lemma 3.1. *If $e_n < 0$, then equation (12) has at least one positive root.*

Proof. Since $h(0) = e_n < 0$ and $lim_{z\to\infty}h(z) = +\infty$. Hence, there exists a $z_0 > 0$ such that $h(z_0) = 0$. □

Now, from (15) we get

$$\tau_k^{(j)} = \begin{cases} \frac{1}{\omega_k}(cos^{-1}(\frac{-A}{b}) + 2j\pi) \text{ if } \frac{B}{b} \geq 0 \\ \frac{1}{\omega_k}(2\pi - cos^{-1}(\frac{-A}{b}) + 2j\pi) \text{ if } \frac{B}{b} < 0 \end{cases} (j = 0, \pm1, \pm2, ...; k = 1, 2, ..., n), \tag{30}$$

where $w_k = \sqrt{z_k^*}$, and without loss of generality, z_k^* $(k = 1, ..., n)$ are the positive roots of (28). Therefore, we can define

$$\tau_0 = \tau_{k_0}^{(0)} = \min_{k\in\{1,2,...,n\}} \{\tau_k^{(0)}\}, \quad \omega_0 = \omega_{k_0}. \tag{31}$$

So, with the help of relations (17) and (31), ω_0 and τ_0 are obtained. Now, we show that system (1) undergoes Hopf bifurcation at the origin when $\tau = \sum_{i=1}^n \tau_i$ passes through τ_0. In all cases, the stability and Hopf bifurcation can be analyzed analogously.

Let

$$\lambda(\tau) = \alpha(\tau) + i\omega(\tau), \tag{32}$$

be the root of equation (8) near $\tau = \tau_k^{(j)}$ satisfying $\alpha(\tau_k^{(j)}) = 0$, $\omega(\tau_k^{(j)}) = \omega_k$. Then, the following lemma holds.

Lemma 3.2. *Suppose $h'(z_k^*) \neq 0$, where $h(z)$ is defined by (4) and $z_k^* = \omega_k^2$. Then $\pm i\omega_k$ is a pair of simple purely imaginary roots of equation (3) when $\tau = \tau_k^{(j)}$. Moreover,*

$$\frac{dRe\big(\lambda(\tau)\big)}{d\tau}|_{\tau=\tau_k^{(j)}} \neq 0.$$

Proof. From (17) and (12), equation (26) can be transformed into the following form

$$p(i\omega)\overline{p(i\omega)} - b^2 = 0. \tag{33}$$

From (29), we have

$$h(\omega^2) = p(i\omega)\overline{p(i\omega)} - b^2. \tag{34}$$

Differentiating both sides of equation (34) with respect to ω, we obtain

$$2\omega h^{'}(\omega^2) = i[p^{'}(i\omega)\overline{p(i\omega)} - p(i\omega)\overline{p^{'}(i\omega)}]. \tag{35}$$

If $i\omega_k$ is not simple, then ω_k must satisfy

$$\frac{d}{d\lambda}[p(\lambda) + be^{-\lambda\tau_k^{(j)}}]\big|_{\lambda=i\omega_k} = 0,$$

that implies

$$p^{'}(i\omega_k) + b(-\tau_k^{(j)})e^{-i\tau_k^{(j)}\omega_k} = 0.$$

Together with equation (10), we have

$$\tau_k^{(j)} = -\frac{p^{'}(i\omega_k)}{p(i\omega_k)}. \tag{36}$$

Thus, by (33), (35) and (36), we obtain

$$Im(\tau_k^{(j)}) = Im\Big\{ -\frac{p^{'}(i\omega_k)}{p(i\omega_k)}\Big\} = Im\Big\{ -\frac{p^{'}(i\omega_k)\overline{p(i\omega_k)}}{p(i\omega_k)\overline{p(i\omega_k)}}\Big\}$$

$$= i\frac{p^{'}(i\omega_k)\overline{p(i\omega_k)} - \overline{p^{'}(i\omega_k)}p(i\omega_k)}{2p(i\omega_k)\overline{p(i\omega_k)}} = \frac{\omega_k h^{'}(\omega_k^2)}{|p(i\omega_k)|^2}.$$

Since $\tau_k^{(j)}$ is real, i.e. $Im(\tau_k^{(j)}) = 0$, we have $h^{'}(z_k^*) = h^{'}(\omega_k^2) = 0$. Therefore, we get a contradiction to the condition $h^{'}(z_k^*) \neq 0$. This proves the first conclusion in the lemma. Differentiating both sides of equation (10) with respect to τ, we obtain

$$\big[p^{'}(\lambda) - b\tau e^{-\lambda\tau}\big]\frac{d\lambda}{d\tau} - b\lambda e^{-\lambda\tau} = 0,$$

which implies

$$\frac{d\lambda(\tau)}{d\tau} = \frac{b\lambda}{p^{'}(\lambda)e^{\lambda\tau} - b\tau} = \frac{b\lambda\big[\overline{p^{'}(\lambda)e^{-\lambda\tau} - b\tau}\big]}{|p^{'}(\lambda)e^{\lambda\tau} - b\tau|^2} = \frac{\lambda\big[-\overline{p^{'}(\lambda)}p(\lambda) - b^2\tau\big]}{|p^{'}(\lambda)e^{\lambda\tau} - b\tau|^2}.$$

It follows together with (35) that

$$\frac{d(Re\lambda(\tau))}{d\tau}\Big|_{\tau=\tau_k^{(j)}} = \frac{Re\Big\{\lambda\big[-\overline{p'(\lambda)}p(\lambda)-b^2\tau\big]\Big\}}{|p'(\lambda)e^{\lambda\tau}-b\tau|^2}\Big|_{\tau=\tau_k^{(j)}}$$
$$= \frac{i\omega_k\big[+p'(i\omega_k)\overline{p(i\omega_k)}-\overline{p'(i\omega_k)}p(i\omega_k)\big]}{2|p'(i\omega_k)e^{i\omega_k\tau_k^{(j)}}-b\tau_k^{(j)}|^2}$$
$$= \frac{\omega_k^2 h'(\omega_k^2)}{|p'(i\omega_k)e^{i\omega_k\tau_k^{(j)}}-b\tau_k^{(j)}|^2}$$
$$= \frac{\omega_k^2 h'(z_k^*)}{|p'(i\omega_k)e^{i\omega_k\tau_k^{(j)}}-b\tau_k^{(j)}|^2} \neq 0.$$

Thus, $\frac{dRe(\lambda(\tau))}{d\tau}\Big|_{\tau=\tau_k^{(j)}} \neq 0$. This completes the proof. □

By the well-known Routh-Hurwitz criteria, we can find the following set of conditions:

$$H_1=det\begin{pmatrix}a_1\end{pmatrix}>0, H_2=det\begin{pmatrix}a_1 & 1\\ a_3 & a_2\end{pmatrix}>0, ..., H_n=det\begin{pmatrix}a_1 & 1 & 0 & 0\dots & 0\\ a_3 & a_2 & a_1 & 0\dots & 0\\ \vdots & \vdots & \vdots & \vdots\dots & \vdots\\ 0 & 0 & 0 & 0\dots & a_n\end{pmatrix}>0. \quad (37)$$

To discuss the distribution of the roots of the exponential polynomial equation (8), we need the following result from Ruan and Wei [14].

Theorem 3.3. *Consider the exponential polynomial*

$$p(\lambda, e^{-\lambda\tau_1}, ..., e^{-\lambda\tau_m}) = \lambda^n + p_1^{(0)}\lambda^{n-1} + .. + p_{n-1}^{(0)}\lambda + p_n^{(0)}$$
$$+[p_1^{(1)}\lambda^{n-1}+...+p_{n-1}^{(1)}\lambda+p_n^{(1)}]e^{-\lambda\tau_1}$$
$$+ ... + [p_1^{(m)}\lambda^{n-1} + ... + p_{n-1}^{(m)}\lambda + p_n^{(m)}]e^{-\lambda\tau_m},$$

where $\tau_i \geq 0$ $(i = 1, 2, ..., m)$, and $p_j^{(i)}$ $(i = 0, 1, 2, ..., m; j = 1, 2, ..., n)$ are constants. As $(\tau_1, \tau_2, ..., \tau_m)$ vary, the sum of the orders of the zeros of $p(\lambda, e^{-\lambda\tau_1}, ..., e^{-\lambda\tau_m})$ on the open right half plane can change only if a zero appears on or crosses the imaginary axis.

Now, we can state the following main theorem:

Theorem 3.4. *Suppose that conditions (1), (5) and $h'(z_k^*) \neq 0$ hold, where $h(z)$ is defined by (4) and $z_k^* = \omega_k^2$. Then, system (1) undergoes Hopf bifurcation at the origin when $\tau = \sum_{i=1}^n \tau_i$ passes through τ_0, and it has a branch of periodic solutions bifurcating from the zero solution near $\tau = \tau_0$, where τ_0 is defined by (31).*

Proof. By the well-known Routh-Hurwitz criteria, we can conclude that when (37) holds, all the roots of equation (8) at $\tau = 0$, have negative real parts.

Hence, by using theorem 3.3, we conclude that the zero solution of system (1) is asymptotically stable when $\tau < \tau_0$. By using lemma 3.2, we can see that the conditions of Hopf bifurcation are satisfied at $\tau = \tau_0$ in system (1), and so Hopf bifurcation occurs at the origin. In addition, a family of periodic solutions appear as τ passes through τ_0. □

4 Numerical simulations

In this section, numerical simulations are presented to support our theoretical results. We will study system (1) for the case $n = 7$. Consider the following system

$$\begin{cases} \dot{x}_1(t) = -2x_1(t) + tanh(x_1(t)) + 2tanh(x_7(t-\tau_7)), \\ \dot{x}_2(t) = -2x_2(t) + tanh(x_2(t)) + tanh(x_1(t-\tau_1)), \\ \dot{x}_3(t) = -2x_3(t) + tanh(x_3(t)) + 1.2tanh(x_2(t-\tau_2)), \\ \dot{x}_4(t) = -2x_4(t) + tanh(x_4(t)) + tanh(x_3(t-\tau_3)), \\ \dot{x}_5(t) = -2x_5(t) + tanh(x_5(t)) - 0.5tanh(x_4(t-\tau_4)), \\ \dot{x}_6(t) = -2x_6(t) + tanh(x_6(t)) + 0.5tanh(x_5(t-\tau_5)), \\ \dot{x}_7(t) = -2x_7(t) + tanh(x_7(t)) + 2tanh(x_6(t-\tau_6)), \end{cases} \tag{38}$$

which has the origin as an equilibrium point. By equations (8) and (9), we obtain the associated characteristic equation for $n = 7$:

$$\lambda^7 + 7\lambda^6 + 21\lambda^5 + 35\lambda^4 + 35\lambda^3 + 21\lambda^2 + 7\lambda + 1 + 1.2e^{-\lambda\tau} = 0. \tag{39}$$

By the equations (30) and (31), we can compute $\tau_0 = 6.7067$. Choosing $\tau_1 = 0.8$, $\tau_2 = 1$, $\tau_3 = 0.6$, $\tau_4 = 1$, $\tau_5 = 1.1$, $\tau_6 = 0.7$ and $\tau_7 = 1$, Figure 2 shows that the origin is asymptotically stable, and the phase portraits for system (38) are shown in Figure 4. When $\tau = \sum_{i=1}^{7} \tau_i$ passes through the critical value $\tau_0 = 6.7067$, the origin loses its stability and Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcate from the origin. Choosing $\tau_1 = 0.9$, $\tau_2 = 1$, $\tau_3 = 0.8$, $\tau_4 = 1$, $\tau_5 = 1.3$, $\tau_6 = 0.7$ and $\tau_7 = 1.1$, Hopf bifurcation happens at the zero solution. In Figure 3, the bifurcating periodic solutions are presented, and the phase portraits for system (38) are shown in Figure 5. Hopf bifurcation is supercritical and the bifurcating periodic solutions exist for $\tau > \tau_0$.

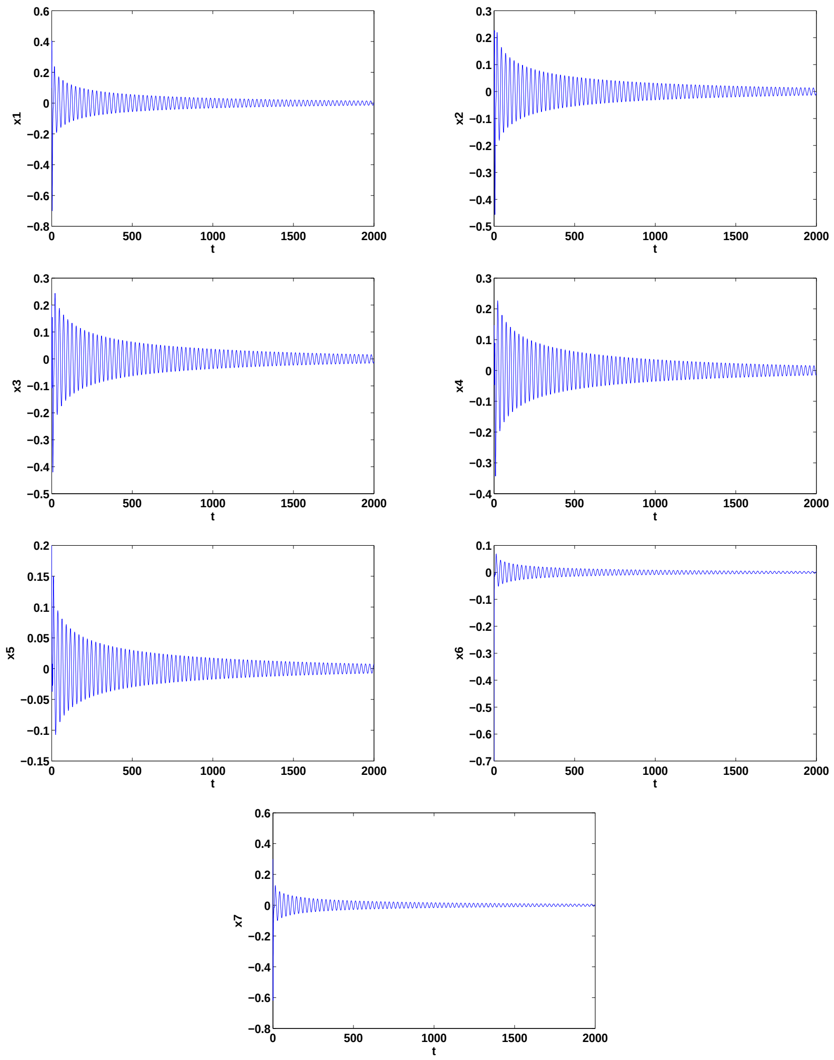

Figure 2: The origin is asymptotically stable while $\tau_1 = 0.8$, $\tau_2 = 1$, $\tau_3 = 0.6$, $\tau_4 = 1$, $\tau_5 = 1.1$, $\tau_6 = 0.7$ and $\tau_7 = 1$

5 Conclutions

In this paper, we investigated the dynamics of a general class of ring networks with n neurons and n time delays. We have chosen $\tau = \tau_1 + \tau_2 + ... + \tau_n$ as a bifurcation parameter and analyzed the corresponding characteristic equation. Then we have proved that the zero solution (the equilibrium point of the

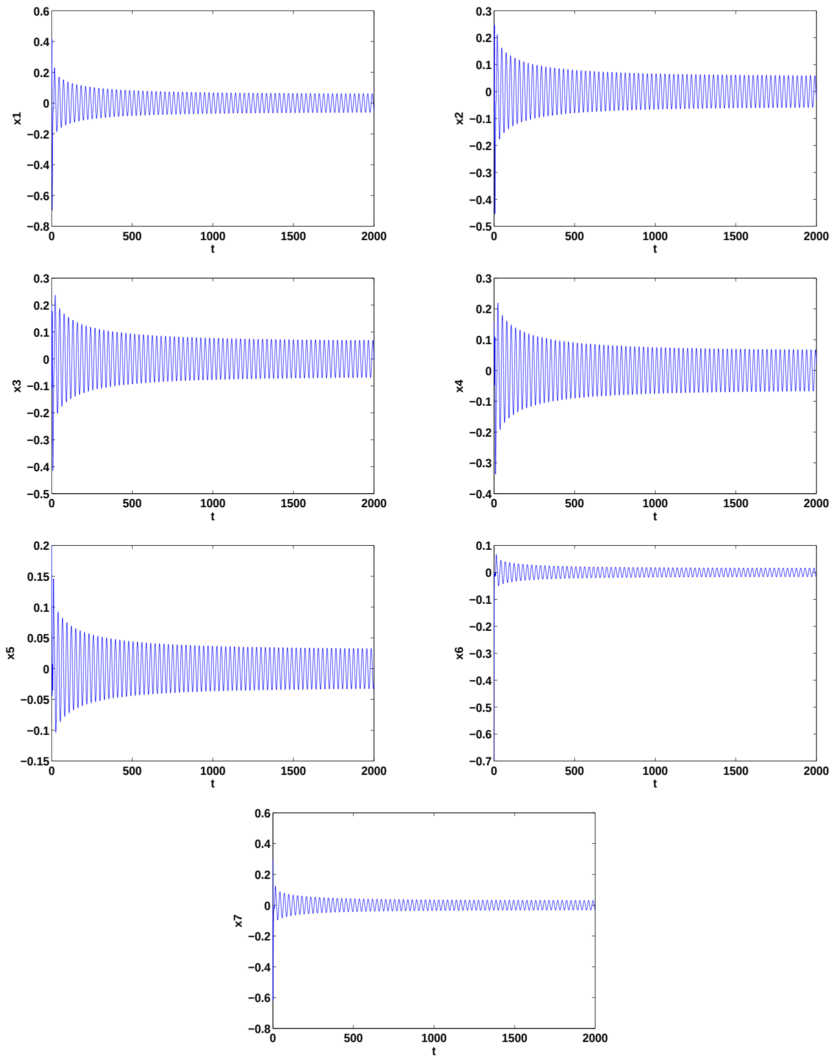

Figure 3: A family of periodic solutions bifurcates from the origin when $\tau_1 = 0.9$, $\tau_2 = 1$, $\tau_3 = 0.8$, $\tau_4 = 1$, $\tau_5 = 1.3$, $\tau_6 = 0.7$ and $\tau_7 = 1.1$

system) loses its stability and Hopf bifurcation occurs. Therefore, a family of periodic solutions bifurcate from the zero solution when τ passes through a critical value τ_0. Finally, the results have been validated by numerical simulations.

At the end, we would like to point out that it is so significant to study the networks in general case, not for a special value of n. Although, in this

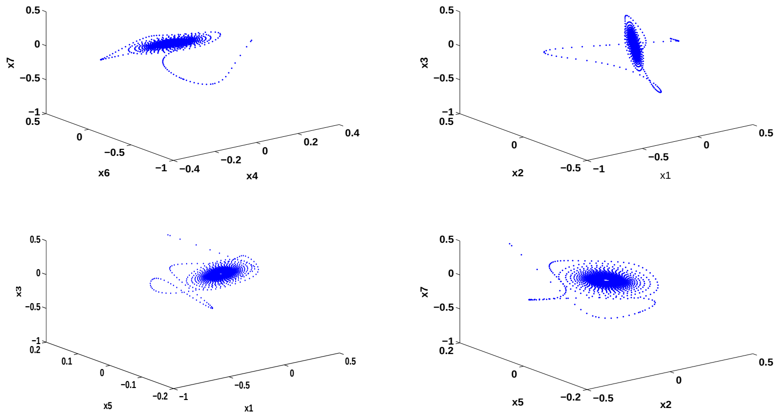

Figure 4: The phase portraits while $\tau_1 = 0.8$, $\tau_2 = 1$, $\tau_3 = 0.6$, $\tau_4 = 1$, $\tau_5 = 1.1$, $\tau_6 = 0.7$ and $\tau_7 = 1$

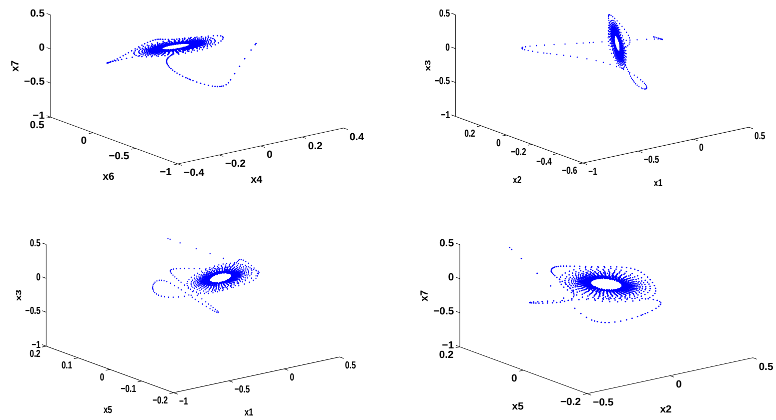

Figure 5: The phase portraits when $\tau_1 = 0.9$, $\tau_2 = 1$, $\tau_3 = 0.8$, $\tau_4 = 1$, $\tau_5 = 1.3$, $\tau_6 = 0.7$ and $\tau_7 = 1.1$

paper, we considered a general kind of ring networks, the methods, we have proposed can be generalized to be applied for other kinds of neural networks. We leave this as the future research.

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بررسی انشعاب هاف در شبکه حلقه ای n نرونی با n تاخیر زمانی

الهام جاویدمنش و محسن خورشیدی

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چکیده : در این مقاله، یک شبکه حلقه ای n نرونی با n تاخیر زمانی را در نظر می گیریم و به مطالعه ی انشعاب هاف سیستم حاصل از این شبکه می پردازیم. با طبقه بندی بر حسب باقیمانده های تقسیم n بر ۴، به مطالعه ی معادله ی مشخصه حاصل از این دسته بندی می پردازیم. مجموع تاخیرها را به عنوان پارامتر سیستم در نظر می گیریم. نشان می دهیم وقتی پارامتر تاخیر از یک مقدار بحرانی می گذرد انشعاب هاف رخ می دهد. در واقع، زمانی که نقطه تعادل سیستم (جواب صفر) پایداری مجانبی اش را از دست می دهد، یک خانواده از جواب های دوره ای از مبدا منشعب می شوند.

کلمات کلیدی : شبکه حلقه ای؛ پایداری؛ جواب های دوره ای؛ انشعاب هاف؛ تاخیر زمانی.